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Puzzles
Blockades, Carrier Missions, Secret Intelligence, and More

Barry Nalebuff

As usual, this feature begins with several speed puzzles; answers can be found at the end of the problems. Following is one longer puzzle for which readers are invited —nay, challenged—to submit their own answer. The puzzles in this issue focus on everything from diversifying strategies in naval warfare to ordering in restaurants to finding the quickest way around Delhi. The column ends with reader mail, including submitted solutions to Schelling's Poison Pill and the fairest way to score a contest.

Please send your answers, comments and favorite puzzles to me directly: Barry Nalebuff, "Puzzles," Department of Economics, Princeton University, Princeton, NJ 08544. Good luck.

Puzzle 1: Sunk Costs

Jack Hirshleifer (UCLA) writes in with the following puzzle: "This one is an almost excessively simple problem in terms of algebra. The puzzle is to explain the seemingly counterintuitive result.

A naval procurement decision involves a choice between two options: large carriers which will operate singly, or small carriers which will operate in pairs. A pair of small carriers costs as much as one large one in all relevant dimensions. The difference between the two options is on the score of vulnerability. Under enemy attack, a large carrier has a probability p of surviving and executing its mission, while each of the smaller carriers has probability $p/2$. However, the small carrier pair can carry out its mission *even if only one survives*. (Assume that for each member of the pair,

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the risks are independent.) If we are only interested in execution of the mission, which option is superior?"

Puzzle 2: Cheaper by the Half-Dozen

Richard Zeckhauser (Harvard-JFK School) tells of a puzzle from dining in *The Ocean Club* restaurant (located in the Charles Hotel, Cambridge, MA). This restaurant offers half-portions of the main course entrees for less than half the price of the full portion. When Richard and his companions used this option, they discovered that the half-portions were at least half the size of the full portions and more than that, they came with full portions of the accompanying vegetables. As economists, they debated possible explanations. Could it be that the restaurant wanted to encourage people to try more than one dish? If diners most clearly remembered their favorite dish, then this incentive to sample would increase the chance that someone would find a "winner" and this would lead to great word of mouth advertising. Or did this half portion ordering reduce the variance in the distribution of orders so that there would be less spoilage and hence lower costs? To settle the dispute, the maître d' was called over. What was his explanation? Hint: No one at the table guessed correctly.

Puzzle 3: The Chicken or the Egg?

Emmett Keeler (Rand) offers the following little puzzle on estimation to get at stock versus flow issues. Question: Are there more hens (the kind of chicken that lays eggs) or broilers (the kind of chicken we eat) alive today in the United States?

To answer this question, you should assume that laying hens and broilers are two different species. You will want to make some back-of-the-envelope estimations for how many eggs and how many chickens are consumed and for the time it takes to produce an egg versus a broiler. Although these numbers may well be outside your area of expertise, the difference in population sizes is a factor of fifteen, so there is plenty of room for rough guesses.

Puzzle 4: War and Profits

Avinash Dixit (Princeton) describes a puzzle from James McPherson's recent best-seller, *The Battle Cry of Freedom.* With the blockade of the South during the Civil War, "Wilmington and Nassau became wartime boom towns.... The chance of profits from a successful voyage outweighed the one in three chance (by 1864) of capture. Owners could make back their investment in one or two round trips, clearing pure profit with every subsequent voyage." Given the returns and the chance of being caught reported by McPherson, was there a potential for profitable entry into this smugglers' market? Remember, smugglers made money and ran the risk of capture on *both* legs of the round-trip from Nassau to Wilmington.

Puzzle 5: The Delhi Dilemma

To rival the best location in Manhattan puzzle, Kaushik Basu (Delhi School) offers his version of a Delhi dilemma. In New Delhi's Connaught Place area, there are two roads which are concentric circles with plenty of radial roads as shown below. All roads are two-way, as shown in Figure 1. In negotiating Connaught Place, drivers in Delhi regularly encounter a small dilemma. While going from one point to another on the outer circle, they face two options: they can drive all the way along the outer circle, or instead, they may move to the inner circle, drive along it and come out at the relevant point on the outer circle.

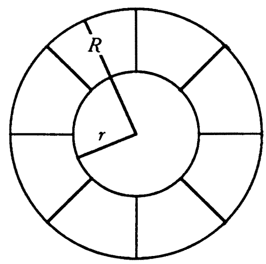

Fig. 1. Connaught Place.

Which route is shorter? To answer this question, you should assume that *there are radial roads everywhere*. If you wish, you may take the radius of the inner road as r and of the outer road as R. How far must one be traveling along the outer road before it becomes worthwhile to take the inner road?

Answers to Puzzles 1–5 follow Puzzle 6.

Puzzle 6: Significance is in the Eye of the Beholder

Once again Jack Hirshleifer comes up with a fascinating puzzle. In World War II, the Allies employed a technique called 'serial number analysis' to estimate military production. For example, if a captured tank of a particular model bore serial number 116, then (assuming away deliberate enemy misperception) at least 116 units must have been produced.

Imagine that, before any captures of a new enemy weapon, we have made plans on the basis of a very reliable espionage report that 1000 units were produced, so that $N = 1000$. Now we learn that the spy's report may have been garbled in transmission, so that there is a 50 percent chance that he may have really reported $N = 10{,}000$. Only two actions are available to us, one being superior if $N = 1000$ and the other if

$N = 10{,}000$. After careful consideration of the costs and benefits, we determine that, in the absence of any additional information, it is best to act as originally planned.

Before this decision is carried out, however, we manage to capture a single enemy unit bearing serial number 992. Using the "tests of significance" approach or the "Bayesian" approach, should we now change our decision?

A statistician working in terms of tests of significance might say: "Clearly the null hypothesis is $H_0 : N = 1000$ while the alternative hypothesis is $H_A : N = 10{,}000$. An 'upper rejection region' should be established corresponding to the acceptable risk α of a type I error, that is, falsely rejecting the null hypothesis. In practice, we statisticians use $\alpha = 0.05$ as a rejection criteria, except that when we are being very careful we sometimes use $\alpha = 0.01$ instead. With $\alpha = 0.05$, serial numbers from 951 upward would be in the rejection region; with $\alpha = 0.01$, the rejection region is from 991 upward. Either way, 992 is in the upper rejection region for the null hypothesis, so we should accept the alternative hypothesis."

A Bayesian statistician replies: "That's ridiculous. If the observed serial number is anywhere in the range from 1 to 1000, that increases our confidence in the null hypothesis that $N = 1000$. Since our prior beliefs already favored acting in accord with the null hypothesis, we should obviously continue to accept the null hypothesis."

Which decision is correct? What are the implications for the two approaches to interpretation of statistical evidence? Dale Poirier's article in Winter 1988 issue of this journal offers some guidance—perhaps even the answer to the clever reader.

Hirshleifer offers a hint: "In using the tests of significance approach, it is incorrect to entirely ignore Type II error. But that raises another question: why do practical statisticians, practically all the time in fact accept or reject hypothesis solely in terms of the $\alpha = 0.05$ or $\alpha = 0.01$ convention?"

Answers to Speed Puzzles

Answer to Puzzle 1

Hirshleifer provides his own answer: "Intuitively, since only one member of a pair of small carriers has to survive for the mission to be carried out, the small-carrier option seems to be superior. But algebra indicates this is not the case. The large carrier executes the mission with probability p. For the small carrier, the corresponding probability of success is the chance that not both are sunk: $1 - (1 - p/2)^2 = p - p^2/4 < p$. To explain the seemingly counterintuitive result, notice that the mathematical expectation of carrier survival is exactly the same for each option. But, from the point of view of executing the mission, for the small-carrier pair some of this expectation is 'wasted' in that, with probability $p^2/4$, *both* survive whereas only one is needed."

A second advantage to the large carrier option is the future valuation. Although the expected number of surviving ships is the same (p) in both cases, with the large ship, a single surviving vessel is much more valuable than a small ship. For example, if there are two missions, the chance that the large ship will successfully execute both is p^2 while for the small carriers the chance is $2(p/2)(1 - p/2)(p/2) + (p/2)^2 (p - p^2/4) = p^2/2 - p^4/16$, or less than half the chance with the large carrier option.

Answer to Puzzle 2

The maître d' listened patiently to all the alternatives and rejected each one. The true reason, he explained was that this was an *irrational* thing to do. People noticed this discrepancy quite regularly and told their friends about this inconsistency on the menu. As a result, this generated tremendous word of mouth publicity for the restaurant at very little cost. This puzzle is simply one more feather in their cap. In the spirit of Sherlock Holmes, when all explanations appear to fail, the explanation must be that there is no explanation.

Answer to Puzzle 3

There are roughly 15 times as many broilers as laying hens. A modern hen lays 5 eggs a week, up from 3 eggs a week in the 1920s (see Conniff, 1988). The statistical abstract for 1981 shows egg output at 190 million eggs per day; this implies the average person eats almost an egg a day (including the eggs in cakes, mayonnaise, and so on). The stock of hens needed to satisfy this consumption flow is $(7/5) \times 190$ million or about 265 million hens.

The modern broiler takes only 7 weeks to reach its four pound market weight, down from 16 weeks in the 1920s (Conniff, 1988). Hence the next seven weeks of consumption must all be alive at the present time. Working backwards, we find that the average per capita consumption of chicken is around 2.3 per week. A weekly consumption flow of 2.3 chickens requires a stock of slightly over 16 chickens per person, which is in line with the statistical abstract for 1981 which lists 4.1 billion broilers. While that seems to be a lot of chicken, U.S. consumption of chicken now outweighs beef. Colonel Sanders and Chicken McNuggets are doing their part.

Answer to Puzzle 4

Let K equal the investment in a blockade running ship, including the cost of stocking it with goods. A successful trip leads to net profits of X, which includes the restocking cost at the end of each trip. The probability of capture on each trip is denoted by p. With risk neutral smugglers and free entry, total investment costs should equal expected profits.

With a p chance of capture, the expected number of trips until capture is $1/p$, but on the last trip, no profits are made as the ship and cargo are captured. Hence the number of profitable trips is $1/p - 1$ and this must be enough to offset the literally sunk investment costs: $K = X(1 - p)/p$. If $p = 1/3$, then $K = 2X$, and the investment is recouped in one successful *round* trip. For recoupment in two successful round trips, we have $K = 4X$, which needs $p = 1/5$, which is closer to the average figure for the war as a whole (1/6, given later on page 380).

Answer to Puzzle 5

If one could travel all the way to the center of the circle, it would be possible to get to any point along the circumference in distance $2R$: the distance to the center is R and then back to any point on the circumference is another R. This is shorter provided one is travelling more than $2R$ along the circumference, which corresponds to $1/\pi$ of the way around the circle or an angle of more than 2 radians. (A radian is $360/2\pi$ degrees.) More generally, if one is travelling over an angle of θ radians, the distance along the circumference is $R\theta$. If instead, one first travels a distance $R - r$ towards the center this will reduce the circumference portion to $r\theta$ at a cost of $2(R - r)$ (the roundtrip). This is worthwhile provided $2(R - r) < \theta(R - r)$ or $\theta < 2$, the same condition as before.

It is worthwhile noting that since $R - r$ factors out, the result does not depend on the relative distances between the two circular roads. All that matters is what fraction of the way around one is going. As soon as one is travelling more than 2 radians or $360/\pi$ degrees it is worthwhile heading into the center (and doing so as far as possible).

Mail

Comments on Puzzles from Fall 1987

Puzzle 5. Unhealthy Constitution. Tom Schelling told the tale of a fantastically wealthy company which had an unusual company charter. The firm was owned and managed by thirteen shareholders who, each owning 1/13th of the stock, constituted the board of directors. The board had full authority not only to run the company but also to redistribute ownership shares and to determine board membership. All decisions of the board were by simple majority vote. Voting was open, each member voting yes or no in his turn. The turns progressed clockwise around the table. No motion required a seconder. The voting always began with the director sitting to the left of the proposer. Anyone making a proposal was recorded as voting for his proposal.

Among the provisions in the company's charter, one was unusual. It was designed to discourage "constitutional change," which was interpreted as any change in the membership of the board, or any change in ownership of the shares of the company. The rule was that if anyone offered a motion changing the voting rules, changing ownership of shares, or changing board membership, and this motion failed, he would be deprived of his ownership rights and of his membership on the board. The confiscated shares would be divided evenly among the remaining members of the board. Furthermore, anyone who voted in favor of a motion that failed would suffer the same fate as the proposer; namely, he would lose his shares in the company and his membership on the board.

In spite of all these "poison pills" someone proposed a motion which provided, in a somewhat devious way, for confiscating all of the shares of his fellow board members, making him owner of 100 percent of the company and sole member of the board of directors, all in return for a $1 compensation. His motion passed unanimously. What was the motion that received their twelve yes votes?

Tom Romer (Carnegie-Mellon) writes in with a correct solution. "Solving it cost me the better part of a day—but, at least it was the *better* part of the day. It's obviously a backward-induction puzzle that requires a motion with contingencies tied to the vote outcome. There are many variants of the details that would work, but all motions have to have an element like item (1) to make the pivotal voter prefer passage and, obviously, an element like item (2) to deliver unaimity. Mr. Devious (D) makes the following motion:

1. If this motion passes but not unanimously, the shares of those voting No will be taken away and divided equally among those, *other than* D, who voted Yes. Those who voted No will receive nothing and be removed from the board.
2. If this motion passes unanimously, everyone's shares will be given to D. Everyone other than D will receive $\varepsilon > 0$, and will be removed from the board.

"To verify that this proposal works, we start with the last member to vote. Obviously if there are already 7 or more Yes votes, she will vote Yes (as otherwise she will receive nothing). If the vote is tied at 6–6, then voting Yes gives her a payoff of two shares—her own plus one-sixth of the six shares held by the six No voters. If she votes No, her payoff is only 13/7 shares—her own plus one-seventh of the six shares held by the six Yes voters. So she will vote Yes.

"Now consider the 12th (next-to-last) voter. If there are already six or more Yes votes, then she can predict that the motion will pass, since voter 13 will vote Yes by the above argument. So #12 will vote Yes too. Suppose there are only 5 Yes votes when #12 casts her vote. If she votes Yes, so will #13, and the final outcome will be 7-6 in favor; #12's payoff will be two shares as above. If she votes no, the motion cannot pass, so #13 will also vote No, leading to a 5-8 outcome; #12's payoff is 13/8 shares—her own share plus 1/8 of the five confiscated shares. So #12 will vote Yes.

"We can carry this same argument back down to the eighth voter. If before her vote, 2 or more Yes votes have been cast, she can predict that the motion will pass regardless of her vote, so she will vote Yes. If only the proposer has so far voted Yes, then a Yes vote by 8 will lead to a 7-6 passage and a payoff of 2. Voting No will doom the motion to a 1-12 failure and #8's payoff will be a mere 13/12 shares. So #8 will vote Yes in response to Mr. Devious's proposal.

"For voters #2 through #7, there is really no choice. No matter what they do, #8 through #13 will vote Yes so that passage is assured. Voting No will result in a payoff of 0, while voting Yes guarantees at least $\varepsilon > 0$. So they will all vote Yes. The final result: Unanimous passage or murder (probably serial murders, at that)."

When I gave this problem to my students, Eliot Robin came up with an ingenious alternative solution. One of the features that makes his solution different is the last voters will not necessarily vote to pass the resolution when pivotal. Yet, in equilibrium, the resolution always passes. Mr. Devious makes the following proposal:

1. If someone votes No and the next person votes Yes, then the yes voter gets to keep their own shares plus the shares of the previous No voter.
2. If this motion passes unanimously, everyone's shares will be given to D. Everyone other than D will receive $\varepsilon > 0$, and will be removed from the board.

This proposal works via *pair-wise* backwards induction. Imagine that when it comes to the last two voters there are 6 Yes votes. If #12 says No then #13 will want to say Yes in order to get two shares rather than 13/7ths. If #12 says Yes, then the proposal passes.

Next go back to the #10 and #11 pair. If #10 faces 6 or more Yes votes, he can predict the proposal will pass, so he votes Yes. If there are only 5 Yes votes and he votes No, then #11 will want to vote Yes in order to take #10's shares—he can do so with confidence as this will leave the #12 and #13 pair with 6 yes votes so that one of them will vote Yes and the proposal will pass.

Working backwards, we see that in each of the six pairs [(#2, #3), . . . , (#12, #13)] at least one of the two will always vote Yes provided each of the previous pairs has come up with at least one Yes vote. Since everyone can then predict that the proposal will pass, they will all vote Yes in order to get ε rather than 0.

Although these proposals may seem a little far-fetched, in fact they are a simple extension of the same coercive bid involved in two-tiered tender offers. As described in Grossman and Hart (1980), imagine that the fair market value of a share is $1.00. Let a raider make an unconditional offer of $1.01 for the first 50 percent of the shares tendered and zero thereafter. If more than 50 percent of the shares are tendered, the $1.01 will be prorated; each sharehlder will receive $1.01 × (50 percent)/(percent tendered). In this case, tendering is a dominant strategy: if the offer fails, tendering is profitable as you gain a penny per share; if the offer succeeds, again tendering is a good idea as you get something as opposed to nothing.

Puzzle 6. Close Counts in Hand Grenades, Ballroom Dancing, and Game Design. This puzzle asked how one can design scoring rules to ensure that two players of equal ability will have an equal chance of winning even when they face differential conditions (for example, in ping-pong the two sides of the table are often uneven). To restore equity, the two players should switch sides sometime during the match. When?

Emmett Keeler (Rand) writes in with the solution: "The most interesting case is when you allow the sum of the scores to be used in determining the switch. Here tennis and beach volleyball do it right, by forcing each team to play half the time on each side. For ping-pong, the very nearly optimal rule is to switch at 20.5 total points. This can be seen using the trick of assuming we continue to play to 41 points even if a winner has already been determined. If we switch after point 20, the distribution of scores after point 40 are symmetric for equal players. Hence, the scores $(20 + x, 20 - x)$ and $(20 - x, 20 + x)$ are equally likely. When $x > 0$, the first case implies a win for player 1 while the second implies a win for player 2. The only place for imbalance is the resolution of a 20–20 tie. Let the chance of a 20–20 match be $2p$ so that each player has a $50 - p$ chance of already winning. Since the player with the good side wins the next point with probability q, he wins the game with chance $50 + (2q - 1)p$.

"On the other hand, suppose you switch after point 21. Then the scores after 42 points are symmetric. Again everything is balanced except for the resolution of a 21–21 tie. Let the chance of a tie at 21–21 be $2\hat{p}$. Then the second-half good-side player is at a disadvantage in the 21–21 game, since it is more likely than not that he won the previous point. Reasoning as before, he will have a probability of $50 + (1 - 2q)\hat{p}$ of winning the game. By choosing 20 or 21 point as the switch number via a coin flip, we achieve $50 + (2q - 1)(p - \hat{p})$, which would be very close to 50 percent.

"This also suggests that the winning by two point rule may actually simplify rather than complicate the optimal switching rule. Switch after 20 points. Then if the score is tied at 21–21, switch after the next point and switch again after every second point thereafter until one side is ahead by two."

Keeler continues with a comment on optimal scoring rules. He describes the work of Carl Morris, a statistician at University of Texas. "In most cases, to maximize the chance of the better player winning, while constraining the expected number of points played, you should play until one side gets N points ahead. If the server in squash has an advantage, then it is more informative to change the rule so that the person who loses the point gets the next serve, and you only win points when you are the receiver." Since they have changed the ball in squash 3 times in the last 15 years, maybe they will catch on and change the rules.

■ *Kaushik Basu, Avinash Dixit, Jack Hirshleifer, Emmett Keeler, Eliot Robin, Tom Romer, and Richard Zeckhauser share the credit for this issue—my heartfelt thanks. Behind-the-scenes, comments and kibitzing from Timothy Taylor and Carl Shapiro ensure that each puzzle isn't unnecessarily puzzling.*

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