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Puzzles
Choose a Curtain, Duel-ity, Two Point Conversions, and More

Barry Nalebuff

In presenting economic puzzles, I have three goals in mind: some puzzles are chosen to stimulate research; others offer examples that will help undergraduate and graduate teaching; all should provide quality distractions during seminars. As usual, this feature begins with several speed puzzles, with answers provided at the end of the problems. Following are several longer puzzles, for which readers are invited—nay, challenged—to submit their own answers. The responses will be discussed in a future issue.

I also encourage readers to share their favorite teaching problems and research puzzles (with answers, please). Please send your answers and favorite puzzles to: Barry Nalebuff, "Puzzles," c/o *Journal of Economic Perspectives*, Woodrow Wilson School of Public and International Affairs, Princeton University, Princeton, NJ 08544. Good luck.

Puzzle 1. Free to Choose

The TV game show "Let's Make a Deal" provides Bayesian viewers with a chance to test their ability to form posteriors. Host Monty Hall asks contestants to choose the prize behind one of three curtains. Behind one curtain lies the grand prize; the other two curtains conceal only small gifts. Once the costumed contestant has made a choice, Monty Hall reveals what is behind one of the two curtains that was not chosen. Now, Monty must know what lies behind all three curtains, because never in the history of the show has he ever opened up an unchosen curtain to reveal the

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grand prize. Having been shown one of the lesser prizes, the contestant is offered a chance to switch curtains. If you were on stage, would you accept that offer and change your original choice?

Puzzle 2. Triple Duel

Here is a problem in "duelity" theory. Three economists, Anita, Bob, and Larry, are engaged in a three-way duel. Each is given an opportunity to shoot at the others in turn: first Anita, then Bob, and then Larry. After the first round, any survivors are given a second chance to shoot each other, again beginning with Anita, then Bob, and then Larry. And so it goes until only one person remains. Larry is a perfect shot; he never misses. Bob is a very good shot, achieving 80 percent accuracy. Anita is a poor shot, with only a 30 percent chance of hitting the person at whom she aims.

Each person's objective is to be the sole survivor. What is Anita's optimal strategy in the first round? Whom would you most prefer to be in this problem?

Puzzle 3. No Tee for Two, but Two for the Team

In college football, a team that scores a touchdown next runs one play from a hashmark two and one-half yards from the goal line. The team has a choice between trying a running or passing play to advance the ball into the end zone, which scores two additional points, or trying the far less risky strategy of kicking the ball through the uprights, which scores one extra point. Put yourself in the cleats of a coach who is behind by 14 points. The team scores a touchdown (6 points) and must decide whether to risk going for the two extra points or stay with the safer one extra point. The coach decides that for now, the team will play it safe and try to kick the single extra point. But the coach vows that if the team manages to score a second touchdown, it will try to win the game by trying for the two point conversion rather than kicking another extra point and settling for a tie. What do you think of this coach's logic?

Puzzle 4. Prisoner in a Cacophonous Cell

The "Puzzles" in the first issue of this journal presented a strategy puzzle where readers were invited to submit programs to play in a prisoners' dilemma tournament with noise. (A more thorough explanation of the game can be found in that issue.) To review the specifics of the dilemma, suppose the payoffs in each period are as follows: if both players cooperate then each receives a payoff of 3; if both defect then they each receive a payoff of only 1; if one defects and the other cooperates then the defector receives a payoff of 5 and the cooperator receives zero.

The game is repeated over a number of periods, thus providing each player with a chance to respond according to the moves of the other player in previous periods. The aggregate payoff for a player is simply the average of all payoffs that player received in previous periods; no discounting exists. While the payoffs are based on the true actions, each player sees only a distorted signal of his opponent's move in this game. What do you predict would be the best strategy if each player correctly perceives the other player's previous decision with only a 0.5 probability?

Answers to Puzzles 1, 2, 3, and 4 appear following Puzzle 7.

Puzzle 5. Unhealthy Constitution

Tom Schelling tells the tale of a fantastically wealthy company that had an unusual company charter. The firm was owned and managed by thirteen shareholders who each owned one-thirteenth of the stock and constituted the board of directors. The board had full authority not only to run the company but also to redistribute ownership shares and to determine board membership. All decisions of the board were by simple majority vote. Voting was open, each member voting yes or no in turn. The turns progressed clockwise around the table. No motion required a seconder. The voting always began with the director sitting to the left of the proposer. Anyone making a proposal was recorded as voting for the proposal.

Among the provisions in the company's charter, one was unusual. It was designed to discourage constitutional change. "Constitutional change" was interpreted as any change in the membership of the board or any change in ownership of the shares of the company. The rule was that if anyone offered a "constitutional change"—a motion changing the voting rules, changing ownership of shares, or changing board membership—and this motion failed, the proposer would be deprived of all company stock and of membership on the board. The confiscated shares would be divided evenly among the remaining members of the board. Furthermore, those who voted in favor of a motion that failed would suffer the same fate as the proposer; namely, all those who voted in favor of a motion that failed would lose their shares in the company and their membership on the board.

The rule was designed to prevent a majority of the board from ganging up on a minority and passing a discriminatory rule that somehow confiscated the wealth of the minority or otherwise took advantage of the unlimited power contained in the majority vote. In this regard, the rule was effective. Frivolous changes were never proposed. In fact, no constitutional changes were ever proposed. It was so risky to vote "yes" and so safe to vote "no," that no one dared offer even a compellingly attractive constitutional proposition. After all, a majority of the board had the power to vote down an attractive proposal, remove the member and any supporters from the board, divide their holdings, and then consider the same proposal at a later date.

At one of the regular meetings, the board went through its usual procedures for giving everybody an opportunity to offer proposals. The final turn fell to a member of the board who surprised his colleagues by offering a drastic proposal for constitutional change.

The motion provided, in a somewhat devious way, for confiscating all of the shares of fellow board members, making this board member owner of 100 percent of the company and sole member of the board of directors, all in return for a one dollar compensation to be divided among the others. His motion passed unanimously. Thereafter, the former members of the board, having neither wealth nor any power to vote themselves wealth, lived off such charity as their former colleague on the board was willing, from time to time, to provide them.

What exactly was the motion that received their twelve "yes" votes?

Puzzle 6. Close Counts in Hand Grenades, Ballroom Dancing, and Game Design

The rules of sporting contests often have two objectives: one is to ensure that the better player or team wins; the second is to keep the match close. Clearly, these two objectives will conflict from time to time.

In ping-pong, the two sides of the table are often uneven. The differential conditions give one of the two players an advantage. Let us say that playing on the left-hand side gives an otherwise evenly matched player some probability $q > 0.5$ of winning each point. To restore equity and excitement, the two players should switch sides at some time during the match. The question is: when?

Traditionally, the game of ping-pong is played to 21 points, with every rally resulting in a point being scored. As soon as one of the players reaches 11 points, the two contestants switch sides. They do not switch sides again.

The present rule seems to be unfair to the person who is first to play on the good side. If after switching, the scoring patterns are reversed, then the initially disadvantaged player will typically win. For example, if the switch occurs at a score of $(11, x)$ and then the advantage is reversed, we could expect the game to be tied at $(11 + x, 11 + x)$. But with no further switching, the player with the better side has the edge. This suggests that the players are switching too soon.

What do fair switching rules look like? When should the switching occur if it depends only on the leader's score? Imagine that the advantage of the table is unknown. Is there still some fair switching rule (which now might depend on the total number of points played and the relative score)? Remember to focus on rules in which only one switch takes place. To keep the problem simple, you should ignore the possibility of win-by-two overtime or a 7-0 shutout and declare the first player to reach 21 the winner.

A related question arises in squash. In American squash, every serve results in a point being scored and the game is played to fifteen points. In English squash, the game is played to nine points but only the server can score a point. If the player receiving serve wins the exchange, that player gets to serve. Which of these two scoring rules leads to closer matches? Under which one is the better player more likely to win?

Puzzle 7. The Smuggler Who Fled With Grace on the Sea

The Coast Guard reaches the site of a drug drop-off one hour after the transfer has been made. The smugglers are travelling at a known constant speed (X) and in a constant direction. The Coast Guard's boat speed (Y) is faster than the smugglers'. What is the optimal searching strategy for the Coast Guard? How much faster must its boat be to ensure that it will eventually catch the smugglers?

Hint: You should assume that the initial point is somewhere in the middle of an infinitely large ocean so that boundary constraints do not play any role. Note that those on the Coast Guard boat have no vision (a heavy fog has settled); they can only find the smugglers if the boats bump together.

If you wish to make this question more difficult, imagine that the smugglers have as their goal reaching the mainland. Let the drop-off point be a distance D from the mainland where the distance is measured as the sea gull flies. The smugglers have an hour headstart on the Coast Guard. They are sufficiently far away from the shore that if they head directly towards the shore and the Coast Guard comes straight after them, the Coast Guard will catch them. If the smugglers head towards the shore at some angle, then they have the advantage of hiding their route while having the disadvantage of lengthening the time it takes to reach the safety of the shore. Again, the smugglers are constrained to travel in a constant direction at a constant speed. The question is: what is the optimal (mixed) strategy for the smugglers to follow in choosing their route to the shore and how should the Coast Guard search to find them?

I will report the answers to Puzzles 5, 6, and 7 in a subsequent issue.

Answers to Speed Problems

Answer to Puzzle 1

This puzzle is one of those famous probability problems, in which, even after hearing the answer, many people still do not believe it is true. You should always switch curtains. Here are two ways of understanding why.

First, with probability $1/3$ you picked the correct door. In this case, if you switch curtains you will certainly lose. With probability $2/3$, you picked the wrong door. In this case, Monty Hall does not have a choice of which curtain to show you if he wants to keep the grand prize hidden. If you switch, you will certainly win it.

Since your chance of initially picking the correct curtain was $1/3$, if you never switch your chance of winning must remain at $1/3$. In contrast, if you always switch, you win whenever your first choice was mistaken and this gives you a $2/3$ chance of winning.

Second, before Monty Hall showed you anything, you had a 1/3 chance of having picked the correct curtain. Since two wrong choices are possible, Monty Hall can always open up one curtain and not reveal the grand prize. After he has done this, your chance of winning is still 1/3, since you have learned no information about the door you chose. That leaves only one other unopened door which correspondingly must have a 2/3 chance of being the correct one (since probabilities always add up to one).

Repeated experiments have shown that individuals make systematic violations from behavior predicted by von Neumann-Morgenstern expected utility theory. Since many if not most individuals choose to stay with their original choice, does this suggest we should look for alternatives to Bayes rule?

Answer to Puzzle 2

Examine each of Anita's options in turn. If she shoots at Bob and hits then she signs her own death warrant, as it becomes Larry's turn to shoot and he never misses. If she shoots at Larry and hits, then it is Bob's turn and he will shoot at her. Hence, her chance of survival is less than 20 percent (the chance that Bob misses). However, if Anita misses, then it does not matter at whom she aimed. In this case, Bob will shoot at Larry and if he misses, Larry will shoot and kill Bob. Then it becomes the second round and it is Anita's turn to shoot again. Since only one other person remains, she has at least a 30 percent chance of survival since she might win on her first shot in this second round, and if Bob kills Larry she might get another shot in a third round.

What should Anita do initially? She should miss intentionally. By firing straight up in the air, she ensures that Bob and Larry will spend the first round firing at each other, and increases her survival chance from below 20 percent (when she hits her target) to above 30 percent.[1]

Larry, although he is the most accurate, has the lowest chance of survival—only 14 percent. Bob has a 52.1 percent chance of winning, while this strategy allows Anita to turn her 30 percent accuracy into a 33.9 percent chance of winning.

Answer to Puzzle 3

The coach has chosen a dominated strategy. As it currently stands, if the team makes the one point attempt following the first touchdown but misses the try for two points following the second touchdown, then it loses the game. The coach has put the team in a position such that when they try the risky two point attempt, they can only win or lose the game. He would have done better to try the riskier strategy first. That way, if the two point attempt fails after the first touchdown, the team has the option to try another two point attempt following the second touchdown and salvage a tie rather than a loss.[2]

[1] Literalists may object to this proposed answer. As the question is posed, even if she tries to shoot up into empty space, her chance of hitting this target is 30 percent and may suffer some risk of shooting herself. This objection is ruled out of court.

[2] Of course, this conclusion ignores the fact that the probability of scoring a two-point conversion may not be independent of when in the game the attempt is made.

This situation actually occurred in the 1984 Orange Bowl game. The Nebraska Cornhuskers entered the game undefeated, needing a victory over the once-beaten Miami Hurricanes to finish the season in first place in the polls. But Nebraska fell behind by 31–17 in the fourth quarter. After scoring a touchdown to make the score 31–23, Nebraska successfully kicked for one extra point rather than trying for two. Then, after scoring another touchdown to make the score 31–30, Nebraska disdained kicking another extra point for a tie, tried for a victory with a two-point conversion, but failed. On the strength of that victory, Miami finished the season with first place in the polls. If Nebraska was planning to try for victory, they would have been better off to try a two point conversion after the first touchdown. Even if the try failed, they might have salvaged a tie by managing a two point conversion after the second touchdown. Since Miami already had one loss, a tie in that game might well have won Nebraska the national championship.

Answer to Puzzle 4

The key to solving this problem is to recognize that a 50 percent chance of reversal is as bad as perception can get. If the chance of a reversal were 100 percent, the signals would be equivalent to perfect information since each player could simply reverse whatever signal was received. When there is a fifty percent probability of the signals getting crossed, the optimal strategy is always to play defect. Remember that the actual payoff depends on the true actions. In each period, defection is a dominant strategy.

The reason defection is not necessarily dominant in the repeated game is that by playing cooperation, you may also induce your opponent to cooperate in later rounds. But when signaling is only 50 percent accurate, your opponent's distribution of signals will always be 50-50 no matter what you do. In this case, your opponent should ignore these signals since they carry no information. Further, you shouldn't change your actions since nothing you can do will change the distribution of signals that your opponent receives. In essence, a 50 percent probability of a reversal reduces the multi-period repeated prisoners' dilemma back to a one-shot game.

■ *Bill Samuelson shared his expertise as a Monday morning quarterback. Bob Summers related both the drug-running problem and the duelists. Richard Zeckhauser posed the ping-pong problem; unfortunately, he does not know the answer. For the squash problem, I have been hard at work personally conducting lengthy empirical studies; much more work remains to be done and I am in training to do it.*