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Puzzles
Cider in Your Ear, Continuing Dilemma, The Last Shall Be First, and More

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In presenting economic puzzles, I have three goals in mind: some puzzles are chosen to stimulate research; others offer examples that will help undergraduate and graduate teaching; all should provide quality distractions during seminars. As usual, this feature begins with several speed puzzles; answers can be found at the end of the problems. Following are several longer puzzles, for which readers are invited — nay, challenged, to submit their own answers. The puzzles in this issue focus on betting and voting, and once again include a prisoners' dilemma problem. The column ends with reader mail, including the submitted solutions to "The Best Location in Manhattan," Puzzle 4 in the Summer 1987 issue of this journal.

As always, please send your answers and favorite puzzles to: Barry Nalebuff, "Puzzles," c/o *Journal of Economic Perspectives*, Woodrow Wilson School of Public and International Affairs, Princeton University, Princeton NJ 08544. Good luck.

Puzzle 1: Here's Cider in Your Ear

Consider the following excerpt from Huff and Geis (1959) *How to Take a Chance*.

> Says Joe then: "It's a good night to play matching pennies—too hot for anything more strenuous. In fact, it's pretty warm to flip actual coins, so let's just lean back and *say* 'Heads' or 'Tails' instead." [*Assume these are called out simultaneously.*]
>
> "All right so far," you agree.
>
> "And to put a little variety into the game," Joe goes on, "I'll give you $3 when I call tails and you call heads. I'll give $1 when it is the other way around

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(i.e., I call heads and you call tails). And when we match, you give me \$2 to make it even."

At this point you do a little figuring. On the basis of pure chance, which would apply if this were actual coin tossing, the offer is fair enough. In the long run, out of each four tosses you would win \$3 once and \$1 once and lose \$2 twice, for an even break.

If you play against Joe, what do you expect to earn (and how would you play)? So would you play?

Puzzle 2: When You Can't Beat Them One-on-One

A serious problem with voting under 50 percent majority rule is that there may fail to exist a proposal which beats all other proposals in pairwise competition. The standard Condorcet cycle is but one of many examples: if there are three individuals with strict preferences $A > B > C$; $B > C > A$; and $C > A > B$, then in pairwise votes A beats B, B beats C, and C beats A.

When no pairwise victor emerges, a possible alternative might be to choose a proposal (for example, A) that beats all other proposals in pairwise competition, either directly (as in A beats B) or indirectly in *one* step (as in A beats B and B beats C so A indirectly beats C in one step).

Does such a proposal always exist?

Puzzle 3: The Dilemma Continues

Abraham Neyman offers the following twist on the prisoners' dilemma. This provides a good test for your understanding of game theory. It is well known that there is a unique subgame perfect equilibrium in the finitely repeated prisoners' dilemma. It is also true that there is a unique Nash equilibrium in the finitely repeated prisoners' dilemma. How is this argument different from that of subgame perfection?

Answers to Puzzles 1, 2, and 3 appear following Puzzle 5.

Puzzle 4: The Other Person's Envelope is Always Greener

You have two envelopes. In one, you place a hidden amount of money and give the envelope to Ali. Then you flip a hidden coin. If it comes up heads, you place twice the original amount of money in the second envelope. If it comes up tails you only put half the original in the second envelope. You give this second envelope to Baba. So far, the contents of both envelopes are hidden, as is the outcome of the coin toss. Ali and Baba are allowed to look privately at the amount of money in their own envelopes. They are then given an opportunity to trade envelopes if they both agree.

Suppose, for the sake of argument, that Ali finds $10.00 in her envelope. Ali reasons that Baba is equally likely to have $5.00 or $20.00. Trading envelopes gives her an expected gain of $2.50 (or 25 percent). Acting in a risk-neutral manner, she would want to switch.

Now Baba looks inside his envelope. Whatever amount he finds (either $5.00 or $20.00), he too reasons that Ali is equally likely to have half or double his amount. The expectation is $0.5[0.5X + 2X] = 1.25X$, so he too expects a 25 percent gain from switching envelopes.

But this is paradoxical. The sum of the amounts in the envelopes is whatever it is. Trading envelopes simply cannot make *both* students better off. Yet they both expect to make a 25 percent gain. Where did they go wrong?

I learned about this puzzle from Hal Varian who heard it from S. Zabell at Northwestern who heard it from Steve Budrys of the Odesta Corporation. Although it may not have originated with Budrys, this is as far back as we have been able to trace it. I would be delighted to hear more about the history of this puzzle. My telling above involves a slight variation on the "original" version, where there is no coin toss. We are simply told that one envelope is twice as nice as the other, but not which is which. So before the envelopes are handed out, the two students should be indifferent as to which they get. But once they open their envelopes, each appears to be eager to trade with the other. Just as above, both expect a 25 percent return from the trade. In fact, even before they look in their envelope, they both want to trade. How can that be? For those readers who are impatient, Zabell (1987) provides a short exposition of the puzzle and his proposed solution. I will discuss his and your solutions in a later column.

Puzzle 5: The Last Shall be First?

One of my all-time favorite puzzles is to figure out what to do in Tom Schelling's paradox of the departing voter below.

Schelling describes the set-up: "A five-man board is to elect one of its members chairman by a procedure involving successive majority votes. Anderson, first in alphabetical order, will be paired against Barnes; the winner of that vote will be paired against Carlson, the winner then paired against Davis, and the winner of that paired against Evans. The winner of this fourth and final ballot will be declared chairman.

Everyone knows everyone else's preferences. Everyone wants to be chairman. Reading (initial letter only) left to right in the table below are the ways that each of the five members ranks all five as candidates for the job. Anderson, for example, ranks them all in alphabetical order. Barnes, for example, ranks Evans third and Carlson last.

Anderson	A	B	C	D	E
Barnes	B	A	E	D	C
Carlson	C	D	A	E	B
Davis	D	B	A	E	C
Evans	E	D	B	C	A

All five committee members have foresight and vote strategically. Who will win? So far, the answer is straightforward, but I'll let you figure it out.

Now for the Schelling fillip. Anderson is forced to miss the meeting. He is allowed to transfer his vote to someone else. This transfer must be unconditional in that Anderson is not allowed to specify how the receiver must vote. To whom should Anderson give his vote and what do you expect will happen?

Answers to Speed Puzzles

Answer to Puzzle 1

In *Guys and Dolls*, gambler Sky Masterson relates this valuable advice from his father: "Son, one of these days in your travels a guy is going to come to you and show you a nice brand-new deck of cards on which the seal is not yet broken, and this guy is going to offer to bet you that he can make the jack of spades jump out of the deck and squirt cider in your ear. But son, do not bet this man, for as surely as you stand there you are going to end up with cider in your ear."

Joe has just offered you one of those cider in your ear bets. If he were to randomize 50–50, as with a fair coin, then you would expect to break even. But Joe can do even better by playing a mixed strategy, calling heads with chance 5/8 and tails 3/8. Whether you call Heads or Tails, Joe has guaranteed an expected payoff of 1/8; you have cider in your ear (and an expected $-1/8$ payoff) if you accepted his bet.

Answer to Puzzle 2

This question was first answered by Landau (1953) in the context of determining hierarchies in animal colonies. Think of each proposal as one chicken in a flock. In pairwise competition, the two chickens are ranked based on who can peck the other. Landau demonstrated the existence of a Chicken King in the sense that this chicken can beat (up) every other chicken either directly or via some third chicken. In particular, the chicken which beats up the greatest number of other chickens has this property. The proof is straightforward. Call the chicken which beats the most other chickens K. If K beats all other chickens then we are done. If K loses to some chicken L then it must be the case that K beats something which beats L. Otherwise, L would beat everything K beats and K too: this would give L more victories than K, contradicting the assumption that no chicken has more victories than K.

Returning to the question of ranking social alternatives, Nicholas Miller (1980) offers an elegant solution in his article on the uncovered set. Miller defines the "Cover" relationship by A covers B if (1) A beats B in a pairwise majority vote and (2) A also beats everything that B beats (in pairwise majority votes). Miller shows that this covering relationship is transitive; if A covers B and B covers C this implies that A covers C. Since a transitive relationship cannot cycle in any finite collection of alternatives, there must be at least one "Uncovered" proposition. I leave it to the reader to show that a proposal uncovered by anything else has the "Chicken King" property, namely it beats everything else either directly or in one step.

The importance of the uncovered set goes beyond the existence of a "Chicken King." As emphasized by Ordershook (1986) it allows us to restrict the problem of social choice to the set of uncovered proposals; why pick B if A covers B? Although this may not reduce the choice to a singleton, it is a stronger selection criteria than the Pareto principle alone. Second, Miller (1980) demonstrates that sophisticated voting under *amendment procedure* can never lead to an outcome outside the uncovered set. Hence, the ability to manipulate an agenda is restricted to the set of alternatives that are uncovered.

Answer to Puzzle 3

First, the subgame perfection argument. In the final period, there is a unique Nash equilibrium: both players defect. Hence behavior in the penultimate period cannot affect the final play, so again both players should defect. And so it goes.

The Nash argument is more subtle. Again, at the final period both players must defect in the Nash equilibrium outcome. But that does not immediately imply that in the penultimate period both players must defect in any Nash equilibrium outcome. If in the penultimate period, player 1 deviates from the predicted equilibrium behavior, player 2 is no longer required to behave rationally in final period. But in the prisoners' dilemma example, there is nothing worse that he can do than defect. Player 2's equilibrium strategy already *minimizes* player 1's final period payoff. There is no off-equilibrium punishment that can be used to generate cooperation in earlier periods.

This difference is easier to see in Table 1 where the prisoners' dilemma is expanded to include a third strategy, mutual destruction (confession to a crime that

Table 1
Prisoners Dilemma with Mutual Destruction

		Prisoner 2		
		Cooperate	Defect	Mutual Destruction
	Cooperate	1, 1	5, 0	−99, −99
Prisoner 1	Defect	0, 5	3, 3	−99, −99
	Mutual destruction	−99, −99	−99, −99	−99, −99

will give both prisoners the death penalty). Mutual destruction is clearly a dominated strategy and will never be played in any subgame perfect equilibrium. But in a Nash equilibrium, the prisoners could use this "incredible" threat of mutual destruction to sustain cooperation in all but the last period of a finite repetition of the game above. This is accomplished by the pair of equilibrium strategies: cooperate in period $t + 1$ if the other has cooperated in periods 1 to t, else mutual destruction.

Mail

Comments on Puzzles from Fall 1987

Puzzle 3. Two for the Team, but not Tea for Two. It turns out that not all college football teams made Nebraska's mistake. In the 1962 Yale–Colgate football game, the Elis were trailing 14–0 in the final quarter. After their first touchdown, they went for the two point conversion but failed. But all was not lost. They scored a second touchdown, made their second attempt at a two point conversion and tied the contest (See Porter, 1967).

Richard Porter (Michigan) is the expert on this topic and has a wonderful two-page article "Extra-Point Strategy in Football." His note is ideal for introducing revealed preference, expected utility, and dominated strategies to undergraduates. It is packed with fascinating examples from college football. As for those who haven't read Porter, they end up condemned to rewrite it, either by ignoring his conclusions (Nebraska) or rediscovering them (me). It's better in the original, both the sin and the solution.

Comments on Puzzles from Summer 1987

Answer to Puzzle 4: The Best Location in Manhattan. This puzzle asked for the most efficient pattern of firm locations when consumers are uniformly distributed over a two-dimensional plane, and consumers shop at the "closest" store, where closest is based on the Manhattan metric. Under the Manhattan metric, the distance between A and B equals the sum of the vertical and horizontal differences; if $A = (a_1, a_2)$ and $B = (b_1, b_2)$ then the distance between A and B, denoted by $\|A - B\|$, equals $|a_1 - b_1| + |a_2 - b_2|$. With this metric, what is the most efficient location pattern and what is the resulting shape of market areas?

No, the answer is not squares! But if you guessed squares, you are in good company. The correct answer is Diamonds. Along the border of a diamond, all customers travel exactly the same distance to the center. This shape minimizes the average (Manhattan) distance travelled for a given market area. For example, we see in Figure 1 that the Manhattan metric distance to any point on the border of a diamond is $\sqrt{2}\,Y/2$ when the market area per store is Y^2. In contrast, if the same density of firms were to locate in a square pattern, then wherever the square sticks out over the diamond (the four shaded corners) these customers will have to travel farther, all the way up to Y at the corners. The same "overlapping" argument demonstrates that over all market areas of Y^2 the diamond shape minimizes the average Manhattan distance travelled.

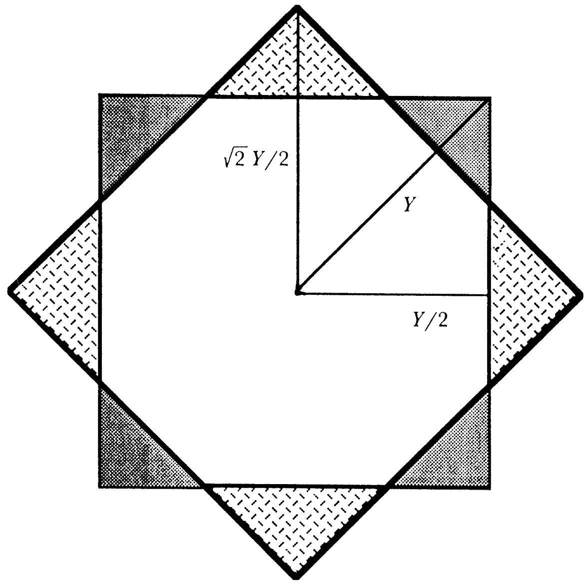

FIGURE 1

In order received, correct answers were submitted by John Brown (Albion College), R.D. Cairns (McGill), James Dow (Univ. of Penn.), and Curtis Eaton (Simon Fraser University).

But Curtis Eaton wins the prize for sending in a proof. One cannot just assume as above that the solution is symmetric; this has to be demonstrated. Curtis went on to write: "What about configurations that are not area-symmetric? If there are any non-diamond shaped markets, total and hence average distance from the nearest firm would decrease if we could 'diamond' them, leaving all the other markets the same [for the reason given above]. In general, we could not 'diamond' them, leaving all the other markets the same. But let's suppose that we could. Now consider any pair of diamond markets with unequal area. We could again reduce total and hence average distance if we could produce identical diamond markets for them (leaving all other markets the same), since total distance from a firm with a diamond market with side Y is $\sqrt{2}\,Y^3/3$, which is a convex function of Y. Again, we could not obviously produce the identical diamond market areas, but suppose that we could. Replication of the 'diamonds of unequal sizes' argument leads us in a series of imagined steps, each of which reduces average distance, to the conclusion that all diamond market areas must be identical. We can, of course, implement this configuration."

More on Answer to Puzzle 2: In Fact, It's a Gas. The controversy over the leaded-unleaded gasoline price differential continues. John Lott Jr. (Rice University) and Russell Roberts (UCLA) write to offer an alternative explanation of the price premium for unleaded gasoline that does not depend on differing price elasticities nor loss leader effects. They argue that the price differential is due to gasoline stations attempting to cover their fixed costs such as checking the oil, turning on the pump, making change, and so on. Consumers of leaded gas typically have cars with bigger gasoline tanks since their cars are older and less fuel efficient. Since their purchases are bigger, the fixed cost can be covered with a smaller mark-up on the larger transaction. They draw an interesting analogy to price discrimination in restaurants. "The most important factor explaining the retail over wholesale mark-up in a restaurant is the time it takes to eat an item. This explains why wine and coffee are priced seemingly so far above 'marginal cost.' People linger over these items, so their price must include the cost of renting the table." While I find the restaurant analogy helpful, I think their explanation is not entirely convincing. If the markup on coffee is an attempt to cover the rental cost of a table, why then are refills typically free and why is there no discount for take out? As for gasoline, the fixed cost associated with self-service must be lower. This would imply a smaller price differential between grades at the unleaded pumps. Is that true? Lott and Roberts are aware of these issues and have promised to drive from coffee shop to gas station until they find the answer.

As for New Jersey, Burt Malkiel (Yale) recalls the legislative debate when self-service gasoline was outlawed. The "argument" was that self-service pumps discriminate against women.

More on Puzzle 1: Turn Out the Lights. Peter Albin (CUNY—John Jay College) asked if he "might shed additional darkness on Puzzle 1 before Consolidated Edison takes the matter as settled doctrine and mails out my priority vouchers." The problem with the solution as given is that it requires that brownouts have no significant rationing or signalling effects on their own. This might not be the case if in the event of a brownout "certain equipment is turned off entirely because it cannot function if plug current does not meet design specifications: e.g. computers without independent power supplies are switched off." Since the priority system might allow others to continue using their computers, this could exacerbate the total shortage. "Although the net value of transferred priority vouchers will always be zero, the status need not be quo." I plead guilty as charged.

■ *Behind the scenes Carl Shapiro and Timothy Taylor have been improving the puzzles and the answers. Thomas Schelling at Harvard's JFK School has been kind enough to share this paradox of the departing voter. The idea of an "uncovered set" is based on an article by Nicholas Miller (1980). Finally, many thanks to all of those writing in with answers and new puzzles.*

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