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Puzzles: Queues, Coups and More

Barry Nalebuff

This is my ninth and last column of puzzles. I want to thank my readers for their support; without their contributions I would never have made it this long. The puzzles in this column look at queues, revolutions, squash, and more Pareto improvements, all with answers provided at the end. We continue to award T-shirts for the best new puzzles and most innovative answers. The new winners are Amahai Glazer, Peter Karpoff, John Londregan, William Hurley, Hal Varian, and one for me.

The editors are currently searching both for a new puzzles editor and for new puzzles. If you are interested in the job, this is your chance to apply by sending in a sample column with some of your favorite puzzles (with answers, please). Until the torch is actually passed, please keep sending your answers, comments, favorite puzzles and T-shirt size to me: Barry Nalebuff, "Puzzles," Yale School of Organization and Management, Box 1A, New Haven, CT 06520. Once again, I thank you and wish you good luck in solving all your puzzles, not just the ones below.

Puzzle 1: Crowd Teaser

Amahai Glazer (U.C.–Irvine) and Refael Hassin offer some thoughts about queues that might help pass the time when next you are waiting in line. They begin with the following observation: "In many circumstances, consumers encounter the phenomenon of length bias that is well known to statisticians. The essential idea is that the longer some individual stays in some sample population, the more likely he is to be sampled. Thus, because few passengers are on a flight that is uncrowded, the average

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airline passenger experiences flights that are more crowded than the average flight. Similarly, students may not understand why their tuition does not cover the university's expenses: their classes are almost always very large, though professors teach mostly small classes."

If any population is randomly split into two groups, it follows by definition that the majority of people must find themselves in the bigger group. This suggests that the representative or average person will have an upwardly-biased view about group size. How big is this bias? To answer this question, Glazer and Hassin offer the following puzzle.

"Let two identical roads connect Paris and Lyon. One of the roads has just been opened for public use (after complaints that the old one could not accommodate the traffic). The roads are of identical length and quality. Every morning, 2000 persons drive from Paris to Lyon. A driver chooses the road to travel, and once on it cannot change his decision. The travel time increases with the load on the road, and therefore a driver wants to be on the less crowded road. There is nothing to do about the following fact: *a majority of drivers will use the more crowded road*! This will be true no matter how decisions are made. Suppose no driver knows beforehand the number of others using the road that day, though he may reasonably suppose that other persons are as likely to choose one road as the other and these decisions are made independently.

"Consider the decision for a typical driver, say Monsieur Malheureux. Whichever road he chooses, with probability greater than 1/2, his road will be more crowded than the other road. Moreover, with 2000 cars travelling the two roads, the number of cars on the more crowded road is on average 36 more than the number on the other road."[1]

Since M. Malheureux is on the more crowded road more often than not and the more crowded road has 1018 travellers on average (and always more than 1000), M. Malheureux's experience would seem to diverge from the true average of 1000 per road. Thus we ask, to what extent does interviewing a random driver, such as M. Malheureux, provide a biased view of the actual extent of congestion?

Puzzle 2: Apocalypse Later

John Londregan (Carnegie Mellon, GSIA) has been investigating the relationship between coups and poverty. His research with Keith Poole (Londregan and Pool, 1990) indicates that the probability a government is ousted by a coup is a decreasing function of per capita income. This relationship may be an important consideration

[1]The expected difference between the more crowded and less crowded roads equals *twice* the expectation of $|1{,}000 - x|$ where x is the number of cars on one of the roads. Numerical calculation yields the value 36. To get a ballpark estimate, note that the standard deviation provides a reasonable approximation for the absolute value of the deviation from the mean. Here the variance is 500 and the standard deviation is thus 22. Our quick approximation of the expected maximum − minimum by 2σ gives 44, which is a slight overestimate of the true value, 36.

for the government of an otherwise poor country endowed with an exhaustible natural resource. How will the government's desire to remain in power affect the extraction rate of this resource? Londregan offers the following revolutionary puzzle:

"Imagine that you are the mayor of a besieged medieval town, with a finite stock of food. There is no probability that the besieging army will either quit or be driven off. The probability that your citizens revolt, and the town thereby falls, depends entirely on the rate at which food is consumed. To keep matters simple, suppose that the current probability of revolt is a *linearly* decreasing function of the current rate of food consumption."

More formally, the probability of a revolt during a given week is one if no food is consumed that week, and declines linearly with the level of food consumption during the week, up to the point where the probability of a revolt is zero (and remains at zero if still more food is consumed). Given an initial stock of food, the challenge is to calculate the time path of weekly food consumption that maximizes the expected longevity of the regime before the revolt. Is it better to consume a lot now and avoid the chance of a near-term revolt? Or is it better to consume less now, thus stretching out supplies and postponing the day of zero food? Will food consumption rise, fall, or stay constant over time?

Puzzle 3: Stamp of Approval

Peter Karpoff offers a combinatorics puzzle for stamp collectors. "In recent years, the Postal Service has occasionally issued stamps in a colorful sheet in which each of the 50 stamps has a different design. Collectors of used stamps like to get all 50 different, but often it's quite a challenge. Assume that collectors do not resort to sending themselves letters and that stamps received are a random sample of the 50 designs. How many stamps, including duplicates, will the typical collector accumulate before she can expect to have a complete set of 50?

"Hey, you say, where's the economics in this? Some stamp traders advertise that they will provide 7 required designs in exchange for any 10 stamps from the same group. Now what is the expected number of stamps for the collector to achieve the set of 50 (using the 7:10 trades)?"

Puzzle 4: Our Last Squash Problem, I Promise

William Hurley (Royal Military College of Canada) writes in with an interesting observation about the optimal choice of a tie-breaking strategy in squash. English squash is played to nine points. However, if the score is tied at 8 apiece, then the receiver has the choice of playing to 9 or 10. Whichever choice is made, the player to reach that score first wins.

Normally in these circumstances, it seems that the weaker player should always pick the shorter contest. Surely, the chance that an inferior player will be the first to score two points must be less than the chance of scoring one? But in English squash, winning a rally and scoring a point are not the same. The receiver is at a disadvantage because *only the server scores a point when winning a rally.* (The receiver earns the serve for winning a rally.) This asymmetry suggests that the receiver would like to extend the tie-breaker so as to reduce the server's advantage. How much worse must the receiver be before it becomes advantageous to pick the one-point tie-breaker?

Puzzle 5: Caveat Imitator

A serious problem for software vendors and recording artists is that the consumer can make copies of their product and then resell it. Hal Varian (1989) looks at how the potential for copying affects a monopolist's profitability. Imagine that the number of copies any customer can make is finite and equal to N (and one can't make copies of copies). Does the monopolist do better or worse when the potential for duplication, N, increases (and eventually becomes unlimited)? How do monopolists' profits change as the quality of customer copying goes up?

Answers to Puzzles

Answer to Puzzle 1

The bias rests on a difference between interviewing a random road (how many cars on a particular road?) and interviewing a random driver (how many cars were there on your road?). Interviewing drivers places a biased weight on congested roads since that is where, by definition, the majority of drivers are found.

Consider an example with three cars and two roads, where each driver is equally likely to take either road. The division will either be $(3,0)$, $(2,1)$, $(1,2)$ or $(0,3)$, with the first and last situations having a $1/8$ chance of happening, while the middle two each have a $3/8$ chance. Someone who interviews drivers will never interview anyone who is on a road with no cars. Yet $1/4$ of the time, one road will be empty. Averaging across roads, the number of cars is always $1\frac{1}{2}$. But averaging across drivers, the numbers of cars per road is 2. With probability $1/4$, all drivers will see three cars. With probability $3/4$, one driver will see just his own car while the other two drivers will see two cars; thus the average driver will observe $5/3$rds cars (a $1/3$ combination of seeing one car and a $2/3$rds combination of seeing 2). Combining the two cases shows that the average driver sees $3 \times 1/4 + 5/3 \times 3/4 = 2$ cars. Since the true average is $1\frac{1}{2}$, interviewing drivers gives an estimate of congestion on the road that is biased upwards by $1/2$.

What is surprising is that the bias remains at $\frac{1}{2}$ *independent* of the number of drivers. The reason is that the sole source of bias is due to the fact that by choosing to interview drivers, one will limit the sample to cases where there is at least one car on the road. Thus, if drivers were asked "how many *other* cars did you see on your road?" the answer would be an unbiased estimate of the congestion arising on any particular road from a population of $(N-1)$ commuters using two roads. A driver's own perception is off by $\frac{1}{2}$, because the driver fails to take into account the fact that the expected number of cars on the road he travels is greater than the expected number on the other road by exactly one—namely himself. The reported number of cars per road would be $(N-1)/2+1=(N+1)/2$ while the true average is $N/2$.

However, a bias of size $\frac{1}{2}$ is unlikely to explain the typical driver's perception that his road is the much more crowded or the students' view that their classes are all bigger than the average. One explanation is that drivers (or students) do not choose their roads (or courses) with equal probability. The sample bias becomes much more significant when the two roads are chosen with unequal weights. Thus imagine that one of the two roads is chosen with probability p and the other with probability $1-p$. Note that for all values of p, the average road still has $N/2$ cars since between the two roads there are N cars. But the sample based on the experience of drivers is now

$$1+(N-1)\left[p^2+(1-p)^2\right]>1+(N-1)/2 \quad \text{for } p\neq 1/2;$$

each driver sees himself and additionally, fraction p are on the road with $(N-1)p$ others while fraction $(1-p)$ are on a road with $(N-1)(1-p)$ others. Since the average is greater than $(N+1)/2$, the bias is even bigger than $\frac{1}{2}$. The worst case for a biased observation occurs when $p=1$ and all drivers pick the same road: no driver will report the empty road. One must interview the roads rather than the drivers in order to discover the fact that the excess congestion is due to a failure of drivers to coordinate which road they will take.

As a final thought, imagine that there is a coordination failure and we want to measure its severity. How should we estimate a random driver's probability, p, of picking one road? Let us continue to hold that the decisions are made independently. While the sample mean on each road would be an ideal estimator, the only data we have might be average experience across all drivers. If the average driver saw X cars on his road (including his own) then my guess is that we should use the p^* that solves

$$X=1+(N-1)\left[p^{*2}+(1-p^*)^2\right].$$

The solution is naturally symmetric in p^* and $(1-p^*)$ since we haven't specified which road is which. Dare I ask what are the large sample properties of this estimator as N becomes large?

Answer to Puzzle 2

The optimal consumption strategy is to consume a constant amount of food per week, just enough to keep the revolt probability at zero, and continue doing so until the supply runs out. To understand why this must be so, consider the situation where there are a finite (but large) number of periods.[2] If some other strategy were optimal

[2]The result remains true even with an infinite number of periods, but the argument is a little more tricky. In sending in the problem, Londregan included a formal proof for the case with an infinite number of periods.

then there must be some *first* period in which the chance of revolt becomes positive while food remains. Eventually there will be some later period in which the food supplies are exhausted and revolt becomes certain. Call these two periods A and B respectively, and consider trading a marginal amount of food between the two periods. Food spent in B period has little effect in delaying the revolt since at best it delays the revolt by one period. Moreover, there is some chance that period B will not be reached (since the probability of making it past A is less than 1), so the expected marginal delay of revolt from using food in period B is strictly less than Δ, where Δ is the rate at which food reduces revolt probability. In contrast, period A is always reached since it is the first period with a positive probability of a revolt. Thus, the expected marginal delay of revolt from using food in period A is at least Δ from the direct effect in period A and greater than Δ if there is any positive chance of lasting through period $A + 1$. Since the marginal contribution in A is greater than B, a path of increasing consumption over time cannot maximize the expected duration before revolt.

The general flavor of this result does not depend on the linear transformation between food and probability of revolt. It is always more beneficial to reduce the probability of a revolt sooner rather than later. Thus more resources are used early rather than later. With a nonlinear tradeoff between food consumption and revolt, the optimal policy employs a (weakly) decreasing level of consumption until food runs out.

Londregan concludes, "Although this example may seem fanciful . . . governments that control reserves of exhaustible natural resources, such as petroleum, can raise per capita income, and thereby increase their survival probability by hastening the pace of extraction. While the survival probability does not appear to be a linear function of income, the solution to the above problem suggests that the desire to remain in power could induce a policy of rapid resource depletion by governments facing a substantial risk of a coup."

Answer to Puzzle 3

The expected number of draws needed to get the $k + 1^{\text{th}}$ distinct stamp (out of N) is $N/[N - K]$; this is just the inverse of the chance of drawing one of the $N - k$ missing stamps from the pool of N. Thus the expected number of draws needed to get all 50 stamps is the sum of this series from K equals 0 to 49. Using the approximation that

$$S \sum_{k=0}^{49} \frac{1}{(N - k)} = \ln(N),$$

we have that the expected number of draws needed to get all 50 stamps is roughly $50 \ln 50 = 195$.

When trade is possible, even at 7 specified stamps for 10 arbitrary stamps, this number falls quite dramatically. However, the answer is rather complicated. One cannot simply look for a number of distinct stamps such as 34, so that the expected number of stamps needed to reach 34 distinct (which is 57) leads to 23 duplicates which at the stated exchange rate fills in the missing number (16). The reason is that it is not appropriate to sum the independent waiting times for the first 34 distinct stamps and then evaluate the number of duplicates at this expectation; since the waiting time is a nonlinear function, the expectation of this function need not equal the function of its expectation. What is the right number?

Answer to Problem 4

Let the inferior player's probability of winning a rally be p. This probability is assumed independent of who serves. What is the receiver's chance of winning for each of the two tie-breaking options? If the receiver is the inferior player and chooses a 1-point tie-breaker then we may calculate the probability of winning, W, as follows. The receiver may win two rallies in a row (p^2) and thus the game. Or the receiver may win the first rally and lose the second in which case the score remains at 8 all, the inferior player is once again receiving and we are back to the original situation where the chance of winning is W. This defines the recursive relationship:

$$W = p^2 + p(1 - p)W = p^2/[1 - p(1 - p)].$$

The case of a two-point tie-breaker is more complicated. Using a similar argument, Professor Hurley shows the receiver's chance of winning the game is

$$W' = \{p/[1 - p(1 - p)]\}^3[2 - p - p^2 + p^3].$$

These two probabilities are equal for $p = 0.382$! The weaker player must have less than a 38.2 percent chance of winning a rally before going for 1 point is optimal. Thus Hurley seems quite justified in his conclusion that when his squash opponent picks the one-point tie-breaker, "I can only infer that my opponent's reaction is rather complimentary." But Carl Shapiro, the top-ranked Princeton Economics squash player, adds a caveat. If the weaker player is in fact that much weaker, how likely is it that an 8-8 tie would have ever come about? In light of the fact that they reached an 8-8 tie, perhaps Hurley's opponent simply made the wrong calculation.

Answer to Question 5

If the monopolist anticipates the customer copying and this copying is of equal quality (and expense) to the monopolist's production, then Varian shows there is no effect on monopoly profits. Instead of selling the monopoly output Q^* at monopoly price p^*, the firm sells $Q^*/[N + 1]$ at price $[N + 1]p^* - C(N)$, where $C(N)$ is the customer's cost of making N copies. Each customer then makes N copies, sells them at price p^*, and the total quantity put up for sale in the market will equal Q^*. As a monopolist, the firm is able to extract all the expected resale profits from the customer's copying. Thus the price of the $Q^*/[N + 1]$ units is inflated upwards to account for the profits of the N resales. It is worth noting that this result depends critically on the restriction that one cannot make copies of copies. Otherwise, we obtain the competitive solution as the monopolist cannot control the total output level and thus restrict it to be at Q^*. In this light, it is interesting to note that the recording industry seems to have reached a compromise with the manufacturers of digital tape recorders; the digital tape decks will be installed with electronic circuitry so that customers will be able to make digital recordings from compact disks but will be prevented from making copies of these tapes.

The problem becomes more complicated if output is subject to an integer constraint (such as is the case with records and compact disks). If $Q^*/[N + 1]$ is restricted to be an integer, this limits the monopolist's ability to undo the effects of customer copying; but, even if N becomes very large, when copies can't be made of copies, the monopolist does not have to worry that the customer will use his large copying capacity to reach the competitive price. The customer becomes the monopoly seller and will restrict total output to Q^* (and the monopolist producer will sell one unit at price $pQ^* - C(Q^* - 1)$, reflecting the fact that the customer will only make $Q^* - 1$ copies, not N).

In this problem, the monopolist is effectively subcontracting production to the consumers. Consequently, the monopolist would like the consumer copies to be produced as cheaply as possible and to be of the maximum quality. Inferior quality and high copying cost simply lower the expected profits that the monopolist can extract from its consumer resellers.

Reader Mail

A previous column discussing optimal tolls as a way of solving the congestion over the Bay Bridge suggested tolls that varied by time of day. My Yale SOM colleague, Steve Heston, takes that proposal one step farther in his suggestion for a more efficient toll policy. Instead of having a constant toll price for all lanes, why not have clearly marked *differential* prices for different lanes even at the same time. Someone who didn't want to wait in line could pass quickly through the $5 lane while a driver with a high patience and low opportunity cost might choose to enter the presumably longer line for the free lane. This is reminiscent of the Paris subway where first-class cars and regular-class cars are identical: the only difference is the price. The higher priced first-class cars are less crowded and thus worth more. The reduced congestion at the higher-priced lanes would both justify the higher cost to many commuters while providing a voluntary progressive tax that improves efficiency.

Heston continued his toll proposals with the question of how to give the commuter a price break while maintaining the correct marginal incentives for the traveller not to use peak travel times. He suggests that the toll authority should "sell nontransferable annual rush hour tokens at quantity discounts. However, the quantity should be restricted to 80 percent of the number of annual work days. Workers with a fairly inelastic travel demand could purchase most of their tokens cheaply. But since most workers would run out of tokens by year end, they would face the same marginal travel cost decisions as non-commuters." Might it be possible for the toll authority to issue a universal price code type sticker on the side of a car that was valid for 200 round-trips?

Julian Simon (Univ. of Maryland, College Park) has done one better than propose a Pareto improvement: his proposal has come to pass. In 1966, Simon provided what he called "An Almost Practical Solution to Airline Overbooking." The essential idea was that when overbooking made it necessary to "bump" some passengers, the decision of whom to bump should be determined by auction. Those who are willing to accept the least amount are compensated by their stated demand until enough passengers have been bribed so that the number of passengers no longer exceeds capacity. Although Simon despaired that his proposal would not be implemented because, "It just isn't done..." today both bumped and unbumped passengers benefit from the present auction-like compensation system. However, there is a question of whether airlines are better off and even how that question would be tested.

Professor S. C. Littlechild (Birmingham, U.K.) offers two elegant alternative explanations for those who still remain unconvinced by the puzzle of whether to switch curtains, "Free to Choose," which appeared in the Fall 1987 issue. "(A) There are a million curtains. After the contestant chooses one, the host opens all but one of the others. Would you switch now? (B) A bag contains two [or a million] white balls and one black one. The first contestant chooses one ball without looking in the bag. The host then chooses one of the remaining balls *after* looking in the bag. The object is to get the black ball. Who would you rather be, the contestant or host?"

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