Online Appendix A to “Which Early Withdrawal Penalty Attracts the Most Deposits to a Commitment Savings Account?”

Supplementary Tables and Figures

John Beshears, Harvard University and NBER
James J. Choi, Yale University and NBER
Christopher Harris, University of Cambridge
David Laibson, Harvard University and NBER
Brigitte C. Madrian, Harvard University and NBER
Jung Sakong, University of Chicago

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Appendix Table A1. Percent of Decisions Allocating Strictly Positive Amount to Commitment Account: Experiment 1

For each experimental condition, this table reports the percent of decisions that allocate a strictly positive amount to the commitment account. There are three observations for every participant: one observation for each possible endowment amount. Standard errors clustered at the participant level are in parentheses. The table also gives p-values from tests comparing pairs of conditions, as indicated.

<table>
<thead>
<tr>
<th>Withdrawal restrictions on commitment account prior to commitment date</th>
<th>Commitment account interest rate</th>
<th>p-value for test of equality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>21%</td>
<td>22%</td>
</tr>
<tr>
<td>10% early withdrawal penalty</td>
<td>0.681</td>
<td>0.722</td>
</tr>
<tr>
<td></td>
<td>(0.053)</td>
<td>(0.050)</td>
</tr>
<tr>
<td>20% early withdrawal penalty</td>
<td>--</td>
<td>0.789</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.043)</td>
</tr>
<tr>
<td>No early withdrawals</td>
<td>--</td>
<td>0.823</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.046)</td>
</tr>
</tbody>
</table>

p-value for test of equality

- 10% penalty vs. 20% penalty: 0.310, 0.570
- 10% penalty vs. no early w/d: 0.142, 0.841
- 20% penalty vs. no early w/d: 0.590, 0.445
Appendix Table A2. Regression Analysis of Percent of Endowment Allocated to Commitment Account and Dollar-Weighted Days to Commitment Date: Experiment 1

This table reports the results of ordinary least squares regressions that use the sample of all allocation decisions in the first experiment. There are three observations for every participant: one observation for each possible endowment amount. In the first column, the outcome variable is the percent of endowment allocated to the commitment account. In the second column, the outcome variable is the dollar-weighted days to commitment date, which is the fraction of the endowment initially allocated to the commitment account multiplied by the number of days separating the initial allocation decision date and the commitment date. The explanatory variables are indicator variables for different interest rates, indicator variables for different withdrawal restrictions on the commitment account prior to the commitment date, and the interactions of those indicator variables. The omitted category is the condition featuring a 22% interest rate and a 10% early withdrawal penalty for the commitment account. Standard errors clustered at the participant level are in parentheses.

<table>
<thead>
<tr>
<th>Explanatory variable</th>
<th>% of endowment allocated to commitment account</th>
<th>Dollar-weighted days to commitment date</th>
</tr>
</thead>
<tbody>
<tr>
<td>21% interest rate</td>
<td>-11.3* (4.4)</td>
<td>-17.4 (11.6)</td>
</tr>
<tr>
<td>23% interest rate</td>
<td>19.3** (4.8)</td>
<td>47.8** (13.9)</td>
</tr>
<tr>
<td>20% early withdrawal penalty</td>
<td>5.9 (4.8)</td>
<td>18.7 (14.1)</td>
</tr>
<tr>
<td>No early withdrawals</td>
<td>17.1** (5.3)</td>
<td>50.0** (16.5)</td>
</tr>
<tr>
<td>23% interest rate × 20% early withdrawal penalty</td>
<td>-2.9 (6.8)</td>
<td>-21.3 (21.5)</td>
</tr>
<tr>
<td>23% interest rate × no early withdrawals</td>
<td>-15.4* (7.2)</td>
<td>-61.7** (22.6)</td>
</tr>
<tr>
<td>Constant (22% interest rate, 10% early withdrawal penalty is omitted category)</td>
<td>38.9** (3.4)</td>
<td>81.8** (9.0)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.128</td>
<td>0.060</td>
</tr>
<tr>
<td>$N$</td>
<td>1,485</td>
<td>1,485</td>
</tr>
</tbody>
</table>

* Significant at the 5% level. ** Significant at the 1% level.
Appendix Table A3. Percent of Participants Allocating Strictly Positive Amount to Commitment Account: Experiment 2

For each experimental condition, this table reports the percent of participants allocating a strictly positive amount to a commitment account. For the condition offering two commitment accounts, the table also reports the percent of participants allocating a strictly positive amount to each individual commitment account. Standard errors are in parentheses.

<table>
<thead>
<tr>
<th>Withdrawal restrictions on commitment account prior to commitment date</th>
<th>% of participants using commitment account</th>
</tr>
</thead>
<tbody>
<tr>
<td>Safety valve (withdrawals only in financial emergencies)</td>
<td>75.3</td>
</tr>
<tr>
<td></td>
<td>(3.5)</td>
</tr>
<tr>
<td>10% early withdrawal penalty</td>
<td>83.0</td>
</tr>
<tr>
<td></td>
<td>(3.8)</td>
</tr>
<tr>
<td>No early withdrawals</td>
<td>90.7</td>
</tr>
<tr>
<td></td>
<td>(2.4)</td>
</tr>
<tr>
<td>Two commitment accounts: strictly positive allocation to either 10%</td>
<td>80.7</td>
</tr>
<tr>
<td>early withdrawal penalty account or no early withdrawals account</td>
<td>(3.2)</td>
</tr>
<tr>
<td>10% early withdrawal penalty account</td>
<td>56.0</td>
</tr>
<tr>
<td></td>
<td>(4.1)</td>
</tr>
<tr>
<td>No early withdrawals account</td>
<td>75.3</td>
</tr>
<tr>
<td></td>
<td>(3.5)</td>
</tr>
</tbody>
</table>
Appendix Figure A1. Description of the Liquid Account

The **Freedom Account** is designed to let you access your money whenever you want. You can withdraw money from this account any time over the next year, starting one week from today.

Money in the Freedom Account will grow at an interest rate of 22% per year until you withdraw it. When you withdraw money from the Freedom Account, you don't have to withdraw all of it. Whatever you leave in the account will continue to earn 22% interest until the end of the experiment, one year from today.
Appendix Figure A2. Description of the 22% Interest Rate, 10% Early Withdrawal Penalty Commitment Account

The Goal Account is designed to help you save. You can withdraw money from this account without penalty any time after a goal date that you pick. Setting a goal for yourself and picking the right goal date can help you avoid the temptation to spend your money too soon.

Money in the Goal Account will grow at an interest rate of 22% per year, both before and after the goal date, until you withdraw it. When you withdraw money from the Goal Account, you don’t have to withdraw all of it. Whatever you leave in the account will continue to earn 22% interest until the end of the experiment, one year from today.

As explained earlier, if you withdraw money from the Goal Account before your goal date, you will incur a penalty equal to 10% of the amount you withdraw.

To use the Goal Account, you will need to pick a goal date. You might want to pick a date based on something you want to save money for, like a birthday gift, holiday presents, vacation, or any other special purchase that you plan to make. You can also use the Goal Account as a way to help you save, even if you don’t have a special purchase in mind.
Suppose you receive $50. How would you like to divide it between the two accounts?

<table>
<thead>
<tr>
<th>Freedom Account</th>
<th>Goal Account</th>
</tr>
</thead>
<tbody>
<tr>
<td>• No goal date</td>
<td>• You pick the goal date, no earlier than one week from today</td>
</tr>
<tr>
<td>• Withdraw money any time you want to, starting one week from today</td>
<td>• If you choose to withdraw money before the goal date you will incur a penalty of 10%</td>
</tr>
<tr>
<td>• 22% interest per year</td>
<td>• 22% interest per year</td>
</tr>
</tbody>
</table>

| $ ______.00 | $ ______.00 |

Remember, if you receive $50, it will be divided between the accounts based on this decision.

If you have decided to put some money into the Goal Account, please choose a goal date below.

Click here  Click here  Click here

Would you like to share your goal with us (e.g. birthday gift, holiday presents, vacation, general saving)? If yes, enter it here:

[Box for entering goal]

[Buttons: Next>>  Instructions]
Appendix Figure A4. Sample Weekly Email to Participant

Dear Participant,

This is a breakdown of your current balances:

Freedom Account: $24.25  
Goal Account: $53.18  
Goal Date: July 20th, 2010

If you wish to withdraw any money from your accounts, please go to your panel pages and click on the "Savings Game" button: https://mmic.rand.org/panel

If you have any questions about this game or your accounts, please feel free to contact us at webhelp@rand.org or 866.591.2909

Thanks!  
www.rand.org/alp
Appendix Figure A5. Withdrawal Interface

Freedom Account
remaining balance: $100.70

Goal Account
remaining balance: $105.47
goal date: July 20th, 2010

* If you make a withdrawal, a check will be mailed to you within the next three business days.
Appendix Figure A6. Balance Ratios by Experimental Condition: Experiment 1

For each experimental condition, these figures show withdrawal patterns over the course of the experiment. For each participant and for each day, we calculate the sum of the liquid account and commitment account balances that the participant would have had if no withdrawals had been requested. This hypothetical total balance takes as given the participant’s initial allocation between the liquid account and the commitment account, and it uses the allocation decision that applies to the ex post realization of the endowment amount ($50, $100, or $500). We then calculate the ratio of the participant’s actual balance to the hypothetical total balance, and we plot the mean of this ratio against the number of days since the initial deposit into the participant’s accounts.

10% withdrawal penalty conditions

22% commitment account interest rate conditions

23% commitment account interest rate conditions

Days since endowment received
Appendix Figure A7. Balance Ratios by Experimental Condition: Experiment 1
For each experimental condition, these figures show withdrawal patterns over the course of the experiment. For each participant and for each day, we calculate the sum of the liquid account and commitment account balances that the participant would have had if no withdrawals had been requested. This hypothetical total balance takes as given the participant’s initial allocation between the liquid account and the commitment account, and it uses the allocation decision that applies to the ex post realization of the endowment amount ($50, $100, or $500). We then calculate the ratio of the participant’s actual balance to the hypothetical total balance, and we plot the mean of this ratio at four points in time: the day of the initial deposit into the participant’s accounts, three days before the participant’s commitment date, three days after the participant’s commitment date, and three days before remaining account balances were automatically disbursed to the participant. For participants who did not allocate any funds to a commitment account, we use the balance ratio on the initial deposit date as the balance ratio three days before the commitment date, and we use the balance ratio three days after the initial deposit date as the balance ratio three days after the commitment date.
Appendix Figure A8. Description of Two Commitment Accounts Offered Simultaneously

The Goal Accounts are designed to help you save. You can withdraw money from these accounts any time on or after goal dates that you pick. Setting goals for yourself and picking the right goal dates can help you avoid the temptation to spend your money too soon.

There are two types of Goal Accounts:

- Goal Account A (10% Penalty) allows you to withdraw your money before its goal date, but you will be charged a 10% penalty on early withdrawals. For example, if you withdraw $10 before your goal date, your account balance will be reduced by $11.
- Goal Account B (No Withdrawal) does not allow withdrawals before its goal date.

If you choose to use both Goal Accounts, you can pick a different goal date for each Goal Account, or you can pick the same goal date.

Money in both Goal Accounts will grow at an interest rate of 22% per year, both before and after the goal date, until you withdraw it. When you withdraw money from a Goal Account, you don’t have to withdraw all of it. Whatever you leave in the accounts will continue to earn 22% interest until the end of the experiment on September 1, 2011.
Appendix Figure A9. Description of the Safety Valve Commitment Account

The **Goal Account** is designed to help you save. You can withdraw money from this account any time **on or after** a goal date that you pick. Setting a goal for yourself and picking the right **goal date** can help you avoid the temptation to spend your money too soon.

You cannot withdraw from this account **before** the goal date, except in the case of a financial emergency. If you have a financial emergency, you can make an early withdrawal. We are relying on you to be honest in judging whether you have a financial emergency.

Money in the Goal Account will grow at an interest rate of 22% per year, both before and after the goal date, until you withdraw it. When you withdraw money from the Goal Account, you don't have to withdraw all of it. Whatever you leave in the account will continue to earn 22% interest until the end of the experiment on September 1, 2011.
Appendix Figure A10. Withdrawal Patterns for Own versus All Liquid Allocation: Experiment 2

For each experimental condition, these figures show withdrawal patterns over the course of the experiment for participants who were randomly assigned to receive their chosen allocations and for participants who were randomly assigned to receive their entire endowment in the liquid account. For each participant and for each day, we calculate the sum of the liquid account and commitment account balances that the participant would have had if no withdrawals had been requested. We then calculate the ratio of the participant’s actual balance to this hypothetical total balance, and we plot the mean of this ratio against the number of days since the initial deposit into the participant’s accounts.
Appendix Figure A11. Withdrawal Patterns for Own versus All Liquid Allocation: Experiment 2

For each experimental condition, these figures show withdrawal patterns over the course of the experiment for participants who were randomly assigned to receive their chosen allocations and for participants who were randomly assigned to receive their entire endowment in the liquid account. For each participant and for each day, we calculate the sum of the liquid account and commitment account balances that the participant would have had if no withdrawals had been requested. We then calculate the ratio of the participant’s actual balance to this hypothetical total balance, and we plot the mean of this ratio at four points in time: the day of the initial deposit into the participant’s accounts, three days before the participant’s commitment date (three days before the participant’s earliest commitment date in the case of participants who had more than one), three days after the participant’s commitment date (three days after the participant’s latest commitment date in the case of participants who had more than one), and three days before remaining account balances were automatically disbursed to the participant. For participants who did not allocate funds to a commitment account, we use the actual balance and the hypothetical total balance on the initial deposit date when calculating the withdrawal measure for three days before the commitment date, and we use the actual balance and the hypothetical total balance three days after the initial deposit date when calculating the withdrawal measure for three days after the commitment date.
To study the tradeoff between commitment and flexibility in a consumption/savings context, Amador, Werning, and Angeletos (2006; hereafter AWA) use a model with three conceptual ingredients.

First, AWA assume dynamically inconsistent preferences generated by the present-biased discount function

\[ D(\tau) = \begin{cases} 
1 & \text{if } \tau = 0 \\
\beta & \text{if } \tau \geq 1 
\end{cases} \]

where \( 0 < \beta < 1 \) (Phelps and Pollak, 1968; Laibson, 1997).\(^1\) This discount function implies that, from the perspective of period 0, the agent is more patient about tradeoffs between periods 1 and 2 than she will be when period 1 actually arrives:

\(^1\) The analysis that follows would be nearly identical if we were to use the more general quasi-hyperbolic discount function given by \( D(0) = 1 \) and \( D(\tau) = \beta \delta^\tau \) for \( \tau \geq 1 \), where \( 0 < \beta < 1 \) and \( 0 < \delta \leq 1 \). For simplicity, we follow AWA and set \( \delta = 1 \).
Dynamically inconsistent preferences generate a motivation for precommitment.

Second, they assume that the agent experiences transitory taste shocks that are not observable in advance and are not contractable. Such taste shocks generate a motivation to give future selves flexibility in choosing the consumption path.

Third, they assume that the agent has a very general commitment technology. Specifically, she can manipulate the choice sets of future selves, trading off the benefits of commitment (preventing later selves from overconsuming) and the costs of commitment (preventing later selves from responding flexibly to the taste shocks).

We enrich AWA’s analysis by placing a bound on the strength of the commitment technology. We show that, in this more general setting, the agent can still achieve the (second-best) optimum using a simple commitment mechanism. Furthermore, we vary the bound and explore the implications for the choice of commitment mechanism. These comparative statics enable us to compare the model’s predictions with the behavior of our experimental participants.

We briefly describe the key properties of the model below. Online Appendix C Sections 1-16 provide a complete exposition and analysis of the model.

\[
\frac{D(1)}{D(2)} = \frac{\beta}{\beta} < 1 = \frac{D(0)}{D(1)}.
\]

Dynamically inconsistent preferences generate a motivation for precommitment.

The simplest model that elicits a tradeoff between commitment and flexibility has three periods: an initial period in which some degree of commitment is created with respect to future decisions; a following period in which a consumption/savings choice is made with immediate utility consequences; and a final period in which residual wealth is consumed.

**Period 0.** Self 0 chooses the commitment mechanism that will govern the choices of selves 1 and 2. (There is no consumption in period 0.)

**Period 1.** A taste shock \( \theta \in \Theta = [\underline{\theta}, \bar{\theta}] \) is realized. Self 1 observes \( \theta \) and makes a consumption/savings decision, subject to the constraints imposed by the commitment mechanism chosen by self 0.

**Period 2.** Self 2 consumes all remaining wealth.

Section B below describes the set of commitment mechanisms available to self 0, and Section C sets out our assumptions on the distribution of \( \theta \). Note that the three-period structure maps...
directly onto our experimental setup, with period 0 corresponding to the initial allocation
decision, period 1 corresponding to the time between the allocation decision and the commitment
date (which was tailored by each participant according to the time horizon over which the
temptation to overspend was relevant), and period 2 corresponding to the time after the
commitment date.

Let \( c_1 \) and \( c_2 \) denote the consumption levels of selves 1 and 2. Then underlying
preferences at dates 0, 1, and 2 can be specified as follows:

\[
\text{utility of self 0} = \beta \theta U_1(c_1) + \beta U_2(c_2) \\
\text{utility of self 1} = \theta U_1(c_1) + \beta U_2(c_2) \\
\text{utility of self 2} = U_2(c_2)
\]

Here \( U_t \) is the utility function at time \( t \). We assume that: \( U_t : [0, \infty) \rightarrow [-\infty, \infty) \); \( U_t' > 0 \); \( U_t'' < 0 \) on
\( (0, \infty) \); and \( U_t'(0+) = \infty \). \(^2\)

We also assume that self 0 fully understands and anticipates the preferences of self 1.
That is, we assume that the agent is sophisticated.

B. Commitment technology

A commitment mechanism is modeled as a budget set \( B \) chosen by self 0. This \( B \) is the
set of consumption pairs \((c_1, c_2)\) that can be chosen by self 1. Recall that the taste shock is not
yet observable in period 0, and that it will only be privately observable in period 1, so \( B \) cannot
be conditioned on the realization of the taste shock.

Let \( y > 0 \) be the agent’s exogenous budget and, without loss of generality, let the gross
interest rate be unity. To map to our experimental design, \( y \) can be interpreted as the agent’s total
wealth if the participants integrate the experimental windfall with their other wealth, or it can be
interpreted as only the windfall itself if the participants psychologically code the windfall as part
of a separate mental account. The main implications of the model apply in both cases.

Let the “ambient budget set” \( A \) be the set of all consumption pairs \((c_1, c_2)\) such that
\( c_1, c_2 \geq 0 \) and \( c_1 + c_2 \leq y \). Fix a parameter \( \pi \in [0, \infty) \), which will bound the strength of the

---

\(^2\) For example, it could be that \( U_t \) has constant relative risk aversion \( \rho_t > 0 \). In that case: if \( \rho_t \in (0,1) \), then \( U_t(0) > -\infty \); and if \( \rho_t \in [1, \infty) \), then \( U_t(0) = -\infty \). In particular, we do not require \( U_t(0) = -\infty \).
commitment mechanism. Then the budget set $B$ chosen by self 0 must satisfy the following two constraints:

**Constraint 1.** $B$ is a non-empty compact subset of $A$.

**Constraint 2.** The penalty for transferring consumption from period 2 to period 1 is no greater than $\pi$.\(^3\)

In other words, self 0 can choose a budget set of almost any size and shape. The only restriction on size is that $B$ must be small enough to fit inside $A$. The only restriction on shape is that, starting from any consumption pair $(c_1, c_2) \in B$ such that $c_2 > 0$, self 1 must always be able to transfer a small amount $\Delta > 0$ of consumption from period 2 to period 1. She may face a penalty for doing so, in the form of a reduction in consumption in period 2 (over and above that resulting from the transfer itself). However, this penalty will never be greater than $\pi \Delta$.\(^4\)

A wide variety of budget sets satisfy Constraints 1 and 2. For example, the budget set shown in Appendix Figure B1 consists of: (i) a downward sloping budget curve that begins on the $c_2$ axis and ends on the $c_1$ axis; and (ii) all the points of $A$ that lie below or to the left of the budget curve. It obviously satisfies Constraint 1. It satisfies Constraint 2 if $\pi = 0.5$, but not if $\pi = 0.1$. Indeed, the slope of the budget curve at the encircled point is $-1.3$. This is greater than $-1.5$ (the most negative slope that is permissible when $\pi = 0.5$), but less than $-1.1$ (the most negative slope that is permissible when $\pi = 0.1$).\(^5\)

---

\(^3\) To be precise, we require that, for all $(c_1, c_2) \in B$ and all $\bar{c}_i \in [c_1, c_1 + \frac{1}{1+\pi} c_2]$, there exists $\hat{c}_i$ such that: (i) $(\hat{c}_1, \hat{c}_2) \in B$; and (ii) $\hat{c}_i \geq c_i - (1+\pi)(\hat{c}_i - c_i)$. In other words, if we take any point $(c_1, c_2)$ in $B$, and if we draw a line $L$ of slope $-(1+\pi)$ through $(c_1, c_2)$, then: associated with every point $(\hat{c}_1, \hat{c}_2)$ on the segment of $L$ that joins the points $(c_1, c_2)$ and $(c_1 + \frac{1}{1+\pi} c_2, 0)$, there is a point $(\tilde{c}_1, \tilde{c}_2) \in B$ that either coincides with $(\hat{c}_1, \hat{c}_2)$ or else lies vertically above it. (Note that $(c_1 + \frac{1}{1+\pi} c_2, 0)$ is the point at which $L$ crosses the $c_1$ axis.)

\(^4\) Perhaps more intuitively, this property can also be expressed in differential terms: starting from any consumption pair $(c_1, c_2) \in B$ such that $c_2 > 0$, it must be possible to draw a curve in $B$ of slope at least $-(1+\pi)$, and to continue that curve up to the point at which it first hits the $c_1$ axis.

\(^5\) Notwithstanding its obvious generality, this budget set is still special in a number of respects. We give four examples. First, there is nothing in Constraints 1 and 2 that requires that the budget curve be downward sloping. Indeed, these constraints place no upper bound at all on the slope of the budget curve. Second, there is no reason why the budget curve need begin on the $c_1$ axis. It could perfectly well begin at some $(c_1^0, c_2^0) \in A$ for which $c_1^0 > 0$. (Constraint 2 does, however, require that the budget curve end on the $c_1$ axis.) Third, there is no reason why points below or to the left of the budget curve need be included. Fourth, there is no reason why the budget curve need be connected. It could perfectly well consist of two or more components. For example, a first component might begin at some $(c_1^0, c_2^0) \in A$ for which $c_1^0 > 0$ and end at some $(c_1^4, c_2^4) \in A$ for which $c_2^4 = 0$. A second
However, as we shall see, the optimum can be obtained using a particularly simple kind of budget set, namely a two-part budget set. Such budget sets consist of: (i) a budget curve that has slopes of $-1$ and $-(1 + p)$ to the left and right of a kink at $(c_1^*, c_2^*)$; and (ii) none, some, or all of the points of $A$ that lie below or to the left of the budget curve. For example, the budget set shown in Appendix Figure B2 consists of just such a budget curve, together with all of the points of $A$ that lie below or to the left of it.

Better still, two-part budget sets arise naturally in practical applications. Indeed, suppose that self 0 sets up two separate accounts: (i) a fully liquid account with balance $c_1^*$; and (ii) a partially illiquid account with balance $c_2^*$ and an early withdrawal penalty $p$. Then self 1 will face a two-part budget set.

C. Distribution of the taste shock

AWA show that their problem can be reduced to a problem in the class of optimization problems identified and analyzed by Luenberger (1969). We follow AWA’s lead. We make the following assumptions on the distribution function $F$ of the taste shock $\theta$.

A1 Both $F$ and $F''$ are functions of bounded variation on $(0, \infty)$.\(^7\)

A2 The support of $F'$ is contained in $[\underline{\theta}, \bar{\theta}]$, where $0 < \underline{\theta} < \bar{\theta} < \infty$.

A3 Put $G(\theta) = (1 - \beta) M F'(\theta) + F(\theta)$. Then there exists $\theta_M \in [\underline{\theta}, \bar{\theta}]$ such that: (i) $G' \geq 0$ on $(0, \theta_M)$; and (ii) $G' \leq 0$ on $(\theta_M, \infty)$.\(^8\)

---

\(^6\) By saying that self 1 pays a penalty $p$ on early withdrawals from the second account, we mean that if she consumes $\Delta$ from the second account then that account is debited $(1 + p)\Delta$.

\(^7\) A sufficient condition for A1 is that: (i) $F'$ and $F''$ both exist; and (ii) $\int_0^\infty |F'(\theta)| d\theta$ and $\int_0^\infty |F''(\theta)| d\theta$ are both finite. In other words, if one walks along the graph of $F$ or $F'$, then the total vertical distance travelled (both up and down) is finite. We do not use this stronger condition because we want to allow for densities, like that of the uniform distribution, that have jumps at $\underline{\theta}$ and $\bar{\theta}$. Indeed, a good way of generating examples is to take a standard distribution and truncate it at suitable points $\underline{\theta}$ and $\bar{\theta}$. This procedure typically results in discontinuities in $F$ and $F'$ at $\underline{\theta}$ and $\bar{\theta}$.

\(^8\) A3 is slightly stronger than the analogous assumption in AWA, namely their Assumption A. However: (i) it is not clear that our results for the model with $\pi < \infty$ actually hold under AWA’s A; (ii) A3 is easier to state than AWA’s A; and (iii) it is easier to check whether a given distribution satisfies A3 than to check whether it satisfies AWA’s A.
We now comment on these assumptions. A function \( f : (0, \infty) \rightarrow \mathbb{R} \) is of bounded variation if and only if it is the difference of two bounded and non-decreasing functions \( f_1, f_2 : (0, \infty) \rightarrow \mathbb{R} \). Since \( F \) is a distribution function, it is automatically a function of bounded variation. The substance of A1 is therefore the requirement that \( F \) has a density \( F' \) that is a function of bounded variation. A2 means that \( F' = 0 \) on \((0, \infty) \setminus [\underline{\theta}, \bar{\theta}]\). Notice that \( F' \) need not be continuous. In particular, it can jump up at \( \underline{\theta} \) and down at \( \bar{\theta} \). A3 means that \( G \) is first increasing and then decreasing. It implies that the support of \( F' \) is connected. It is preserved under truncation: if a distribution function \( F \) satisfies A3, then so too does the distribution function obtained by truncating \( F \) at \( \underline{\theta} \) and \( \bar{\theta} \).

A3 is satisfied by many of the distributions that one encounters in practice. To illustrate this point, we have made a list of all the distributions occurring in either of two leading statistics textbooks: Rice (1995) and Hogg, McKean, and Craig (2005). This list contains 18 distributions. Of these, 14 satisfy A1-A3 for all parameter values (including \( \underline{\theta} \) and \( \bar{\theta} \)). More precisely, we have:

**Remark** Suppose that \( D \) is one of the Burr, Chi-squared, Exponential, Extreme Value, \( F \), Gamma, Gompertz, Log-Normal, Maxwell, Normal, Rayleigh, \( t \), Uniform, and Weibull distributions. Then, for any \( 0 < \underline{\theta} < \bar{\theta} < \infty \), the distribution function \( F \) obtained by truncating \( D \) at \( \underline{\theta} \) and \( \bar{\theta} \) satisfies Assumptions A1-A3.\(^9\)

The four exceptions are the Beta, Cauchy, Log-Gamma, and Pareto distributions. In the form in which it appears in both Rice and Hogg, McKean, and Craig, the Beta distribution does in fact satisfy A1-A3. However, for our purposes, it is more natural to consider a generalization of the Beta distribution for which the support is a compact interval contained in \((0, \infty)\). For this distribution, A3 is not always satisfied.\(^{10}\) Similarly, the standard Cauchy distribution, which is

\(^9\) Notice that 5 of these 14 distributions (namely the Burr, Chi-squared, F, Gamma, and Weibull distributions) are unbounded at zero for some parameter values. However, the truncated distributions all satisfy A1 because \( \underline{\theta} > 0 \).

\(^{10}\) The density of the generalization of the Beta that we consider is proportional to \((x - a)^{\zeta - 1}(b - x)^{\eta - 1}\) on the interval \((a, b)\), where \( 0 < a < b < \infty \) and \( \zeta, \eta > 0 \). Exceptions to A3 occur when \( \zeta < 1 \).
the form of the Cauchy distribution considered in both Rice and Hogg, McKean, and Craig, satisfies A1-A3. However, in its general form, the Cauchy distribution fails A3 for some choices of the parameter values.\textsuperscript{11} Next, the Log-Gamma distribution appears only in Hogg, McKean, and Craig. This distribution may or may not satisfy A3, depending on the parameters.\textsuperscript{12} Finally, Rice and Hogg, McKeen, and Craig each consider a special case of the Pareto distribution. Both of these special cases satisfy A1-A3. However, in general, the Pareto distribution fails A3 for some choices of the parameter values.\textsuperscript{13} For additional discussion of these exceptional cases, see Online Appendix C Section 17.

D. Theorems and relationship to experimental results

AWA show that, when there is no bound on the strength of the commitment technology, an optimal choice for self 0 is a minimum-savings rule. In our terminology, this can be expressed by saying that an optimal choice for self 0 is to divide her endowment \( y \) between two accounts: (i) a fully liquid account that places no penalty on withdrawals in either period 1 or period 2; and (ii) a fully illiquid account that disallows any withdrawals in period 1 but places no penalty on withdrawals in period 2.\textsuperscript{14,15} Our first result generalizes AWA’s result to the case in which there is a bound \( \pi \) on the strength of the commitment mechanism.

\textsuperscript{11} The density of the general form of the Cauchy distribution is proportional to \( \left( 1 + \left( \frac{x - \mu}{\sigma} \right)^2 \right)^{-1} \) on \( \mathbb{R} \), where \( \mu \in \mathbb{R} \) is a location parameter and \( \sigma > 0 \) is a scale parameter. Exceptions to A3 occur when \( \mu/\sigma \) is large and positive.

\textsuperscript{12} The density of the Log-Gamma distribution is proportional to \( x^{-\zeta} \left( \log(x) \right)^{\eta-3} \) on \((1,\infty)\), where \( \zeta, \eta > 0 \). Exceptions to A3 occur when \( \zeta < 1 \) and \( \eta > 1 - \beta \).

\textsuperscript{13} For example, the density of the Pareto type II distribution is proportional to \( \left( 1 + \frac{x - \mu}{\sigma} \right)^{-\zeta} \) on \((\mu,\infty)\), where \( \mu \in \mathbb{R} \) is a location parameter, \( \sigma > 0 \) is a scale parameter and \( \zeta > 0 \) is a shape parameter. Exceptions to A3 occur when \( \zeta \) is small and \( \mu/\sigma \) is large and positive.

\textsuperscript{14} There is a small technical difference between a fully illiquid account and a partially illiquid account with penalty \( p = \infty \). If self 0 places \( y_{\text{liquid}} \) in a fully liquid account and \( y - y_{\text{liquid}} \) in a fully illiquid account, then she is effectively choosing a budget set that consists of the line segment joining the two points \((0, y)\) and \((y_{\text{liquid}}, y - y_{\text{liquid}})\). On the other hand, if she places \( y_{\text{liquid}} \) in a fully liquid account and \( y - y_{\text{liquid}} \) in a partially illiquid account with penalty \( p = \infty \), then she is effectively choosing a budget set that consists of all points on or vertically below the line segment joining the two points \((0, y)\) and \((y_{\text{liquid}}, y - y_{\text{liquid}})\). (In effect, an illiquid account with penalty \( p = \infty \) gives self 1 the possibility of free disposal, whereas a fully illiquid account does not.) Of course, these two mechanisms are equivalent from the point of view of self 0, since self 1 will always choose from the line segment joining the two points \((0, y)\) and \((y_{\text{liquid}}, y - y_{\text{liquid}})\). Therefore, we shall not distinguish between them in what follows.

\textsuperscript{15} Ambrus and Egorov (2013) provide additional analysis of AWA’s model.
Theorem 1 Suppose that $U_1 = U_2 = U$, and that $U$ has constant relative risk aversion $\rho > 0$. Then an optimal choice for self 0 is to divide her endowment $y$ between two accounts: (i) a fully liquid account with no penalty on withdrawals in either period 1 or period 2; and (ii) a partially illiquid account, with a penalty $p = \pi$ on withdrawals in period 1 and no penalty on withdrawals in period 2.\textsuperscript{16}

See Online Appendix C Sections 2 through 9.

Theorem 1 implies that there is no advantage to self 0 in using more than two accounts, in using accounts with more complex conditions attached to them, in using accounts with a penalty $p < \pi$, or in using some commitment mechanism other than accounts.

Our next result gives a sufficient condition under which self 0’s allocation to the partially illiquid account is monotonic in $p$.

Theorem 2 Suppose that $U_1 = U_2 = U$, and that $U$ has constant relative risk aversion $\rho = 1$ (i.e., that $U = \log$).\textsuperscript{17} Suppose further that self 0 must divide her endowment $y$ between two accounts: (i) a fully liquid account with no penalty on withdrawals in either period 1 or period 2; and (ii) a partially illiquid account with a penalty $p$ on withdrawals in period 1 and no penalty on withdrawals in period 2. Denote the optimal allocations to the two accounts by $y_{\text{liquid}}$ and $y_{\text{penalty}}$. Then the correspondence mapping $p$ to the set of possible choices for $y_{\text{penalty}}$ is weakly increasing. Specifically:\textsuperscript{18}

\textsuperscript{16} As the wording of the Theorem implies, the optimal choice of self 0 is not unique. Indeed, as long as one thinks of self 0 as choosing a budget set $B$, her optimal choice is inherently non-unique. This is because, starting from any given $B$ (optimal or not), one can make equivalent budget sets by adding or removing consumption pairs that would not be chosen by any type. This particular form of non-uniqueness can be eliminated if, instead of thinking of self 0 as choosing a budget set $B$, we think rather of her as choosing an incentive-compatible consumption curve $c = (c_1, c_2): [\tilde{\theta}, \tilde{\theta}] \rightarrow A$.

\textsuperscript{17} We have conducted a number of simulations for the case $\rho \neq 1$. These simulations suggest that, in practice, $y_{\text{penalty}}$ is non-decreasing in $p$ for reasonable calibrations of the model. However, our analytic results reveal that the simulation results are not completely generalizable, and we can construct counterexamples to the claim that $y_{\text{penalty}}$ is non-decreasing in $p$ when $\rho \neq 1$. We recover the result that $y_{\text{penalty}}$ is non-decreasing in $p$ when $\rho = 1$ if we impose further restrictions on the distribution function $F$ beyond Assumptions A1-A3 (see Online Appendix C Section 15).

\textsuperscript{18} In the current case, there exist two cutoffs $\pi_0 \geq 0$ and $\pi_1 > \pi_0$. We say that $p$ is small if $p \in (0, \pi_0)$, intermediate if $p \in (\pi_0, \pi_1)$, and large if $p \in (\pi_1, \infty)$. The significance of $\pi_0$ is that the optimal allocation to the partially illiquid account is non-unique when $p < \pi_0$ and unique when $p > \pi_0$. The significance of $\pi_1$ is that the maximum-penalty constraint is strictly binding when $p < \pi_1$ and strictly slack when $p > \pi_1$. We typically have $\pi_0 = 0$. In that case, non-uniqueness occurs only when $p = 0$, and then only for the trivial reason that the liquid account and the partially illiquid account are indistinguishable (and self 0 is therefore indifferent as to how she divides $y$ between the two
1. For small values of $p$, there may be a non-trivial interval $[y_{\text{penalty}}, \bar{y}_{\text{penalty}}]$ of possible choices for $y_{\text{penalty}}$.\footnote{The form of non-uniqueness discussed here arises even if (as in Footnote 16) we think of self 0 as choosing an incentive-compatible consumption curve. It can be eliminated, as we show in Online Appendix C Section 10, by introducing an additional assumption, namely: (A4) $G$ is strictly increasing on $(0,\bar{p}_p)$. However: (i) A4 is a little out of keeping with the rest of our framework; (ii) we can obtain satisfactory comparative-statics results without A4; and (iii) the form of non-uniqueness described rarely arises even in the absence of A4. We do not therefore include A4 among our basic assumptions.} However, $y_{\text{penalty}}$ is strictly increasing in $p$, and $\bar{y}_{\text{penalty}}$ is independent of $p$.

2. For intermediate values of $p$, there is a unique choice for $y_{\text{penalty}}$.\footnote{In other words, if one thinks of self 0 as taking the two accounts (including the penalty $p$ on the partially illiquid account) as given, then there is only one way in which she can allocate $y$ between the two accounts.} This $y_{\text{penalty}}$ is strictly increasing in $p$.

3. For large values of $p$, there is again a unique choice for $y_{\text{penalty}}$.\footnote{Self 0’s choice is actually unique in a stronger sense than that given in Footnote 20. Suppose that we confine her to using at most two accounts, and that we place an upper bound $\pi$ on the penalty that she can attach to either of these two accounts. Then she will attach a penalty of 0 to one account and the maximum penalty of $\pi$ to the other. Moreover there is only one way in which she can allocate $y$ between the two accounts.} This $y_{\text{penalty}}$ is independent of $p$.

See Online Appendix C Sections 10 through 15.

Theorem 2 aligns with our empirical results when the liquid account and the illiquid account have the same interest rate (although log utility is a knife-edge case). Specifically, as $p$ rises exogenously across these experimental arms, the allocation to the partially illiquid account rises. This is true in both the original experiment (where $p$ rises from 10% to 20% to $\infty$) and the second experiment (where $p$ rises from 10% to $\infty$).

**Theorem 3** Suppose that $U_1 = U_2 = U$, and that $U$ has constant relative risk aversion $\rho > 0$.

Suppose further that self 0 must divide her endowment $y$ among three accounts: (i) a fully liquid account with no penalty on withdrawals in either period 1 or period 2; (ii) a partially illiquid account). There is, however, a very specific scenario in which $\pi_0 > 0$. In that scenario, non-uniqueness occurs for all $p$ in the non-trivial range $(0,\pi_0)$.

 accounts). There is, however, a very specific scenario in which $\pi_0 > 0$. In that scenario, non-uniqueness occurs for all $p$ in the non-trivial range $(0,\pi_0)$.

19 The form of non-uniqueness discussed here arises even if (as in Footnote 16) we think of self 0 as choosing an incentive-compatible consumption curve. It can be eliminated, as we show in Online Appendix C Section 10, by introducing an additional assumption, namely: (A4) $G$ is strictly increasing on $(0,\bar{p}_p)$. However: (i) A4 is a little out of keeping with the rest of our framework; (ii) we can obtain satisfactory comparative-statics results without A4; and (iii) the form of non-uniqueness described rarely arises even in the absence of A4. We do not therefore include A4 among our basic assumptions.

20 In other words, if one thinks of self 0 as taking the two accounts (including the penalty $p$ on the partially illiquid account) as given, then there is only one way in which she can allocate $y$ between the two accounts.

21 Self 0’s choice is actually unique in a stronger sense than that given in Footnote 20. Suppose that we confine her to using at most two accounts, and that we place an upper bound $\pi$ on the penalty that she can attach to either of these two accounts. Then she will attach a penalty of 0 to one account and the maximum penalty of $\pi$ to the other. Moreover there is only one way in which she can allocate $y$ between the two accounts.

22 Self 0’s choice is once again unique in a stronger sense than that given in Footnote 20. Suppose that, as in Footnote 21, we confine her to using at most two accounts. Then she will attach a penalty of 0 to one account and any penalty $p$ in the range $[\pi_1, \pi]$ to the other. (The point here is that any penalty of at least $\pi_1$ is sufficient to deter self 1 from making any withdrawal from the partially illiquid account.) So her choice of accounts is no longer unique. However, there is only one way in which she can allocate $y$ between the two accounts, and this allocation is independent of the penalty on the partially illiquid account.
account with a penalty $p$ on withdrawals in period 1 and no penalty on withdrawals in period 2; and (iii) a fully illiquid account that disallows any withdrawals in period 1 but places no penalty on withdrawals in period 2. Denote the optimal allocations to the three accounts by $y_{\text{liquid}}$, $y_{\text{penalty}}$, and $y_{\text{illiquid}}$. Then her liquid allocation $y_{\text{liquid}}$ is unique and independent of $p$. By the same token, her total illiquid allocation $y_{\text{penalty}} + y_{\text{illiquid}}$ is unique and independent of $p$. Furthermore, self 0 weakly prefers the fully illiquid account to the partially illiquid account. Specifically:

1. For small and intermediate values of $p$, self 0 strictly prefers the fully illiquid account to the partially illiquid account. More precisely: self 0 places her total illiquid allocation $y_{\text{penalty}} + y_{\text{illiquid}}$ in the fully illiquid account; $y_{\text{penalty}} = 0$; and $y_{\text{illiquid}}$ is unique and independent of $p$.

2. For large values of $p$, self 0 is indifferent between the fully illiquid account and the partially illiquid account. More precisely: self 0 does not care how her total illiquid allocation $y_{\text{penalty}} + y_{\text{illiquid}}$ is divided between the partially illiquid account and the fully illiquid account.

See Online Appendix C Sections 13 and 14.2.

The logic behind Theorem 3 runs as follows. First, it follows directly from the formulation of our problem that the maximum expected utility of self 0 is weakly increasing in $\pi$. Second, if $U_1 = U_2 = U$ and $U$ has constant relative risk aversion, then self 0 can achieve this maximum using two accounts, namely a fully liquid account and a partially illiquid account with penalty $p = \pi$. Hence, if we restrict self 0 to dividing her endowment between a fully liquid account and a partially illiquid account with penalty $p$, then she will always weakly prefer a higher $p$. In particular, she will like $p = \infty$ best of all. In other words, she weakly prefers the fully illiquid account to the partially illiquid account.

Now suppose that the optimal allocation to a fully illiquid account is $y_{\text{illiquid}}$. If self 0 deposits $y_{\text{illiquid}}$ in a fully illiquid account, then there will be a $\theta_1$ such that: (i) any self 1 of type $\theta \leq \theta_1$ will choose freely from the line segment joining the points $(0, y)$ and $(y - y_{\text{illiquid}}, y_{\text{illiquid}})$; and (ii) any self 1 of type $\theta \geq \theta_1$ will end up choosing the endpoint $(y - y_{\text{illiquid}}, y_{\text{illiquid}})$. If,

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23 We say that $p$ is small if $p \in (0, \pi_0)$, intermediate if $p \in (\pi_0, \pi_1)$ and large if $p \in (\pi_1, \infty)$. Cf. Theorem 2 and Footnote 18.
however, self 0 deposits $y_{\text{illiquid}}$ in a partially illiquid account with penalty $p$, then the behavior of self 1 will not change if and only if $p \geq \pi_i$, where $\pi_i$ is the minimum penalty necessary to deter the $\bar{\Theta}$ type of self 1 from increasing consumption above $y_{\text{liquid}} = y - y_{\text{illiquid}}$.

Hence, if $p \geq \pi_i$, then self 0 can attain the maximum expected utility associated with a fully illiquid account by using a partially illiquid account with a penalty $p$ instead. She will therefore be indifferent between these accounts. On the other hand, if $p < \pi_i$, then a penalty of $p$ is no longer sufficient to deter the $\bar{\Theta}$ type of self 1 from increasing consumption above $y_{\text{liquid}}$. Hence the behavior of self 1 will change if self 0 deposits $y_{\text{illiquid}}$ in a partially illiquid account with penalty $p$. Furthermore, it can be shown that, even when self 0 makes the optimal allocation to the partially illiquid account, her expected utility will still be strictly lower than the expected utility that she can obtain from the fully illiquid account. She will therefore strictly prefer the fully illiquid account.

This prediction of an overall weak preference for the fully illiquid account over the partially illiquid account is consistent with our empirical results in the experimental arm in which participants allocated their endowments across three accounts: a liquid account, a partially illiquid account with a 10% penalty, and a fully illiquid account. Among the participants in this experimental arm, 37 allocated money to the fully illiquid account but not to the partially illiquid account, while only 8 allocated money to the partially illiquid account but not to the fully illiquid account (76 allocated money to both illiquid accounts, and 29 allocated money to neither). The average allocations to the accounts follow a similar pattern: the partially illiquid account attracts 16% of the endowment, while the fully illiquid account attracts 34% of the endowment. The decision to allocate money to the partially illiquid account is not inconsistent with the model. Theorem 3 predicts that participants who allocate money to the partially illiquid account do so because the 10% penalty is above $\pi_i$ and therefore sufficient to deter early withdrawals. There were 42 participants who allocated money to the partially illiquid account and were randomly assigned to receive their chosen allocation (instead of having all of their endowment placed in the liquid account), and out of those 42 participants, only one made a withdrawal from the partially illiquid account before the goal date.
Thus, the data tend to support Theorem 3. However, it would be necessary to extend the model to accommodate some of the nuances of the experimental design. Most importantly, participants in the study were allowed to set different goal dates for the fully illiquid account and the partially illiquid account, and 55 out of the 76 participants who allocated money to both accounts took advantage of this flexibility. Among the experimental participants who chose to allocate money to both the partially illiquid account and the fully illiquid account, the average goal horizon for the partially illiquid account was 116 days, and the average goal horizon for the fully illiquid account was 145 days, a difference that is statistically significant at the 1% level in a paired t-test. Hence, participants tended to use the partially illiquid account to create short-run commitments and the fully illiquid account to create long-run commitments. We do not know whether participants would prefer to use the fully illiquid account to create commitments at all horizons if they were given the option to do so.

Finally, it is important to emphasize that our theoretical analysis considers a sophisticated agent, who in period 0 fully anticipates the difference between the current self’s preferences and preferences as of period 1. There is evidence that many individuals in the population are only partially sophisticated—they understand that there is a divergence between current and future preferences but underappreciate the full extent of that divergence (John, 2018). From a descriptive perspective, our theoretical analysis of commitment account allocations also applies to the case of a partially sophisticated agent. However, the welfare implications may not apply. In particular, because a partially sophisticated agent makes commitment account allocation decisions in period 0 based on an incorrect forecast of consumption decisions in period 1, the period 0 decisions may not be optimal in the sense that welfare from the period 0 perspective may be improved by selecting different commitment account allocations. As the current analysis focuses on descriptive issues, we leave welfare analysis to other work (see Galperti, 2015; Beshears et al., 2017; and Moser and Olea de Souza e Silva, 2017).
References
Appendix Figure B1. Illustrative Budget Set

This figure shows a budget set that illustrates Constraints 1 and 2 of the model. The budget set is a non-empty compact subset of the ambient budget set and therefore satisfies Constraint 1. It satisfies Constraint 2 if \( \pi = 0.5 \), but not if \( \pi = 0.1 \). The slope at the encircled point is \(-1.3\). This is greater than \(-1.5\) (the minimum slope that is permissible when \( \pi = 0.5 \)), but less than \(-1.1\) (the minimum slope that is permissible when \( \pi = 0.1 \)).

\[
\begin{align*}
C_2 & \quad \text{Ambient budget set} \\
\text{slope} &= -1.3 \\
\text{Budget set} & \quad \text{slope} = -1.3 \\
C_1 & \quad y
\end{align*}
\]

Appendix Figure B2. Two-Part Budget Set

This figure shows a two-part budget set. Such budget sets consist of: (i) a budget curve that has slopes of \(-1\) and \(-(1 + p)\) to the left and right of a kink at \((c_1^*, c_2^*)\); and (ii) none, some or all of the points of the ambient budget set that lie below or to the left of the budget curve.

\[
\begin{align*}
\text{slope} &= -1 \\
\text{slope} &= -(1 + p) \\
\left(c_1^*, c_2^*\right)
\end{align*}
\]
1. Introduction

In this appendix we provide a complete analysis of the mechanism-design problem described in the main body of the paper.

2. Preliminaries

2.1. Functions of Bounded Variation. We begin by discussing the concept of bounded variation. This concept will be used to formulate our assumptions on the distribution function $F$ in the subsection immediately following this one, namely Section 2.2. More importantly, it plays an essential role in our proof of sufficiency in a much later section, namely Section 16.

The simplest definition of a function of bounded variation is probably that given in the main text: a function $f : (0, \infty) \to \mathbb{R}$ is of bounded variation iff it is the difference of two bounded and non-decreasing functions $f^+, f^- : (0, \infty) \to \mathbb{R}$. This definition forms the starting point for the definition that we shall use. However, it needs to be developed into a form that is more convenient for the Lagrangean analysis below.

The first step is to collect the functions of bounded variation into equivalence classes. Intuitively speaking, two functions of bounded variation are equivalent iff they differ only at their points of discontinuity. This step is analogous to the first
step in defining spaces of Lebesgue integrable functions. (In that case, one collects the Lebesgue integrable functions into equivalence classes. Two Lebesgue integrable functions are equivalent if they differ only on a set of measure 0.)

The second step is to place a norm on the resulting equivalence classes in such a way that the limit of a sequence of equivalence classes is again a suitable equivalence classes. (This step is analogous to the second step in defining spaces of Lebesgue integrable functions.) The main idea here is to note that, since $f^+$ and $f^-$ are non-decreasing, they are effectively the distribution functions of a pair of non-negative bounded measures $df^+$ and $df^-$. Of course, neither $df^+$ nor $df^-$ is unique. But their difference $df = df^+ - df^-$ is. The main component of the norm is therefore the total variation $\|df\|_{TV}$ of $df$. The other idea is to note that, while $\|df\|_{TV}$ effectively controls the derivative of $f$, it does not control the level of $f$. The remaining component of the norm can therefore be taken to be $|f_R(1)|$, where $f_R(1)$ is the limit of $f$ from the right at 1.

The best way of understanding how these ideas work is to note that we can easily reconstruct $f$ from $df$ and $f_R(1)$. For all $\theta \in (0, 1)$, we have

$$f_R(\theta) = f_R(1) - df((\theta, 1])$$

and

$$f_L(\theta) = f_R(1) - df([\theta, 1]),$$

where $f_R(\theta)$ and $f_L(\theta)$ are the limits of $f$ from the right and left at $\theta$. And for all $\theta \in (1, \infty)$, we have

$$f_R(\theta) = f_R(1) + df((1, \theta])$$

and

$$f_L(\theta) = f_R(1) + df((1, \theta)).$$

We also need to work with the space $\mathcal{BV}(\Theta, \mathbb{R})$ of functions of bounded variation on $\Theta = [\underline{\theta}, \overline{\theta}]$. By analogy with our discussion of the space $\mathcal{BV}((0, \infty), \mathbb{R})$, it should be clear that we can endow $\mathcal{BV}(\Theta, \mathbb{R})$ with the norm

$$\|f\|_{\mathcal{BV}} = |f_R(\theta_0)| + \|df\|_{TV},$$

2
where $\theta_0$ is a fixed element of $(\underline{\theta}, \overline{\theta})$ and $d\!f$ is a bounded measure on $\Theta$. There is, however, one surprise: a function $f \in BV(\Theta, \mathbb{R})$ has a limit on the left at $\underline{\theta}$ and a limit on the right at $\overline{\theta}$. Indeed, we have

$$f_L(\underline{\theta}) = f_R(\theta_0) - d\!f([\underline{\theta}, \theta_0])$$

and

$$f_R(\overline{\theta}) = f_R(\theta_0) + d\!f((\theta_0, \overline{\theta}]).$$

To summarize, we denote the space of functions of bounded variation on $(0, \infty)$ by $BV((0, \infty), \mathbb{R})$, and we denote the space of functions of bounded variation on $\Theta = [\underline{\theta}, \overline{\theta}]$ by $BV(\Theta, \mathbb{R})$. Unless explicitly stated to the contrary, we shall always use the right-continuous representative of a function of bounded variation. We will usually denote this representative simply by $f$, but we will occasionally denote it by $f_R$ for emphasis. We will denote the left-continuous representative of $f$ by $f_L$.

### 2.2. Assumptions on $F$

We are now in a position to introduce our assumptions on the distribution function $F$ of the taste shock $\theta$. They are:

**A1** Both $F$ and $F'$ are functions of bounded variation on $(0, \infty)$.

**A2** The support of $F'$ is contained in $[\underline{\theta}, \overline{\theta}]$, where $0 < \underline{\theta} < \overline{\theta} < \infty$.

**A3** There exists $\theta_M \in [\underline{\theta}, \overline{\theta}]$ such that: (i) $G' \geq 0$ on $(0, \theta_M)$; and (ii) $G' \leq 0$ on $(\theta_M, \infty)$.

Here $G$ is given by the formula $G(\theta) = (1 - \beta) \theta F'(\theta) + F(\theta)$. If A1 holds then $G$, like $F$ and $F'$, is a function of bounded variation on $(0, \infty)$.

### 2.3. The Support of $F'$ is Connected

Fourth, we note that either $\beta = 1$, in which case the analysis is trivial, or $\beta < 1$, in which case the support of $F'$ is connected.\(^\dagger\) More precisely, we have:

\(^\dagger\)Notice that $F$ is a distribution function, not a distribution. Also, $F'$ has a dual interpretation. It can be regarded as: either (i) the non-negative finite measure with distribution function $F$; or (ii) the density of that measure with respect to Lebesgue measure. By the same token, the support of $F'$ has a dual interpretation. It can be regarded as: either (i) the support of the non-negative finite measure $F'$; or (ii) the support of the non-negative function of bounded variation $F''$. It makes no difference which of these two interpretations is adopted.
Proposition 1. Suppose that $\beta < 1$ and that A1-A3 are satisfied. Then there exist $\kappa, \pi \in [\underline{\theta}, \overline{\theta}]$ such that: (i) $\kappa < \pi$; (ii) $F' > 0$ on $(\kappa, \pi)$; and (iii) $F' = 0$ on $(0, \infty) \setminus [\kappa, \pi]$.

In what follows we shall therefore take it that $\beta < 1$, and that the support of $F'$ is $[\underline{\theta}, \overline{\theta}]$.

Proof. Note first that there exists $\kappa_1 \in (\underline{\theta}, \overline{\theta})$ such that $F'(\kappa_1) > 0$. Otherwise we would have $F' = 0$ everywhere on $(0, \infty)$, by right-continuity of $F'$. Next, there exists $\kappa_2 \in (\kappa_1, \overline{\theta})$ such that $F' > 0$ on $(\kappa_1, \kappa_2)$, again by right-continuity of $F'$. Third, put $\kappa = \inf \{ \theta \mid F'(\theta) > 0 \}$ and $\pi = \sup \{ \theta \mid F'(\theta) > 0 \}$. Then certainly $\underline{\theta} \leq \kappa < \pi \leq \overline{\theta}$.

Fourth, put $\alpha = \frac{2 - \beta}{1 - \beta}$. Then $G' \geq 0$ iff $(\theta^o F')' \geq 0$ and $G' \leq 0$ iff $(\theta^o F')' \leq 0$. There are therefore two possibilities. If $G' \geq 0$ at $\theta_M$ (i.e. $\Delta G(\theta_M) \geq 0$), then we must have $\theta^o F' > 0$ on $(\kappa, \theta_M]$ (because $(\theta^o F')' \geq 0$ on this interval) and $\theta^o F' > 0$ on $(\theta_M, \pi]$ (because $(\theta^o F')' \leq 0$ on this interval); and if $G' \leq 0$ at $\theta_M$ (i.e. $\Delta G(\theta_M) \leq 0$), then we must have $\theta^o F' > 0$ on $(\kappa, \theta_M]$ and $\theta^o F' > 0$ on $[\theta_M, \pi)$.\(^2\) Either way, we see that: (i) $\theta^o F' > 0$, and hence $F' > 0$, on $(\kappa, \pi)$; (ii) $\theta_M \leq \pi$, for otherwise we would have $F' > 0$ on the non-empty interval $(\pi, \theta_M)$, and this contradicts the choice of $\pi$; and (iii) $\theta_M \geq \kappa$, for otherwise we would have $F' > 0$ on the non-empty interval $(\theta_M, \kappa)$, and this contradicts the choice of $\kappa$. \(\blacksquare\)

2.4. Constraints on the Budget Set. Fifth, recall that self 0 chooses a subset $B$ of the ambient action set $A$, and that self 1’s choice of a consumption pair from $B$ can therefore be described by a consumption curve $(c_1, c_2) : [\underline{\theta}, \overline{\theta}] \rightarrow B$. We consider three possible constraints on $B$, namely:

Constraint 1. $B$ is a non-empty compact subset of $A$.

Constraint 2. The penalty for transferring consumption from period 2 to period 1 is no greater than $\pi$.\(^3\)

\(^2\)If $\theta_M \leq \kappa$ then we take the intervals $(\kappa, \theta_M]$ and $(\kappa, \theta_M]$ to be empty. Similarly, if $\theta_M \geq \pi$, then we take the intervals $(\theta_M, \pi]$ and $[\theta_M, \pi]$ to be empty.

\(^3\)In other words, for any given $(c_1, c_2) \in B$ and any $\Delta c_1 \in \left[0, \frac{1}{1+\pi} c_2\right]$, self 1 can increase her own consumption $c_1$ by $\Delta c_1$ at a cost of at most $(1 + \pi) \Delta c_1$ in terms of the consumption $c_2$ of self 2.
Constraint 3. The penalty for transferring consumption from period 1 to period 2 is no greater than $\pi$.\(^4\)

Constraint 1 involves no loss of generality. Indeed, it must be possible for all possible types $\theta \in \Theta$ to find an optimum within $B$. This being the case, we can always take the closure of $B$ without changing the outcome, since the utility function is continuous. Finally, since $A$ itself is compact, so too is the closure of $B$. Constraint 2 is an essential part of the formulation of our problem. We wish to avoid extreme outcomes in which self 0 imposes an infinite penalty on self 1 for increasing her own consumption at the expense of self 2. Constraint 3 is simply the mirror image of Constraint 2. It eliminates extreme outcomes in which self 0 imposes an infinite penalty on self 1 for increasing the consumption of self 2 at her own expense.

Remark 2. If we only impose Constraint 2, then the problem is one sided: Constraint 2 places a limit on the cost, in terms of $c_2$, of increasing $c_1$; but there is no corresponding limit on the cost, in terms of $c_1$, of increasing $c_2$. By imposing Constraint 3, we eliminate this asymmetry.

Now suppose that $B$ must satisfy all three constraints. Then $B$ must take one of two forms: either

1. it consists of the single point $(0,0)$; or

2. its frontier consists of a curve that begins at some $(0,c_2)$ such that $c_2 > 0$, slopes downwards with slope between $-(1 + \pi)$ and $-(1 + \pi)^{-1}$, and ends at some $(c_1,0)$ such that $c_1 > 0$.

Self 0 will never choose the first option, since the optimal pooling point on the frontier of the ambient budget set $A$ is preferable. (By the same token, self 0 will never choose a $B$, the frontier of which is close to $(0,0)$.) But, if she chooses the second option, then the resulting consumption curve $(c_1,c_2)$ will be interior. That is, we will have $c_1,c_2 > 0$ on $\Theta$.\(^5\)

\(^4\)In other words, for any given $(c_1,c_2) \in B$ and any $\Delta c_2 \in \left[0, \frac{1}{1+\pi} c_1 \right]$, self 1 can increase the consumption $c_2$ of self 2 by $\Delta c_2$ at a cost of at most $(1 + \pi) \Delta c_2$ in terms of her own consumption $c_1$.

\(^5\)This follows from our assumption that $U'_0(0+) = \infty$. 

5
The ideal approach to our problem would therefore be to impose all three constraints on $B$, and to solve the optimization problem of self 0 subject to these constraints. One could then verify ex post that Constraint 3 was not binding.\footnote{It turns out that the slope of the optimal budget set is at most $-1$. So Constraint 3 certainly is not binding!}

In practice, we shall take a shortcut. Rather than working explicitly with Constraint 3, we shall instead replace it by the weaker requirement that consumption curves are interior. Our analysis could, of course, be reworked in such a way as to incorporate Constraint 3 explicitly. But, in practice, this would simply involve an additional notational burden.

\textbf{Remark 3.} The situation would be very different if $\beta > 1$. In that case, it would be Constraint 2 that would not bind. We would therefore replace Constraint 2 by the weaker requirement that consumption curves are interior.

\subsection*{2.5. Utility Curves.} Suppose accordingly that we are given a $B$ satisfying Constraints 1 and 2, and that the associated consumption curve is interior. Define a utility curve

$$(u_1, u_2) : [\underline{\theta}, \overline{\theta}] \rightarrow (U_1(0), U_1(\infty)) \times (U_2(0), U_2(\infty))$$

by the formula $(u_1, u_2)(\theta) = (U_1(c_1(\theta)), U_2(c_2(\theta)))$. Then $(u_1, u_2)$ must satisfy the following conditions:

\begin{enumerate}
  \item [N1] $C_1(u_1(\theta)) + C_2(u_2(\theta)) \leq y$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$.
  \item [N2] $u_1$ is non-decreasing and $u_2$ is non-increasing.
  \item [N3] $\theta \, du_1 + \beta \, du_2 = 0$.
  \item [N4] $\beta (1 + \pi) U_2' (C_2(u_2(\theta))) \geq \theta U_1' (C_1(u_1(\theta)))$.
\end{enumerate}

Here: $C_t = U_t^{-1}$; and $du_1$ is a non-negative finite measure and $du_2$ is a non-positive finite measure.

Conversely, suppose that a utility curve $(u_1, u_2)$ is interior, in the sense that it satisfies $u_1 > U_1(0)$ and $u_2 > U_2(0)$ on $\Theta$, and that it satisfies Conditions N1-N4.
Define \((c_1, c_2)\) by the formula \((c_1, c_2)(\theta) = (U_1^{-1}(u_1(\theta)), U_2^{-1}(u_2(\theta)))\). Then there exists a \(B\) with slope at least \(- (1 + \pi)\) such that \((c_1, c_2)\) is the consumption curve arising from \(B\). Moreover \((c_1, c_2)\) is interior.

2.6. The CRRA Case. Suppose now that \(U_1 = U_2 = U\) on \((0, \infty)\), and that \(U\) has constant relative risk aversion \(\rho > 0\). Indeed, suppose for concreteness that \(U\) is given by the formula

\[
U(c) = \begin{cases} 
\frac{c^{1-\rho} - 1}{1-\rho} & \text{if } \rho \neq 1 \\
\log(c) & \text{if } \rho = 1
\end{cases}.
\]

Then \(N4\) is equivalent to

\[
N4' \quad u_2(\theta) \leq -\frac{1}{\rho} a \left(\frac{\theta}{(1+\pi)^{\beta}}\right) + b \left(\frac{\theta}{(1+\pi)^{\beta}}\right) u_1(\theta),
\]

where \(a\) and \(b\) are given by the formulae

\[
a(z) = \begin{cases} 
z^{1-\rho} - 1 & \text{if } \rho \neq 1 \\
\log(z) & \text{if } \rho = 1
\end{cases} \tag{1}
\]

and

\[
b(z) = z^{1-\frac{1}{\rho}}. \tag{2}
\]

Remark 4. It is obvious that \(N4\) becomes weaker as \(\pi\) increases. Since \(N4'\) is equivalent to \(N4\), \(N4'\) likewise becomes weaker as \(\pi\) increases.

3. The Main Problem

Our strategy will be to study a relaxed version of the problem of self 0 in which we maximize self 0’s expected utility \(\int (\theta u(\theta) + w(\theta)) dF(\theta)\) subject to \(N1, N3\) and \(N4'\), but not \(N2\). Following Luenberger (1969, Sections 8.3 and 8.4, pp. 216-221), we shall need:

1. A vector space\(^7\) \(X\), which we take to be \(C(\Theta, \mathbb{R})^2\). Here: \(\Theta = [\theta, \bar{\theta}] \subset (0, \infty)\) is the space of types; and \(C(\Theta, \mathbb{R})\) is the space of continuous functions from \(\Theta\) to \(\mathbb{R}\).

\(^7\)In our case \(X\) is actually a Banach space. For the Lagrangean analysis, we only need the fact that it is a vector space. When we later use calculus to find necessary and sufficient conditions for the maximization of the Lagrangean, we shall need the fact that it is a normed space. Cf. Luenberger (1969, Lemma 1, p. 227).
2. A convex set $\Omega \subset X$, which we take to consist of all

$$(u, w) \in \left( \mathcal{BV}(\Theta, \text{ran}(U)) \cap \mathcal{C}(\Theta, \text{ran}(U)) \right)^2$$

such that

$$\theta du + \beta dw = 0.$$ 

Here: $\text{ran}(U)$ is the range of $U$; $\mathcal{C}(\Theta, \text{ran}(U))$ is the space of continuous functions from $\Theta$ to $\text{ran}(U)$; $\mathcal{BV}(\Theta, \text{ran}(U))$ is the space of all functions of bounded variation from $\Theta$ to $\text{ran}(U)$; and $du$ and $dw$ are in general elements of the space $\mathcal{M}(\Theta, \mathbb{R})$ of finite Borel measures on $\Theta$.

3. A concave function $M : \Omega \rightarrow \mathbb{R}$ (the objective function), which we take to be given by the formula

$$M(u, w) = \int \left( \theta u(\theta) + w(\theta) \right) dF(\theta).$$

4. A normed space $Z$, which we take to be $\mathcal{C}(\Theta, \mathbb{R})$.

5. A closed convex cone $P$ in $Z$ with vertex 0 and non-empty interior, which we take to be $\mathcal{C}(\Theta, [0, \infty))$. With this choice of $P$, $z_1 \geq z_2$ iff $z_1(\theta) \geq z_2(\theta)$ for all $\theta \in \Theta$ and $z_1 > z_2$ iff $z_1(\theta) > z_2(\theta)$ for all $\theta \in \Theta$. In other words, $P$ is the positive cone of $Z$.

6. The space $Z^*$ of continuous linear functionals on $Z$. Since $Z = \mathcal{C}(\Theta, \mathbb{R})$, $Z^*$ can be identified with $\mathcal{M}(\Theta, \mathbb{R})$.

7. The positive cone $P^*$ of $Z^*$. Since $P = \mathcal{C}(\Theta, [0, \infty))$, $P^*$ can be identified with $\mathcal{M}(\Theta, [0, \infty))$ (i.e. the space of non-negative finite Borel measures on $\Theta$).

---

8In our case $\Omega$ is actually a cone, the vertex of which is the constant mapping $\frac{1}{\rho-1}$ when $\rho \neq 1$ and the constant mapping 0 when $\rho = 1$.

9I.e. $\text{ran}(U)$ is $(\frac{1}{\rho-1}, \infty)$ when $\rho < 1$, $(-\infty, \infty)$ when $\rho = 1$ and $(-\infty, \frac{1}{\rho-1})$ when and $\rho > 1$.

10In our case $M$ is actually defined on the whole of $X$ (and not just on $\Omega$), and it is linear (and not just concave).

11In our case, $Z$ is actually a Banach space, and not just a normed space.
8. Concave mappings $G_1, G_2 : \Omega \to Z$ (the constraint mappings),\textsuperscript{12} which we take to be given by the formulae

$$(G_1(u,w))(\theta) = y - C(u(\theta)) - K(w(\theta))$$

and

$$(G_2(u,w))(\theta) = \left(\frac{\theta}{1 + \pi}\right)^\beta u(\theta) - \frac{1}{\rho} a\left(\frac{\theta}{1 + \pi}\right)^\beta - w(\theta),$$

where $C = K = U^{-1} : \text{ran}(U) \to (0, \infty)$, and $a$ and $b$ are given by the formulae (1) and (2).

Our problem is then to

$$\text{maximize } M(x)$$

subject to

$$\begin{cases}
  x \in \Omega \\
  G_1(x) \geq 0 \\
  G_2(x) \geq 0
\end{cases}.$$

4. Characterizing the Optimum

In our context, the Lagrangean is the mapping $L : \Omega \times Z^* \times Z^* \to \mathbb{R}$ given by the formula

$$L(x, \lambda_1, \lambda_2) = M(x) + \langle G_1(x), \lambda_1 \rangle + \langle G_2(x), \lambda_2 \rangle,$$

where $\langle G_i(x), \lambda_i \rangle$ denotes the real number obtained when the linear functional $\lambda_i \in Z^*$ is evaluated at the point $G_i(x) \in Z$.

In view of our assumptions, the maximum is achieved at $x_0 \in \Omega$ if and only if there exist $\lambda_1, \lambda_2 \in Z^*$ such that:

1. $L(x_0, \lambda_1, \lambda_2) \geq L(x, \lambda_1, \lambda_2)$ for all $x \in \Omega$;
2. $G_1(x) \geq 0$, $\lambda_1 \geq 0$, and $\langle G_1(x), \lambda_1 \rangle = 0$.

\textsuperscript{12}In our case $G_1$ is actually defined on $\Xi = C(\Theta, \text{ran}(U))^2$ (and not just on $\Omega$). This will be useful when we later want to do calculus. Furthermore $G_2$ is defined on the whole of $X$ (and not just on $\Omega$), and it is linear (and not just concave).
3. \( G_2(x) \geq 0, \lambda_2 \geq 0 \) and
\[
\langle G_2(x), \lambda_2 \rangle = 0.
\] (5)

In other words, there exists multipliers \( \lambda_1 \) and \( \lambda_2 \) such that: \( x_0 \) maximizes \( L(\cdot, \lambda_1, \lambda_2) \) over \( \Omega \); complementary slackness holds for the first constraint; and complementary slackness holds for the second constraint.

Since \( P^* \) can be identified with \( M(\Theta, [0, \infty)) \), we have the following explicit representations of \( M(x) \), \( \langle G_1(x), \lambda_1 \rangle \) and \( \langle G_2(x), \lambda_2 \rangle \):

\[
M(x) = \int \left( \theta u(\theta) + w(\theta) \right) dF(\theta),
\] (6)

\[
\langle G_1(x), \lambda_1 \rangle = \int \left( y - C(u(\theta)) - K(w(\theta)) \right) d\Lambda_1(\theta)
\] (7)

and

\[
\langle G_2(x), \lambda_2 \rangle = \int \left( b\left( \frac{\theta}{(1+\pi)\beta} \right) u(\theta) - \frac{1}{\rho} a\left( \frac{\theta}{(1+\pi)\beta} \right) - w(\theta) \right) d\Lambda_2(\theta),
\] (8)

where \( \Lambda_1 \) and \( \Lambda_2 \) are the distribution functions of \( \lambda_1 \) and \( \lambda_2 \) respectively.

Remark 5. In the interests of clarity and consistency, all integrals in this Appendix are Lebesgue-Stieltjes integrals, i.e. integrals with respect to functions of bounded variation.

5. The Lagrangean is Fréchet Differentiable

It is immediate from the formulae (6), (7) and (8) that \( L(x, \lambda_1, \lambda_2) \) is in fact well defined for all \( x \in \Xi = C(\Theta, \text{ran}(U))^2 \). Let us consider accordingly any \( x_0 = (u_0, w_0) \in \Xi \) and any \( x_1 = (u_1, w_1) \in X \). Because \( \Xi \) is open, \( x_0 + \varepsilon x_1 \in \Xi \) for all \( \varepsilon > 0 \) sufficiently small. Furthermore, it can be verified that the directional derivative \( \nabla_{x_1} L(x_0, \lambda_1, \lambda_2) \) of \( L \) at \( x_0 \) in the direction \( x_1 \) takes the form

\[
\int \left( \theta u_1 + w_1 \right) dF - \int \left( C'(u_0) u_1 + K'(w_0) w_1 \right) d\Lambda_1 + \int \left( b\left( \frac{\theta}{(1+\pi)\beta} \right) u_1 - w_1 \right) d\Lambda_2.
\] (9)

This is easily seen to define a continuous linear functional

\[
\nabla L(x_0, \lambda_1, \lambda_2) : x_1 \mapsto \nabla_{x_1} L(x_0, \lambda_1, \lambda_2)
\]
on $X$. That is, $L(\cdot, \lambda_1, \lambda_2)$ is Gâteaux differentiable at $x_0$ with gradient $\nabla L(x_0, \lambda_1, \lambda_2) \in X^*$. Finally, $\nabla L(\cdot, \lambda_1, \lambda_2) : \Xi \to X^*$ can be shown to be continuous. It follows that $L(\cdot, \lambda_1, \lambda_2)$ is Fréchet differentiable on $\Xi$.

6. Maximizing the Lagrangean

Since $L(\cdot, \lambda_1, \lambda_2)$ is convex and Fréchet differentiable on $\Xi$, and since $\Omega$ is convex, the maximum of $L(\cdot, \lambda_1, \lambda_2)$ over $\Omega$ is achieved at $x_0 \in \Omega$ iff

$$\nabla_{x-x_0} L(x_0, \lambda_1, \lambda_2) \leq 0$$

for all $x \in \Omega$. In this section we shall identify the restrictions that this places on $\lambda_1$ and $\lambda_2$.

To this end, put

$$Y = \left( BV(\Theta, \mathbb{R}) \cap C(\Theta, \mathbb{R}) \right) \times \mathbb{R};$$

and consider the affine transformation

$$S : Y \to \left( BV(\Theta, \mathbb{R}) \cap C(\Theta, \mathbb{R}) \right)^2$$

that maps $y = (u, r)$ to $x = (u_0 + u, w_0 + w)$, where $w$ is the unique solution of the equation $\theta du + \beta dw = 0$ with boundary condition $w(\bar{\theta}) = r$.

For any $y \in Y$, we have

$$\nabla_{S(y)-x_0} L(x_0, \lambda_1, \lambda_2) = \int (\theta u + w) \, dF - \int \left( C'(u_0) u + K'(w_0) w \right) d\Lambda_1$$

$$+ \int \left( b \left( \frac{\theta}{1+\pi} \right) u - w \right) d\Lambda_2$$

$$= \int (\theta u + w) \, dF - \int \left( \frac{\theta}{R(u_0)} u + w \right) d\tilde{\Lambda}_1$$

$$+ \int \left( b \left( \frac{\theta}{1+\pi} \right) u - w \right) d\Lambda_2$$

(where $d\tilde{\Lambda}_1 = K'(w_0) d\Lambda_1$). Furthermore, integrating by parts, we have

$$\int w \, dF = \left[ w F \right]_{\bar{\theta}}^{\theta} - \int F \, dw = w(\bar{\theta}) F(\bar{\theta}) + \int F \theta \beta \, du$$

11
(because \( F(\theta) = 0 \) and \( dw = -\frac{\theta}{\beta} du \))

\[
= r F(\theta) + \int F \frac{\theta}{\beta} du.
\]

Moreover

\[
\int F \theta du = \int F (\theta du + u d\theta) - \int F u d\theta
\]

\[
= [\theta u F(\theta)]_{\theta} - \int \theta u dF - \int F u d\theta
\]

\[
= \theta u F(\theta) - \int \theta u dF - \int F u d\theta
\]

(where we have again used the fact that \( F(\theta) = 0 \)). Hence

\[
\int w dF = \left( \frac{\theta}{\beta} u(\theta) + r \right) F(\theta) - \frac{1}{\beta} \int u (\theta dF + F d\theta).
\]

Similarly,

\[
\int w d\Lambda_1 = \left( \frac{\theta}{\beta} u(\theta) + r \right) \Lambda_1(\theta) - \frac{1}{\beta} \int u \left( \theta d\Lambda_1 + \Lambda_1 d\theta \right)
\]

and

\[
\int w d\Lambda_2 = \left( \frac{\theta}{\beta} u(\theta) + r \right) \Lambda_2(\theta) - \frac{1}{\beta} \int u \left( \theta d\Lambda_2 + \Lambda_2 d\theta \right).
\]

Overall, then,

\[
\nabla_{S_{(\theta)-x_0}} L(x_0, \lambda_1, \lambda_2) = \left( \frac{\theta}{\beta} u(\theta) + r \right) \left( F(\theta) - \Lambda_1(\theta) - \Lambda_2(\theta) \right)
\]

\[
- \frac{1}{\beta} \int u \left( (1 - \beta) \theta dF + F d\theta \right)
\]

\[
+ \frac{1}{\beta} \int u \left( \left( \theta - \beta \frac{C'(w_0)}{K'(w_0)} \right) d\Lambda_1 + \Lambda_1 d\theta \right)
\]

\[
+ \frac{1}{\beta} \int u \left( \left( \theta + \beta b \left( \frac{\theta}{1+\pi} \right) \right) d\Lambda_2 + \Lambda_2 d\theta \right).
\]
Next, it is easy to see that the mapping

$$y \mapsto \nabla S_{y-x_0} L(x_0, \lambda_1, \lambda_2)$$

defines a continuous linear functional on $Y$. Since it does not depend on the derivatives of $y$, this functional extends uniquely to a continuous linear functional

$$y^* : C(\Theta, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}.$$ 

Indeed, we have

$$y^* = (u^*, r^*) \in \mathcal{M}(\Theta, \mathbb{R}) \times \mathbb{R} = (C(\Theta, \mathbb{R}) \times \mathbb{R})^*,$$

where

$$du^* = -\frac{1}{\beta} \left( (1 - \beta) \theta dF + F d\theta \right) + \frac{1}{\beta} \left( (\theta - \beta \frac{C'(w_0)}{K'(w_0)}) \right) d\tilde{\Lambda}_1 + \tilde{\Lambda}_1 d\theta$$

$$+ \frac{1}{\beta} \left( \theta + \beta b \left( \frac{\theta}{(1+\pi)^{\beta}} \right) \right) d\Lambda_2 + \Lambda_2 d\theta + \frac{\theta}{\beta} \left( F(\bar{\theta}) - \tilde{\Lambda}_1(\bar{\theta}) - \Lambda_2(\bar{\theta}) \right) dI$$

and

$$r^* = F(\bar{\theta}) - \tilde{\Lambda}_1(\bar{\theta}) - \Lambda_2(\bar{\theta})$$

and $I$ is the distribution function of the unit mass at $\bar{\theta}$.

**Remark 6.** In reading (10), note that $F$, $\tilde{\Lambda}_1$, $\Lambda_2$ and $\theta$ are functions of bounded variation (with $\theta$ being simply the identity function). Hence $dF$, $d\tilde{\Lambda}_1$, $d\Lambda_2$ and $d\theta$ are measures, and the equation as a whole holds in terms of measures.

Finally, it is easy to see that there exists $\varepsilon > 0$ such that $S(y) \in \Omega$ for all $y \in Y \cap B_\varepsilon(0)$, where $B_\varepsilon(0)$ is the open ball in $C(\Theta, \mathbb{R}) \times \mathbb{R}$ with radius $\varepsilon$ and centre 0. It follows that $\langle y, y^* \rangle \leq 0$ for all $y \in Y \cap B_\varepsilon(0)$. But $Y \cap B_\varepsilon(0)$ is dense in $B_\varepsilon(0)$.
Hence \( y^* \leq 0 \) for all \( y \in B_x(0) \). Hence \( y^* = 0 \). In other words, we have

\[
0 = -\frac{1}{\beta} \left( (1 - \beta) \theta dF + F d\theta \right) + \frac{1}{\beta} \left( \left( \theta - \beta \frac{\theta^0}{K(\theta^0)} \right) d\tilde{\Lambda}_1 + \tilde{\Lambda}_1 d\theta \right) \\
+ \frac{1}{\beta} \left( \theta + \beta b \left( \frac{\theta}{1 + \pi} \right) \right) d\Lambda_2 + \Lambda_2 d\theta + \frac{\theta}{\beta} \left( F(\theta) - \tilde{\Lambda}_1(\theta) - \Lambda_2(\theta) \right) dI \tag{11}
\]

and

\[
0 = F(\theta) - \tilde{\Lambda}_1(\theta) - \Lambda_2(\theta). \tag{12}
\]

Taking advantage of (12), (11) simplifies to

\[
0 = -G d\theta + \left( \left( \theta - \beta \frac{\theta^0}{K(\theta^0)} \right) d\tilde{\Lambda}_1 + \tilde{\Lambda}_1 d\theta \right) \\
+ \left( \theta + \beta b \left( \frac{\theta}{1 + \pi} \right) \right) d\Lambda_2 + \Lambda_2 d\theta \tag{13}
\]

where \( G \) is given by the formula \( G(\theta) = (1 - \beta) \theta F'(\theta) + F(\theta) \).

7. A One-Parameter Family of Utility Curves

We shall consider a family of utility curves depending on the single parameter \( \theta_1 \in (0, \overline{\theta}] \). For each \( \theta_1 \), we begin by finding the point \( (c^*(\theta_1), k^*(\theta_1)) \) that would be chosen by a self 1 of type \( \theta_1 \) from the ambient budget set \( A \). The utility curve corresponding to \( \theta_1 \) is then the utility curve associated with a two-part budget set with slopes of \(-1\) and \(-1 + \pi\) to the left and right of a kink at \( (c^*(\theta_1), k^*(\theta_1)) \). Notice that we specifically allow for the possibility that \( \theta_1 < \overline{\theta} \).

Put \( \theta_2 = (1 + \pi) \theta_1 \). Knife-edge cases apart, there are then five possible cases arising from the relative positions of the two non-trivial intervals \([\underline{\theta}, \overline{\theta}]\) and \([\theta_1, \theta_2]\):

**Case 1** \([\underline{\theta}, \overline{\theta}]\) contains \([\theta_1, \theta_2]\);

**Case 2** \([\theta_1, \theta_2]\) contains \([\underline{\theta}, \overline{\theta}]\);

**Case 3** the two intervals overlap, with \([\theta_1, \theta_2]\) lying to the left and \([\underline{\theta}, \overline{\theta}]\) to the right;

**Case 4** the two intervals overlap, with \([\underline{\theta}, \overline{\theta}]\) lying to the left and \([\theta_1, \theta_2]\) to the right;

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Case 5 $[\theta_1, \theta_2]$ lies entirely to the left of $[\theta, \bar{\theta}]$.

(The case in which $[\theta, \bar{\theta}]$ lies entirely to the left of $[\theta_1, \theta_2]$ cannot occur, because we are confining $\theta_1$ to the interval $(0, \bar{\theta})$.)

7.1. Case 1. If the utility curve corresponding to $\theta_1$ is to be an optimum, then the associated multipliers $\lambda_1$ and $\lambda_2$ must satisfy the three necessary conditions (4), (5) and (13) (i.e. complementary slackness for the first constraint, complementary slackness for the second constraint and the measure equation). In this section, we show that these three necessary conditions tie down $\lambda_1$ and $\lambda_2$ uniquely. The fourth necessary condition, namely the boundary condition (12), is not needed at this stage. (It will be used below to tie down $\theta_1$.)

By construction, the maximum-penalty constraint is strictly slack on $[\theta_1, \theta_2)$ and the budget constraint is strictly slack on $(\theta_2, \bar{\theta}]$. Hence $d\Lambda_2 = 0$ on the former interval and $d\Lambda_1 = 0$ on the latter. Furthermore (13) implies that

$$(1 - \beta) \theta_2 \Delta F(\theta_2) = (\theta_2 - \theta_1) \Delta \tilde{\Lambda}_1(\theta_2) + \left( \theta_2 + \beta b\left( \frac{\theta_2}{(1+\pi)^2}\right) \right) \Delta \Lambda_2(\theta_2),$$

where $\Delta F(\theta_2)$, $\Delta \tilde{\Lambda}_1(\theta_2)$ and $\Delta \Lambda_2(\theta_2)$ are the atoms of $dF$, $d\tilde{\Lambda}_1$ and $d\Lambda_2$ at $\theta_2$. But Assumption A1 implies that $\Delta F(\theta_2) = 0$. Since all the terms on the right-hand side of the equation are non-negative, it follows that $\Delta \tilde{\Lambda}_1(\theta_2) = \Delta \Lambda_2(\theta_2) = 0$. Hence $\Delta \Lambda_2(\theta_2) = 0$ (and therefore $d\Lambda_2 = 0$ on $[\theta_2, \bar{\theta}]$) and $\Delta \tilde{\Lambda}_1(\theta_2) = 0$ (and therefore $d\tilde{\Lambda}_1 = 0$ on $[\theta_2, \bar{\theta}]$).

Now let us consider the three intervals $[\theta, \theta_1]$, $[\theta_1, \theta_2]$ and $[\theta_2, \bar{\theta}]$ in turn. On $[\theta, \theta_1]$, we have $\frac{C'(w_0)}{K'(w_0)} = \frac{a}{\beta}$, $\Lambda_2 = 0$ and $d\Lambda_2 = 0$. Hence (13) becomes

$$0 = -G d\theta + \tilde{\Lambda}_1 d\theta.$$

It follows that $\tilde{\Lambda}_1 = G$ almost everywhere with respect to Lebesgue measure $d\theta$. Since both $\tilde{\Lambda}_1$ and $G$ are functions of bounded variation, it then follows (bearing in mind the convention that functions of bounded variation are right continuous) that $\tilde{\Lambda}_1 = G$ everywhere on $[\theta, \theta_1]$, and hence that $\tilde{\Lambda}_1(\theta_1-) = G(\theta_1-)$. 

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On \([\theta_1, \theta_2]\), we have \(\frac{C'(\theta) \cdot \kappa'}{K(\theta)} = \frac{\theta}{\beta} \), \(\Lambda_2 = 0\) and \(d\Lambda_2 = 0\). Hence (13) becomes

\[
0 = -G \, d\theta + (\theta - \theta_1) \, d\tilde{\Lambda}_1 + \tilde{\Lambda}_1 \, d\theta.
\]

This implies that \(\tilde{\Lambda}_1\) takes the form

\[
\tilde{\Lambda}_1(\theta) = \frac{1}{\theta - \theta_1} \int_{\theta_1}^{\theta} G(t) \, dt
\]

for all \(\theta \in (\theta_1, \theta_2)\), that \(\tilde{\Lambda}_1(\theta_1) = G(\theta_1)\) (since \(\tilde{\Lambda}_1\) and \(G\) are right continuous) and that \(\tilde{\Lambda}_1(\theta_2) = \tilde{\Lambda}_1(\theta_2-) = \frac{1}{\theta_2 - \bar{\theta}} \int_{\theta_1}^{\theta_2} G(\theta) \, d\theta\) (since \(\tilde{\Lambda}_1\) cannot have a jump at \(\theta_2\)).

On \([\theta_2, \bar{\theta}]\), we have \(d\Lambda_1 = 0\). Hence (13) becomes

\[
0 = -G \, d\theta + \tilde{\Lambda}_1 \, d\theta + \left(\theta + \beta b\left(\frac{\theta}{1+\pi}\right)\right) \, d\Lambda_2 + \Lambda_2 \, d\theta.
\]

Furthermore, we have the boundary condition \(\Lambda_2(\theta_2) = 0\) (since \(\Lambda_2\) cannot have a jump at \(\theta_2\)). Putting \(\Lambda_2 = \Lambda_2 + \tilde{\Lambda}_1(\theta_2)\), this equation simplifies slightly to

\[
0 = -G \, d\theta + \left(\theta + \beta b\left(\frac{\theta}{1+\pi}\right)\right) \, d\tilde{\Lambda}_2 + \tilde{\Lambda}_2 \, d\theta,
\]

with boundary condition \(\tilde{\Lambda}_2(\theta_2) = \tilde{\Lambda}_1(\theta_2)\).

7.2. Cases 2 to 5. In order to handle the remaining cases, we need a unified construction. (This construction includes Case 1 too.) It is more convenient to work in terms of the distribution function \(\Psi = \Psi(\cdot; \theta_1)\) of the total multiplier \(d\tilde{\Lambda}_1 + d\Lambda_2\), and to view this as a function on \([0, \bar{\theta}]\). The construction is then very simple. For all \(\theta_1 \in (0, \bar{\theta}]\):

1. put \(\Psi = G\) on \([0, \theta_1]\);

2. if \(\theta_1 < \bar{\theta}\) (so that \(\Psi\) is not yet defined on the whole of \([0, \bar{\theta}]\)), then let \(\Psi\) be the unique bounded solution of the o.d.e.

\[
0 = -G + (\theta - \theta_1) \, \Psi' + \Psi
\]
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on \((\theta_1, \theta_2 \wedge \overline{\theta})\), i.e. put

\[
\Psi(\theta) = \frac{1}{\theta - \theta_1} \int_{\theta_1}^{\theta} G(t) \, dt;
\]

3. if \(\theta_2 < \overline{\theta}\) (so that \(\Psi\) is still not defined on the whole of \([0, \overline{\theta}]\)), then let \(\Psi\) be the unique solution of the o.d.e.

\[
0 = -G + \left(\theta + \beta b \left(\frac{\theta}{(1 + \pi)}\right)\right) \Psi' + \Psi
\]

on \((\theta_2, \overline{\theta})\) with boundary condition

\[
\Psi(\theta_2) = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} G(t) \, dt.
\]

Then, using arguments similar to those of the preceding section, it is easy to verify that the three necessary conditions (4), (5) and (13) together imply that, for all \(\theta_1 \in (0, \overline{\theta}]\), \(\Psi\) must take the given form.

**Remark 7.** We can easily extend the construction of \(\Psi\) to include the case \(\theta_1 = 0\). Indeed, in line with the construction above, we can let \(\Psi(\cdot; 0)\) be the unique bounded solution of the o.d.e.

\[
0 = -G + \left(\theta + \beta b \left(\frac{\theta}{(1 + \pi)}\right)\right) \Psi' + \Psi
\]

on \((0, \overline{\theta}]\).

**Remark 8.** With this definition of \(\Psi(\cdot; 0)\), \(\Psi(\cdot; \theta_1)\) is independent of \(\theta_1\) for \(\theta_1 \in [0, \frac{1}{1 + \pi} \overline{\theta}]\).

8. **Existence of an Optimum**

For all \(\theta_1 \in (0, \overline{\theta}]\), we have shown that there exists a unique \(\Psi = \Psi(\cdot; \theta_1)\) satisfying the two complementary slackness conditions (4) and (5) and the measure equation (13). This \(\Psi(\cdot; \theta_1)\) does not in general satisfy the boundary condition (12). The
The purpose of the current section is to establish that there is at least one choice of $\theta_1$ for which (12) is satisfied.

For all $\theta_1 \in (0, \overline{\theta})$, put $\psi(\theta_1) = \Psi(\overline{\theta}; \theta_1)$. Now consider $G$. We certainly have $G = 0$ on $(0, \underline{\theta})$ and $G = F(\overline{\theta})$ on $[\overline{\theta}, \infty)$. Furthermore Assumption A3 tells us that $G$ is first increasing (on $(0, \theta_M)$) and then decreasing (on $(\theta_M, \infty)$). It is therefore obvious that there exists $\theta_F \in [\underline{\theta}, \overline{\theta}]$ such that $G < F(\overline{\theta})$ on $(0, \theta_F)$ and $G \geq F(\overline{\theta})$ on $[\theta_F, \overline{\theta}]$. Our first lemma sharpens this observation.

**Lemma 9.** There exists $\overline{\theta}_F \in [\underline{\theta}, \overline{\theta})$ such that $G \leq F(\overline{\theta})$ on $(0, \overline{\theta}_F)$ and $G \geq F(\overline{\theta})$ on $(\overline{\theta}_F, \overline{\theta})$.

**Proof.** Suppose first that $\theta_M < \overline{\theta}$. Suppose further that there exists $\xi_0 \in (\theta_M, \overline{\theta})$ such that $G(\xi_0) = G(\overline{\theta})$. Since $G' \leq 0$ on $(\theta_M, \infty) \supset (\xi_0, \overline{\theta})$, it follows that $G' = 0$ on $(\xi_0, \overline{\theta})$. We also know that $G = F(\overline{\theta})$ on $[\overline{\theta}, \infty)$, and therefore that $G' = 0$ on $(\overline{\theta}, \infty)$. Overall, then, $G' = 0$ on $(\xi_0, \overline{\theta})$. Hence $\theta^\alpha F'(\theta)$ is constant on $(\xi_0, \overline{\theta})$, where $\alpha = \frac{2-\beta}{1-\beta}$. (Cf. the proof of Proposition 1.) Hence $G' = 0$ on $(\xi_0, \overline{\theta})$. But this contradicts the fact that $\overline{\theta}$ is the maximum of the support of $F$. We may therefore conclude that $G > G(\overline{\theta}) = F(\overline{\theta})$ on $(\theta_M, \overline{\theta})$.

Suppose second that $\theta_M = \overline{\theta}$. Then $G' \geq 0$ on $(0, \overline{\theta})$. Hence $\theta^\alpha F'(\theta)$ is non-decreasing on $(0, \overline{\theta})$. In particular, if we fix $\xi_1 \in (\underline{\theta}, \overline{\theta})$, then we will have $\theta^\alpha F'(\theta) \geq \xi_1^\alpha F'(\xi_1)$ for all $\theta \in (\xi_1, \overline{\theta})$. Letting $\theta \uparrow \overline{\theta}$, it follows that $\overline{\theta}^\alpha F'(\overline{\theta}) \geq \xi_1^\alpha F'(\xi_1)$. But $F'(\xi_1) > 0$, since $\xi_1$ lies in the interior of the support of $F$. Hence $F'(\overline{\theta}) > 0$. Hence $G(\overline{\theta}) = (1 - \beta) \overline{\theta}^\alpha F'(\overline{\theta}) + F(\overline{\theta}) > F(\overline{\theta})$. Hence there exists $\varepsilon > 0$ such that $G > F(\overline{\theta})$ on $(\overline{\theta} - \varepsilon, \overline{\theta})$.

Overall, then, we have the following picture: $G = 0$ on $(0, \overline{\theta})$; there exists $\xi_2 \in (\underline{\theta}, \overline{\theta})$ such that $G > F(\overline{\theta})$ on $(\xi_2, \overline{\theta})$; and $G = F(\overline{\theta})$ on $[\overline{\theta}, \infty)$. Furthermore Assumption A3 ensures that $\{\theta \mid G(\theta) > F(\overline{\theta})\}$ is an interval. We may therefore put $\overline{\theta}_F = \inf \{\theta \mid \theta \in (\underline{\theta}, \overline{\theta}), G(\theta) > F(\overline{\theta})\}$. £

Now, it follows from the construction of $\Psi$ given in Section 7.2 that $\Psi(\overline{\theta}; \theta_1)$ is a convex combination of the values of $G$ on the interval $(\theta_1, \overline{\theta})$. Combining this observation with Lemma 9, we obtain:

**Lemma 10.** $\psi > F(\overline{\theta})$ on $[\overline{\theta}_F, \overline{\theta})$. £
We also have:

**Lemma 11.** \( \psi < F(\theta) \) on \((0, \frac{1}{1+\pi} \theta]\).

**Proof.** We begin by noting that \( \Psi = \Psi(\cdot; \theta_1) \) is independent of \( \theta_1 \) for \( \theta_1 \in \left[0, \frac{1}{1+\pi} \theta\right] \). It is therefore the unique bounded solution of the o.d.e.

\[
0 = -G + \left( \theta + \beta b \left( \frac{\theta}{(1+\pi)\beta} \right) \right) \Psi' + \Psi
\]

(14) on \((0, \theta]\). We compare \( \Psi \) with the function \( \Phi \) which is the unique bounded solution of the o.d.e.

\[
0 = -G + \theta \Phi' + \Phi
\]

(15) on \((0, \theta]\). Now

\[
\Phi'(\theta) = \frac{1}{\theta} \int_0^\theta G(t) \, dt = (1 - \beta) F(\theta) + \beta \frac{1}{\theta} \int_0^\theta F(t) \, dt.
\]

Hence: \( \Phi = 0 \) on \((0, \theta]\); and \( 0 < \Phi < F \) on \((\theta, \theta]\). Hence

\[
\Phi' = \frac{G - \Phi}{\theta} \geq \frac{F - \Phi}{\theta} \geq 0
\]

on \((0, \theta]\), with strict inequality on \((\theta, \theta]\). Furthermore, \( \Phi \) is a supersolution of the equation for \( \Psi \). Indeed, we have

\[
-G + \left( \beta b \left( \frac{\theta}{(1+\pi)\beta} \right) + \theta \right) \Phi' + \Phi
\]

\[
= -G + \beta b \left( \frac{\theta}{(1+\pi)\beta} \right) \Phi' + \theta \Phi' + \Phi
\]

(on rearranging)

\[
= \beta b \left( \frac{\theta}{(1+\pi)\beta} \right) \Phi'
\]

(since \( \Phi \) satisfies (15))

\[
\geq 0
\]

on \((0, \theta]\), with strict inequality on \((\theta, \theta]\). That is, \( \Phi \) is a supersolution of the equation for \( \Psi \), and a strict supersolution on \((\theta, \theta]\). Hence \( \Phi > \Psi \) on \((\theta, \theta]\). In
particular, \( \psi(0) = \Psi(\theta) < \Phi(\theta) < F(\theta) \). The general case now follows on noting that \( \Psi(\cdot; \theta_1) = \Psi(\cdot; 0) \) for all \( \theta_1 \in (0, \frac{1}{1+\pi} \theta] \). Cf. the remark at the end of Section 7.2.  

Since \( \psi \) is continuous, we can combine Lemmas 10 and 11 to obtain:

**Proposition 12.** There exists \( \theta_1 \in (\frac{1}{1+\pi} \theta, \theta_F) \) such that \( \psi(\theta_1) = F(\theta) \).  

That is, there exists \( \theta_1 \in (\frac{1}{1+\pi} \theta, \theta_F) \) such that equation (12) is satisfied. However, we still need to verify that the multipliers associated with any such \( \theta_1 \) are non-negative.

### 9. Non-Negativity of the Multiplier

In this section we show that, if \( \theta_1 \in (0, \theta_F) \) is such that \( \psi(\theta_1) \leq F(\theta) \), then \( \Psi = \Psi(\cdot; \theta_1) \) is non-decreasing on \([0, \theta] \). We treat the intervals \([0, \theta_1], (\theta_1, \theta_F)\) and \([\theta_F, \theta] \) in turn. We begin with a simple lemma.

**Lemma 13.** \( \theta_F \leq \theta_M \).

**Proof.** In the light of Lemma 9, \( \sup G > F(\theta) \). Moreover it follows from the definition of \( \theta_M \) that \( \sup G = \max \{ G_L(\theta_M), G(\theta_M) \} \). There are therefore two possibilities. Either \( G_L(\theta_M) > F(\theta) \), in which case there is a left neighbourhood of \( \theta_M \) on which \( G > F(\theta) \), and therefore \( \theta_F < \theta_M \). Or \( G(\theta_M) > F(\theta) \), in which case necessarily \( \theta_F \leq \theta_M \).  

**Lemma 14.** \( G' \geq 0 \) on \([0, \theta_F] \).

**Proof.** From Lemma 13 we know that \([0, \theta_F] \subset [0, \theta_M] \). Hence \( G' \geq 0 \) on \([0, \theta_F] \). However, \( G \leq F(\theta) \) on \([0, \theta_F] \) and \( G > F(\theta) \) on \((\theta_F, \theta) \). Hence \( \Delta G(\theta_F) \geq 0 \). Hence \( G' \geq 0 \) on \([0, \theta_F] \).  

**Proposition 15.** Suppose that \( \psi(\theta_1) \leq F(\theta) \). Then \( \Psi' \geq 0 \) on \([0, \theta_1] \).

**Proof.** Since \( \psi(\theta_1) \leq F(\theta) \), Lemma 10 implies that \( \theta_1 < \theta_F \). Hence \([0, \theta_1] \subset [0, \theta_F] \). But \( \Psi = G \) on \([0, \theta_1] \) by construction of \( \Psi \), and Lemma 14 tells us that \( G' \geq 0 \) on \([0, \theta_F] \). It follows that \( \Psi' \geq 0 \) on \([0, \theta_1] \).
In order to show that $\Psi' \geq 0$ on $(\theta_1, \theta_F]$, we use the fact that $\Psi$ solves

$$0 = -G + (\theta - \theta_1) \Psi' + \Psi$$

(16)
on $(\theta_1, \theta_2 \land \theta_F]$ and

$$0 = -G + \left(\theta + \beta b \left(\frac{\theta}{(1+\pi)^{\beta}}\right)\right) \Psi' + \Psi$$

(17)
on $(\theta_2 \land \theta_F, \theta_F]$. We also make use of the corresponding o.d.e. for $\theta$, namely

$$\dot{\theta} = -(\theta - \theta_1)$$

(18)
on $(\theta_1, \theta_2 \land \theta_F]$ and

$$\dot{\theta} = -\left(\theta + \beta b \left(\frac{\theta}{(1+\pi)^{\beta}}\right)\right)$$

(19)
on $(\theta_2 \land \theta_F, \theta_F]$. Specifically, for all $g, h \in (\theta_1, \theta_F]$ such that $h < g$, let $T(h, g)$ denote the time at which the solution of the o.d.e. (18-19) for $\theta$ starting from $g$ hits $h$, and put $S(h, g) = \exp(-T(h, g))$. Notice that $S(\cdot, g)$ increases from 0 at $\theta_1$ to 1 at $g$.

**Lemma 16.** Suppose that $\psi(\theta_1) \leq F(\theta_F)$. Then $\Psi \leq G$ on $(\theta_1, \theta_M]$.

**Proof.** Since $\psi(\theta_1) \leq F(\theta_F)$, Lemma 10 implies that $\theta_1 < \theta_F$. Furthermore Lemma 13 tells us that $\theta_F \leq \theta_M$. For all $g \in (\theta_1, \theta_M)$, we therefore have

$$\Psi(g) = \int_{\theta_1}^{g} \frac{\partial S}{\partial h}(h, g) G(h) dh \leq \int_{\theta_1}^{g} \frac{\partial S}{\partial h}(h, g) G(g-) dh$$

(with equality iff $G(g-) = G(\theta_1)$)

$$= G(g-).$$

Taking limits from the right (and using the continuity of $\Psi$ and the right continuity of $G$) then yields $\Psi \leq G$. ■

**Lemma 17.** The sign of $\Psi'$ coincides with that of $G - \Psi$ on $(\theta_1, \theta_F]$.

**Proof.** We have

$$\Psi' = \frac{G - \Psi}{\theta - \theta_1}$$
on \((\theta_1, \theta_2 \wedge \bar{\theta})\) (from equation (16)) and
\[
\Psi' = \frac{G - \Psi}{\theta + \beta b \left( \frac{\theta}{(1+\pi)^2} \right)}
\]
on \((\theta_2 \wedge \bar{\theta}, \bar{\theta})\) (from equation (17)). Bearing in mind that we have \(\theta - \theta_1 > 0\) on \((\theta_1, \theta_2 \wedge \bar{\theta})\), it follows that the sign of \(\Psi'\) coincides with that of \(G - \Psi\) on \((\theta_1, \theta_2 \wedge \bar{\theta}) \cup (\theta_2 \wedge \bar{\theta}, \bar{\theta}) = (\theta_1, \bar{\theta})\). \(\blacksquare\)

**Proposition 18.** Suppose that \(\psi(\theta_1) \leq F(\bar{\theta})\). Then \(\Psi' \geq 0\) on \((\theta_1, \bar{\theta}_F)\).

**Proof.** From Lemma 16 we know that \(\Psi \leq G\) on \((\theta_1, \theta_M)\) and therefore on \((\theta_1, \bar{\theta}_F) \subset (\theta_1, \theta_M)\). Lemma 17 then implies that \(\Psi' \geq 0\) there. \(\blacksquare\)

**Proposition 19.** Suppose that \(\psi(\theta_1) \leq F(\bar{\theta})\). Then \(\Psi' > 0\) on \([\bar{\theta}_F, \bar{\theta})\).

**Proof.** For all \(\theta \in [\bar{\theta}_F, \bar{\theta})\), we have
\[
\Psi(\bar{\theta}) = S(\theta, \bar{\theta}) \Psi(\theta) + \int_\theta^{\bar{\theta}} \frac{\partial S}{\partial h}(h, \bar{\theta}) G(h) \, dh. \tag{20}
\]
Since
\[
\int_\theta^{\bar{\theta}} \frac{\partial S}{\partial h}(h, \bar{\theta}) \, dh = 1 - S(\theta, \bar{\theta}),
\]
this means that \(\Psi(\bar{\theta})\) is a convex combination of \(\Psi(\theta)\) and the values of \(G\) on \((\theta, \bar{\theta})\). But \(\Psi(\bar{\theta}) = \psi(\theta_1) \leq F(\bar{\theta})\) and \(G > F(\bar{\theta})\) on \((\theta, \bar{\theta})\). So we must have \(\Psi(\theta) < F(\bar{\theta})\). On the other hand, \(G(\theta) \geq F(\bar{\theta})\) since \(\theta \geq \bar{\theta}_F\). Lemma 17 therefore implies that \(\Psi'(\theta) > 0\). \(\blacksquare\)

10. **Uniqueness of the Optimum**

At this point we have established that the utility curve associated with \(\theta_1\) solves the main maximization problem (3) iff \(\psi(\theta_1) = F(\bar{\theta})\). Furthermore \(\psi < F(\bar{\theta})\) on \([0, \frac{1}{1+\pi} \bar{\theta}]\) and \(\psi > F(\bar{\theta})\) on \([\bar{\theta}_F, \bar{\theta})\). Hence there exists \(\theta_1 \in (\frac{1}{1+\pi} \bar{\theta}, \bar{\theta}_F)\) such that \(\psi(\theta_1) = F(\bar{\theta})\). In the current section, we show that the set of \(\theta_1\) for which \(\psi(\theta_1) = F(\bar{\theta})\) is an interval. Furthermore, if we strengthen Assumption A3 by
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requiring that $G$ is strictly increasing to the left of its peak, then $\psi' > 0$ over the relevant range. It then follows that there is a unique $\theta_1$ for which $\psi(\theta_1) = F(\bar{\theta})$. This result is limited: it shows that – under the strengthened version of A3 – there is exactly one solution to problem (3) within our one-parameter family of utility curves; but it does not show that there is exactly one solution to problem (3) in $\Omega$. It is, however, very suggestive.

The main idea of the proof is to find an explicit formula for $\psi_0$, and then use this formula to determine the sign of $\psi'$. Of course, the formula depends on whether $\theta_1 < \frac{1}{1+\pi} \bar{\theta}$ or $\theta_1 > \frac{1}{1+\pi} \bar{\theta}$. In the former case: $\theta_2 = (1 + \pi) \theta_1 < \bar{\theta}$; the maximum-penalty constraint is strictly binding; and the types in the range $[\theta_2, \bar{\theta}]$ will choose to incur the consumption penalty. In the latter case: $\theta_2 = (1 + \pi) \theta_1 > \bar{\theta}$; the maximum-penalty constraint is strictly slack; and no type will choose to incur the consumption penalty.

In order to state the formula for $\psi'$, it will be helpful to introduce the functions $\phi$, $\chi$, $\zeta$ and $\eta$ given by the formulae

$$\phi(\theta_1) = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} G(\theta) \, d\theta$$

for all $\theta_1 \in (0, \infty)$ (where we have suppressed the dependence of $\theta_2$ on $\theta_1$),

$$\chi(\theta_1) = \frac{1}{\bar{\theta} - \theta_1} \int_{\theta_1}^{\bar{\theta}} G(\theta) \, d\theta$$

for all $\theta_1 \in (0, \bar{\theta})$,

$$\zeta(\theta_1) = \frac{1}{\theta_1} \left( \frac{\beta}{1+\pi} b \left( \frac{\theta_1}{\beta} \right) \right) \left( G(\theta_2) - \phi(\theta_1) \right) + \frac{1}{\pi \theta_1} \left( G(\theta_2) - G(\theta_1) \right)$$

for all $\theta_1 \in (0, \infty)$ (where we have suppressed the dependence of $\theta_2$ on $\theta_1$) and

$$\eta(\theta_1) = \frac{\chi(\theta_1) - G(\theta_1)}{\bar{\theta} - \theta_1}$$

for all $\theta_1 \in (0, \infty)$.
for all $\theta_1 \in (0, \overline{\theta})$.

**Lemma 20.** $\psi'(\theta_1) = S(\theta_2, \overline{\theta}) \zeta(\theta_1)$ for $\theta_1 \in (0, \frac{1}{1+\pi})$.

**Proof.** Equation (20) gives

$$\psi(\theta_1) = \int_{\theta_2}^{\overline{\theta}} \frac{\partial S}{\partial h}(h, \overline{\theta}) G(h) \, dh + S(\theta_2, \overline{\theta}) \phi(\theta_1),$$

where we have used the fact that $\Psi(\overline{\theta}; \theta_1) = \psi(\theta_1)$ and $\Psi(\theta_2; \theta_1) = \phi(\theta_1)$. Hence

$$\psi' = -\frac{\partial S}{\partial h}(\theta_2, \overline{\theta}) G(\theta_2) \phi_2' + \frac{\partial S}{\partial h}(\theta_2, \overline{\theta}) \theta_2' \phi + S(\theta_2, \overline{\theta}) \phi'$$

$$= \exp(-T(\theta_2, \overline{\theta})) \left( \frac{\partial T}{\partial h}(\theta_2, \overline{\theta}) \theta_2' (G(\theta_2) - \phi) + \phi' \right),$$

where we have suppressed the dependence of $\phi$ and $\psi$ on $\theta_1$, and where $\theta_2'$ and $\phi'$ denote the derivatives of $\theta_2$ and $\phi$ with respect to $\theta_1$. Furthermore

$$\phi = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} G(t) \, dt$$

and

$$T(\theta_2, \overline{\theta}) = \int_{\theta_2}^{\overline{\theta}} \frac{dt}{t + \beta b \left( \frac{t}{(1+\pi)\beta} \right)}.$$

Hence

$$\phi' = -\frac{\theta_2' - 1}{(\theta_2 - \theta_1)^2} \int_{\theta_1}^{\theta_2} G(t) \, dt + \frac{1}{\theta_2 - \theta_1} (G(\theta_2) \theta_2' - G(\theta_1))$$

$$= \frac{1}{\theta_2 - \theta_1} (-\pi \phi + ((1+\pi) G(\theta_2) - G(\theta_1)))$$

$$= \frac{1}{\theta_1} (G(\theta_2) - \phi) + \frac{1}{\pi \theta_1} (G(\theta_2) - G(\theta_1))$$

and

$$\frac{\partial T}{\partial h}(\theta_2, \overline{\theta}) = -\frac{1}{\theta_2 + \beta b \left( \frac{\theta_2}{(1+\pi)\beta} \right)}. \quad (25)$$
Overall, then,

\[
\exp(T(\theta_2, \overline{\theta})) \psi' = \frac{\partial T}{\partial \theta}(\theta_2, \overline{\theta}) \theta'_2 (G(\theta_2) - \phi) + \phi'
\]

\[
= -\frac{1 + \pi}{\theta_1 + \beta b(\frac{\theta_1}{1+\pi})} (G(\theta_2) - \phi) + \frac{1}{\theta_1} (G(\theta_2) - \phi) + \frac{1}{\pi \theta_1} (G(\theta_2) - G(\theta_1))
\]

\[
= \left( \frac{1}{\theta_1} - \frac{1}{\theta_1 + \beta b(\frac{\theta_1}{1+\pi})} \right) (G(\theta_2) - \phi) + \frac{1}{\pi \theta_1} (G(\theta_2) - G(\theta_1))
\]

(collecting terms in \((G(\theta_2) - \phi)\) and \((G(\theta_2) - G(\theta_1))\), and using the fact that \(\theta_2 = (1 + \pi) \theta_1\))

\[
= \frac{\beta b(\frac{\theta_1}{1+\pi})}{\theta_1 (\theta_1 + \beta b(\frac{\theta_1}{1+\pi}))} (G(\theta_2) - \phi) + \frac{1}{\pi \theta_1} (G(\theta_2) - G(\theta_1)).
\]

Making \(\psi'\) the subject of this equation, we obtain the required result. ■

The second of the two formulae for \(\psi'\) is given by the following lemma.

**Lemma 21.** \(\psi' = \eta\) on \(\left[\frac{1+\pi}{1+\pi} \overline{\theta}, \overline{\theta}\right]\).

**Proof.** We have \(\psi = \chi\) on \(\left[\frac{1+\pi}{1+\pi} \overline{\theta}, \overline{\theta}\right]\). Moreover it is easy to check that

\[
\chi'(\theta_1) = \frac{\chi(\theta_1) - G(\theta_1)}{\overline{\theta} - \theta_1}
\]

on \((0, \overline{\theta})\). ■

There are now two main cases to consider. The more general of the two main cases occurs when \(\frac{1}{1+\pi} \overline{\theta} < \overline{\theta}_F\). In this case, there are three main subcases to consider:

**Subcase 1** \(\theta_1 \in \left(\frac{1+\pi}{1+\pi} \theta_1, \frac{1+\pi}{1+\pi} \overline{\theta}_F\right]\). In this subcase, the maximum-penalty constraint is strictly binding in the sense that \(\theta_2 < \overline{\theta}\). I.e. all the types in the non-trivial range of \([\theta_2, \overline{\theta}]\) choose to make an early withdrawal from the penalty account.

**Subcase 2** \(\theta_1 \in \left[\frac{1}{1+\pi} \overline{\theta}_F, \frac{1+\pi}{1+\pi} \overline{\theta}\right]\). In this subcase, the maximum-penalty constraint is weakly binding in the sense that \(\theta_2 \leq \overline{\theta}\).
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**Subcase 3** \( \theta_1 \in \left[ \frac{1}{1 + \pi} \bar{\theta}, \bar{\theta}_F \right) \). In this subcase, the maximum-penalty constraint is weakly slack in the sense that \( \theta_2 \geq \bar{\theta} \).

The less general of the two main cases occurs when \( \frac{1}{1 + \pi} \bar{\theta} \geq \bar{\theta}_F \). In this case, the third subcase does not arise.

We deal with both of the two main cases simultaneously. The first subcase is settled by the following lemma.

**Lemma 22.** Suppose that \( \theta_1 \in \left( 0, \frac{1}{1 + \pi} \bar{\theta}_F \right] \). Then \( \psi'(\theta_1) \geq 0 \).

Proof. The proof relies on the formula \( \psi'(\theta_1) = S(\theta_2, \bar{\theta}) \zeta(\theta_1) \) given in Lemma 20. This formula is valid for \( \theta_1 \in \left( 0, \frac{1}{1 + \pi} \bar{\theta}_F \right] \)  

We have \( [\theta_1, \theta_2] \subset \left( 0, \bar{\theta}_F \right] \) and therefore \( G' \geq 0 \) on \( [\theta_1, \theta_2] \). Hence \( G(\theta_2) \geq \phi(\theta_1) \) and \( G(\theta_2) \geq G(\theta_1) \). It then follows from formula (23) that \( \zeta(\theta_1) \geq 0 \) (with equality iff \( G(\theta_2) = G(\theta_1) \)), and thence that \( \psi'(\theta_1) \geq 0 \) (with equality iff \( G(\theta_2) = G(\theta_1) \)).

We now turn to the second subcase.

**Lemma 23.** Suppose that \( \theta_1 \in \left[ \frac{1}{1 + \pi} \bar{\theta}_F, \frac{1}{1 + \pi} \bar{\theta} \right] \) and that \( \psi(\theta_1) \leq F(\bar{\theta}) \). Then \( G(\theta_2) - \phi(\theta_1) \geq 0 \).

Proof. We have \( \theta_2 \in \left[ \bar{\theta}_F, \bar{\theta} \right] \) and therefore

\[
G(\theta_2) \geq F(\bar{\theta})
\]

(with strict inequality if \( \theta_1 \in \left( \frac{1}{1 + \pi} \bar{\theta}_F, \frac{1}{1 + \pi} \bar{\theta} \right) \), because then \( \theta_2 \in \left( \bar{\theta}_F, \bar{\theta} \right) \))

\[
\geq \psi(\theta_1) = \Psi(\bar{\theta}; \theta_1)
\]
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(by assumption and by definition of \( \psi \) respectively)

\[
\geq \Psi(\theta_2; \theta_1)
\]

(with strict inequality if \( \theta_1 \in [\frac{1}{1+\pi F}, \frac{1}{1+\pi \bar{\theta}}] \), because then \( \theta_2 < \bar{\theta} \))

\[
= \phi(\theta_1)
\]

(by definition of \( \phi \)). \qed

**Lemma 24.** Suppose that \( \theta_1 \in [\frac{1}{1+\pi \bar{\theta}_F}, \frac{1}{1+\pi \bar{\theta}}] \) and that \( \psi(\theta_1) \leq F(\bar{\theta}) \). Then \( G(\theta_2) - G(\theta_1) > 0 \).

**Proof.** We have

\[
G(\theta_2) \geq F(\bar{\theta})
\]

(with strict inequality if \( \theta_1 \in (\frac{1}{1+\pi \bar{\theta}_F}, \frac{1}{1+\pi \bar{\theta}}) \))

\[
\geq \psi(\theta_1) = \Psi(\bar{\theta}; \theta_1) > \Psi(\theta_1; \theta_1) = G(\theta_1)
\]

(by assumption, by definition of \( \psi \), because \( \theta_1 < \bar{\theta} \) and by construction of \( \Psi \) respectively). \qed

Combining Lemmas 23 and 24, we obtain the following result about the right-hand derivative of \( \psi \).

**Lemma 25.** Suppose that \( \theta_1 \in [\frac{1}{1+\pi \bar{\theta}_F}, \frac{1}{1+\pi \bar{\theta}}] \) and that \( \psi(\theta_1) \leq F(\bar{\theta}) \). Then \( \psi'(\theta_1) > 0 \).

**Proof.** The proof again relies on the formula \( \psi'(\theta_1) = S(\theta_2, \bar{\theta}) \zeta(\theta_1) \) given in Lemma 20. In view of this formula, \( \psi'(\theta_1) > 0 \) if \( G(\theta_2) \geq \phi(\theta_1) \) and \( G(\theta_2) \geq G(\theta_1) \) with at least one strict inequality. But Lemmas 23 and 24 show that \( G(\theta_2) \geq \phi(\theta_1) \) and \( G(\theta_2) > G(\theta_1) \) respectively. \qed

We also need the corresponding result about the left-hand derivative of \( \psi \).

**Lemma 26.** Suppose that \( \theta_1 \in (\frac{1}{1+\pi \bar{\theta}_F}, \frac{1}{1+\pi \bar{\theta}}] \) and that \( \psi(\theta_1) \leq F(\bar{\theta}) \). Then \( \psi_L'(\theta_1) > 0 \).
Proof. The proof parallels that of Lemma 25, with minor changes. First of all, bearing in mind that $\phi$ is continuous, we have

$$\zeta_L(\theta_1) = \frac{\beta}{\theta_1 \left( \theta_1 + \frac{\beta}{1+\pi} b\left( \frac{\theta_1}{\beta} \right) \right)} \left( G_L(\theta_2) - \phi(\theta_1) \right) + \frac{1}{\theta_1} \left( G_L(\theta_2) - G_L(\theta_1) \right) \quad (26)$$

for all $\theta_1 \in (0, \infty)$. As in Lemma 20, we then have $\psi_L'(\theta_1) = S(\theta_2, \overline{\theta}) \zeta_L(\theta_1)$ for $\theta_1 \in (0, \frac{1}{1+\pi} \overline{\theta}]$. Next, just as the single peakedness of $G$ implies that $G > F(\overline{\theta})$ on $(\overline{\theta}_F, \overline{\theta})$, so it also implies that $G_L > F(\overline{\theta})$ on $(\overline{\theta}_F, \overline{\theta})$. Arguing as in Lemma 23, we can therefore show that $G_L(\theta_2) - \phi(\theta_1) \geq 0$ for $\theta_1 \in \left( \frac{1}{1+\pi} \overline{\theta}_F, \frac{1}{1+\pi} \overline{\theta} \right]$, with strict inequality if $\theta_1 < \frac{1}{1+\pi} \overline{\theta}$. (We cannot however extend this to $\theta_1 \in \left[ \frac{1}{1+\pi} \overline{\theta}_F, \frac{1}{1+\pi} \overline{\theta} \right]$, since we cannot deduce from the fact that $G_L > F(\overline{\theta})$ on $(\overline{\theta}_F, \overline{\theta})$ that $G_L \geq F(\overline{\theta})$ at $\overline{\theta}_F$.) Next, as in Lemma 24, we have $G_L(\theta_2) - G_L(\theta_1) > 0$ on $\left( \frac{1}{1+\pi} \overline{\theta}_F, \frac{1}{1+\pi} \overline{\theta} \right]$. Indeed, as in the proof of that lemma, we can show that

$$G_L(\theta_2) \geq F(\overline{\theta})$$

(with strict inequality if $\theta_1 \in \left( \frac{1}{1+\pi} \overline{\theta}_F, \frac{1}{1+\pi} \overline{\theta} \right)$)

$$\geq \psi(\theta_1) = \Psi(\overline{\theta}; \theta_1) > \Psi(\theta_1; \theta_1) = G(\theta_1).$$

In particular, $G(\theta_1) < F(\overline{\theta})$; and therefore $\theta_1 < \overline{\theta}_F \leq \theta_M$; and therefore $G_L(\theta_1) \leq G(\theta_1)$. Finally, applying (26) yields the required result. \(\blacksquare\)

Next, we prove a lemma that will be needed for the third subcase.

Lemma 27. $\chi > G$ on $[0, \overline{\theta}_F)$.

Proof. For all $\theta_1 \in [0, \overline{\theta}_F)$, we have

$$\chi(\theta_1) = \frac{1}{\overline{\theta} - \theta_1} \int_{\theta_1}^{\overline{\theta}} G(\theta) \, d\theta = \frac{1}{\overline{\theta} - \theta_1} \int_{\theta_1}^{\overline{\theta}_F} G(\theta) \, d\theta + \frac{1}{\overline{\theta} - \theta_1} \int_{\overline{\theta}_F}^{\overline{\theta}} G(\theta) \, d\theta.$$
since $G' \geq 0$ on $[0, \overline{\theta}_F]$ by Lemma 14; and

$$\int_{\overline{\theta}_F}^{\overline{\theta}} G(\theta) d\theta > (\overline{\theta} - \overline{\theta}_F) F(\overline{\theta}) \geq (\overline{\theta} - \overline{\theta}_F) G(\theta_1),$$

since $G > F(\overline{\theta})$ on $(\overline{\theta}_F, \overline{\theta})$ and (by Lemma 9) $G \leq F(\overline{\theta})$ on $(0, \overline{\theta}_F)$. Hence

$$\chi(\theta_1) > \frac{\overline{\theta}_F - \theta_1}{\overline{\theta} - \theta_1} G(\theta_1) + \frac{\overline{\theta} - \overline{\theta}_F}{\overline{\theta} - \theta_1} G(\theta_1) = G(\theta_1),$$

as required. □

We can now deal with the third subcase, which arises only in the first scenario.

**Lemma 28.** Suppose that $\frac{1}{1+\pi} \overline{\theta} < \overline{\theta}_F$ – i.e. that we are in the first scenario – and that $\theta_1 \in \left[\frac{1}{1+\pi} \overline{\theta}, \overline{\theta}_F\right)$. Then $\psi'(\theta_1) > 0$.

**Proof.** Since $\theta_1 \geq \frac{1}{1+\pi} \overline{\theta}$, we may apply Lemma 21 to obtain

$$\psi'(\theta_1) = \frac{\chi(\theta_1) - G(\theta_1)}{\overline{\theta} - \theta_1}. $$

Since $\theta_1 < \overline{\theta}_F$, we may apply Lemma 27 to obtain $\chi(\theta_1) - G(\theta_1) > 0$. The result follows. □

Combining Lemmas 22, 25, 26 and 28, we obtain:

**Proposition 29.** The set of $\theta_1 \in (0, \overline{\theta}_F)$ such that $\psi(\theta_1) = F(\overline{\theta})$ is a closed interval. □

The idea behind the proof of the proposition is straightforward. We know from Proposition 12 that all solutions to the equation $\psi = F(\overline{\theta})$ lie in $\left(\frac{1}{1+\pi} \theta, \overline{\theta}_F\right)$. Hence, to prove the proposition, we need only show that $\psi' \geq 0$ on this interval. Furthermore this is what Lemma 22 (for the interval $\left(\frac{1}{1+\pi} \theta, \frac{1}{1+\pi} \overline{\theta}_F\right)$), Lemmas 25 and 26 (for the interval $\left(\frac{1}{1+\pi} \overline{\theta}_F, \frac{1}{1+\pi} \overline{\theta}\right)$) and Lemma 28 (for the interval $\left[\frac{1}{1+\pi} \overline{\theta}, \overline{\theta}_F\right)$) seem to tell us. The only complication is that Lemmas 25 and 26 both require the side condition $\psi \leq F(\overline{\theta})$. However, they make up for this by providing strict rather than weak inequalities. The proof of the Proposition does therefore go through.
Indeed, we actually have $\psi' > 0$ on the interval $(\frac{1}{1+\pi} \bar{\theta}_F, \bar{\theta}_F)$. Hence, the only way in which non-uniqueness can occur at all is if there is a non-trivial interval, contained in $(\frac{1}{1+\pi} \bar{\theta}, \frac{1}{1+\pi} F)$, on which $\psi = F(\bar{\theta})$. Unfortunately, it is possible to construct an example in which precisely this form of non-uniqueness occurs. The spirit of the example is that there exist $\theta_3, \theta_4 \in (\bar{\theta}, \bar{\theta}_F)$ such that: (i) $\theta_4 > (1 + \pi) \theta_3$ (i.e. it is possible that the entire interval of types associated with the kink lies within $[\theta_3, \theta_4]$); and (ii) $G_L(\theta_4) = G(\theta_3)$ (i.e. $G$ is constant on $[\theta_3, \theta_4]$). It then follows that, if there exists $\theta_1 \in [\theta_3, \frac{1}{1+\pi} \theta_4]$ such that $\psi(\theta_1) = F(\bar{\theta})$, then $\psi(\theta_1) = F(\bar{\theta})$ for all $\theta_1 \in [\theta_3, \frac{1}{1+\pi} \theta_4]$.

There are two ways to eliminate this possibility. The first way is to ensure that $G$ cannot have a “flat” of the type envisaged. The following assumption is more than sufficient to ensure this:

**A4** $G$ is strictly increasing on $[\bar{\theta}, \theta_M]$.\(^{13}\)

We then have:

**Proposition 30.** Suppose that Assumptions A1-A4 hold. Then there is a unique $\theta_1 \in (0, \bar{\theta}_F)$ such that $\psi(\theta_1) = F(\bar{\theta})$. ■

Working with Assumption A4 certainly simplifies our comparative statics. See Sections 11-14 below. However, we can still obtain satisfactory comparative-statics results without it. See Section 15 below.

The second way to eliminate the possibility of non-uniqueness is to ensure that $G$ cannot have a long enough flat:

**Proposition 31.** Suppose that Assumptions A1-A3 hold, and that $\pi > \frac{\theta_4 - \bar{\theta}}{\bar{\theta}}$. Then there is a unique $\theta_1 \in (0, \bar{\theta}_F)$ such that $\psi(\theta_1) = F(\bar{\theta})$. ■

In particular, if $\pi = \infty$, then we there is a unique optimum within our one-parameter family of candidate optima.

\(^{13}\text{Notice that } G \text{ is identically } 0 \text{ on } (0, \bar{\theta}). \text{ It does not therefore make sense to require that } G \text{ is strictly increasing on } (0, \theta_M).\)
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11. Comparative Statics with A4

The analysis of Sections 8-10 shows that, for all \( \pi \in [0, \infty) \), the set of solutions of the equation

\[
\psi(\theta_1, \pi) = F(\bar{\theta})
\]

is a non-empty interval. We denote this interval by \( \tau(\pi) = [\underline{\tau}(\pi), \overline{\tau}(\pi)] \). The purpose of the current section is to investigate the dependence of \( \tau \) on \( \pi \).

In order to simplify the exposition, it will be helpful to assume for the time being that A4 holds. This ensures that the interval \( \tau(\pi) \) collapses to a single point, which we shall denote by \( \tau_1(\pi) \). It also ensures that \( \frac{\partial \psi}{\partial \theta_1}(\tau_1(\pi), \pi) > 0 \).

If we assume further that all the functions involved are sufficiently smooth, then we can apply the implicit function to the equation

\[
\psi(\tau_1(\pi), \pi) = F(\bar{\theta})
\]

to conclude that

\[
\tau_1' = -\frac{\frac{\partial \psi}{\partial \pi}}{\frac{\partial \psi}{\partial \theta_1}}.
\]

In particular: \( \tau_1 \) will be increasing (decreasing) in \( \pi \) iff \( \frac{\partial \psi}{\partial \pi} < 0 \) \( \frac{\partial \psi}{\partial \theta_1} > 0 \); and the allocation to the illiquid account will be increasing (decreasing) in \( \pi \) iff \( \frac{\partial \psi}{\partial \pi} > 0 \) \( \frac{\partial \psi}{\partial \theta_1} < 0 \).

Motivated by these observations, we look first at the case in which the maximum-penalty constraint is strictly binding. More precisely, we put \( \tau_2(\pi) = (1 + \pi) \tau_1(\pi) \), and we consider the case in which \( \tau_2(\pi) < \bar{\theta} \). In other words, there is a non-trivial interval of types \( (\tau_2(\pi), \bar{\theta}) \) who choose to consume out of the illiquid account and therefore pay the penalty for doing so. In this case we begin by finding explicit formulae for \( \frac{\partial \psi}{\partial \pi} \) and \( \frac{\partial \psi}{\partial \theta_1} \). We then go on to find conditions under which \( \frac{\partial \psi}{\partial \pi} > 0 \) and \( \frac{\partial \psi}{\partial \theta_1} > 0 \), thereby ensuring that \( \tau_1' < 0 \) (and hence that the allocation to the illiquid account will be strictly increasing in \( \pi \)).

We look second at the case in which the maximum-penalty constraint is strictly slack. More precisely, we consider the case in which \( \tau_2(\pi) > \bar{\theta} \). In other words, even the highest type is not tempted to consume out of the illiquid account. In this case we again begin by finding explicit formulae for \( \frac{\partial \psi}{\partial \pi} \) and \( \frac{\partial \psi}{\partial \theta_1} \). It turns out that \( \frac{\partial \psi}{\partial \pi} = 0 \).
We therefore concentrate on finding conditions under which $\frac{\partial \psi}{\partial \theta} > 0$, thereby ensuring that $\tau_1' = 0$ (and hence that the allocation to the illiquid account does not change with $\pi$).

We look third at the intermediate case in which $\tau_2(\pi) = \overline{\theta}$. This case is important because it is $\tau_2(\pi)$ that determines whether we are in the strictly binding case $\tau_2(\pi) < \overline{\theta}$ or the strictly slack case $\tau_2(\pi) > \overline{\theta}$. Our analysis of the comparative statics of our problem is not therefore complete until we have understood how the transition between these two cases occurs.

12. The Strictly Binding Case

In this section we focus on the set $V$ of $(\theta_1, \pi)$ such that

1. $\theta_1 \in (0, \overline{\theta})$,
2. $\pi \in (0, \infty)$ and
3. $\theta_2 = (1 + \pi) \theta_1 < \overline{\theta}$.

In other words, we do not impose the requirement that $\theta_1 = \tau_1(\pi)$ (i.e. that $\theta_1$ be optimal for the given $\pi$), but we do require that the maximum-penalty constraint is binding (in the sense that types in the non-empty interval $(\theta_2, \overline{\theta})$ are choosing to pay the penalty).

12.1. The formula for $\frac{\partial \psi}{\partial \pi}$. Consider the o.d.e.

$$\dot{\theta} = -\left(\theta + \beta b\left(\frac{\theta}{(1+\pi)^\beta}\right)\right)$$

on $[\theta_2, \overline{\theta}]$, with initial condition $\theta(0) = \overline{\theta}$. Let $T(h; \pi)$ denote the first hitting time of $h \in [\theta_2, \overline{\theta}]$, and put $S(h; \pi) = \exp(-T(h; \pi))$. Then the formula for $\frac{\partial \psi}{\partial \pi}$ is given by the following proposition.

**Proposition 32.** Suppose that $\theta_2 < \overline{\theta}$. Then

$$\frac{\partial \psi}{\partial \pi}(\theta_1, \pi) = \left(\frac{1}{\pi} S(\theta_2, \pi) - \theta_1 \frac{\partial S}{\partial h}(\theta_2, \pi) - \frac{\partial S}{\partial \pi}(\theta_2, \pi)\right) (G(\theta_2) - \phi(\theta_1, \pi))$$

$$- \int_{(\theta_2, \pi]} \frac{\partial S}{\partial \pi}(h, \pi) dG(h).$$
**Proof.** Equation (20) can be written

$$\psi(\theta_1, \pi) = \int_{[\theta_2, \pi]} \frac{\partial S}{\partial h}(h, \pi) G(h) \, dh + S(\theta_2, \pi) \phi(\theta_1, \pi).$$

Hence

$$\frac{\partial \psi}{\partial \pi} = \int_{[\theta_2, \pi]} \frac{\partial^2 S}{\partial h \partial \pi}(h, \pi) G(h) \, dh - \frac{\partial S}{\partial h}(\theta_2, \pi) G(\theta_2) \frac{\partial \theta_2}{\partial \pi}$$

$$+ \left( \frac{\partial S}{\partial h}(\theta_2, \pi) \frac{\partial \theta_2}{\partial \pi} + \frac{\partial S}{\partial \pi}(\theta_2, \pi) \right) \phi + S(\theta_2, \pi) \frac{\partial \phi}{\partial \pi},$$

(30)

where we have suppressed the dependence of \( \psi \) and \( \phi \) on \( \theta_1 \) and \( \pi \). Now:

$$\int_{[\theta_2, \pi]} \frac{\partial^2 S}{\partial h \partial \pi}(h, \pi) G(h) \, dh = \int_{[\theta_2, \pi]} \frac{\partial^2 S}{\partial \pi \partial h}(h, \pi) G(h) \, dh$$

$$= \left[ \frac{\partial S}{\partial \pi}(h, \pi) G(h) \right]_{\theta_2} - \int_{[\theta_2, \pi]} \frac{\partial S}{\partial \pi}(h, \pi) dG(h)$$

$$= -\frac{\partial S}{\partial \pi}(\theta_2, \pi) G(\theta_2) - \int_{[\theta_2, \pi]} \frac{\partial S}{\partial \pi}(h, \pi) dG(h),$$

where we have used the fact that \( \frac{\partial S}{\partial \pi}(\theta) = 0 \);

$$\frac{\partial \phi}{\partial \pi} = \frac{G(\theta_2) - \phi}{\theta_2 - \theta_1} \frac{\partial \theta_2}{\partial \pi} = \frac{G(\theta_2) - \phi}{\pi};$$

and

$$\frac{\partial \theta_2}{\partial \pi} = \theta_1.$$
Substituting into (30), we therefore obtain
\[
\frac{\partial \psi}{\partial \pi} = -\frac{\partial S}{\partial \pi}(\theta_2, \pi) G(\theta_2) - \int_{(\pi, \theta_2)} \frac{\partial S}{\partial h}(h, \pi) dG(h) - \frac{\partial S}{\partial h}(\theta_2, \pi) G(\theta_2) \theta_1 \\
+ \left( \frac{\partial S}{\partial h}(\theta_2, \pi) \theta_1 + \frac{\partial S}{\partial \pi}(\theta_2, \pi) \right) \phi + S(\theta_2, \pi) \frac{G(\theta_2) - \phi}{\pi} \\
= -\frac{\partial S}{\partial \pi}(\theta_2, \pi) (G(\theta_2) - \phi(\theta_1, \pi)) - \int_{(\pi, \theta_2)} \frac{\partial S}{\partial h}(h, \pi) dG(h) \\
+ \left( \frac{1}{\pi} S(\theta_2, \pi) - \theta_1 \frac{\partial S}{\partial h}(\theta_2, \pi) \right) (G(\theta_2) - \phi(\theta_1, \pi)).
\]

The required formula now follows on rearranging. □

In view of Proposition 32, it is clear that there are three main contributions to \( \frac{\partial \psi}{\partial \pi} \), namely:

1. \( \frac{1}{\pi} S(\theta_2, \pi) - \theta_1 \frac{\partial S}{\partial h}(\theta_2, \pi) - \frac{\partial S}{\partial \pi}(\theta_2, \pi); \)
2. \( G(\theta_2) - \phi(\theta_1, \pi); \)
3. \( -\int_{(\theta_2, \pi)} \frac{\partial S}{\partial h}(h, \pi) dG(h). \)

We discuss these contributions in turn.

The first contribution can be signed quite generally:

**Proposition 33.** Suppose that \( \theta_2 \leq \overline{\theta} \). Then
\[
\frac{1}{\pi} S(\theta_2, \pi) - \theta_1 \frac{\partial S}{\partial h}(\theta_2, \pi) - \frac{\partial S}{\partial \pi}(\theta_2, \pi) > 0.
\]
In other words, Contribution 1 is strictly positive.

**Proof.** Explicit calculation shows that
\[
\frac{1}{\pi} S(\theta_2, \pi) - \theta_1 \frac{\partial S}{\partial h}(\theta_2, \pi) - \frac{\partial S}{\partial \pi}(\theta_2, \pi) = \frac{N}{D},
\]
where

\[
N = 1 + (1 + \pi) \left( \frac{\theta_2}{\beta(1 + \pi)} \right)^{1/\rho} + \left( \frac{\vartheta}{\beta(1 + \pi)} \right)^{1/\rho} + (1 + \pi) \left( \frac{\theta_2}{\beta(1 + \pi)} \right)^{1/\rho} \left( \frac{\vartheta}{\beta(1 + \pi)} \right)^{1/\rho} + \rho \pi \left( \frac{\vartheta}{\beta(1 + \pi)} \right)^{1/\rho} - \left( \frac{\theta_2}{\beta(1 + \pi)} \right)^{1/\rho}.
\]

and

\[
D = \pi \left( 1 + (1 + \pi) \left( \frac{\theta_2}{\beta(1 + \pi)} \right)^{1-\rho} \left( 1 + (1 + \pi) \left( \frac{\vartheta}{\beta(1 + \pi)} \right)^{1/\rho} \right)^{1+\rho} \right).
\]

Now the last term in the formula for \( N \) is non-negative, since \( \theta_2 \leq \vartheta \). Hence \( N > 0 \). Finally, it is clear that \( D > 0 \).

The second contribution can only be signed when \( \theta_1 = \tau_1(\pi) \) (or, more generally, when \( \theta_2 \in (\vartheta, \vartheta) \) and \( \psi(\theta_1) \leq F(\vartheta) \)). This, however, is enough for the purpose of our comparative statics.

**Proposition 34.** Suppose that:

1. \( \theta_2 \in (\vartheta, \vartheta) \);
2. \( \psi(\theta_1) \leq F(\vartheta) \);
3. Assumption A4 holds.

Then \( G(\theta_2) - \phi(\theta_1, \pi) > 0 \). In other words, Contribution 2 is strictly positive.

**Proof.** We break the proof down into the cases \( \theta_2 \in (\vartheta, \vartheta_F) \) and \( \theta_2 \in [\vartheta_F, \vartheta) \). In the first case, the proof parallels that of Lemma 22. We have \( [\theta_1, \theta_2] \subset \left( \frac{1}{1+\pi} \vartheta, \vartheta_F \right) \subset (0, \theta_{3M}) \) and \( \theta_2 > \vartheta \). Assumption A4 therefore implies that \( G(\theta_2) > \phi(\theta_1, \pi) \). In the second case, it follows from the proof of Lemma 23 that \( G(\theta_2) - \phi(\theta_1, \pi) > 0 \).

The third contribution cannot be signed under our primary assumptions. It is, however, worth drawing attention to three special cases in which it can be signed.
In all three cases, the comparative statics end up going the same way: $\frac{\partial \phi}{\partial \pi} > 0$, and therefore the allocation to the illiquid account will be increasing in $\pi$. We state these three cases as separate propositions, corresponding to the cases $\rho < 1$, $\rho = 1$ and $\rho > 1$.

**Proposition 35.** Suppose that:

1. $\rho < 1$;
2. $G' \leq 0$ on $(\bar{\theta}, \infty)$ (i.e. $\theta_M = \bar{\theta}$);
3. $\theta_2 \in (\theta, \bar{\theta})$.

Then $-\int_{[\theta_2, \bar{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) dG(h) \geq 0$. In other words, Contribution 3 is non-negative.

**Proof.** It is easy to show that we have

$$\frac{\partial S}{\partial \pi}(h, \pi) \begin{cases} > 0 & \text{if } \rho < 1 \\ = 0 & \text{if } \rho = 1 \\ < 0 & \text{if } \rho > 1 \end{cases}$$

for all $h \in [\theta_2, \bar{\theta})$. Furthermore, we have

$$\frac{\partial S}{\partial \pi}(\bar{\theta}, \pi) = 0 \text{ for all } \rho,$$

because $S(\bar{\theta}, \pi) = 1$. We can therefore proceed as follows.

First, we know that $\theta_2 \in (\theta, \bar{\theta})$. Hence $G' \leq 0$ on $(\theta_2, \bar{\theta}] \subset (\theta, \infty)$. Second, $\rho < 1$. Hence $\frac{\partial S}{\partial \pi}(\cdot, \pi) \geq 0$ on $(\theta_2, \bar{\theta}] \subset [\theta_2, \bar{\theta}]$. Putting these two observations together gives us the required conclusion. ■

**Remark 36.** If $G' \leq 0$ on $(\bar{\theta}, \infty)$ then necessarily $\Delta G(\bar{\theta}) > 0$. Hence it is essential for the proof of Proposition 35 that we restrict attention to $\theta_2 > \bar{\theta}$.

**Proposition 37.** Suppose that:

1. $\rho = 1$;
2. \( \theta_2 \in (0, \overline{\theta}) \).

Then \( -\int_{[\theta_2, \overline{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) \, dG(h) = 0 \). In other words, Contribution 3 is zero.

**Proof.** This follows at once from the fact that \( \frac{\partial S}{\partial \pi}(\cdot, \pi) = 0 \) on \([\theta_2, \overline{\theta}]\). ■

**Proposition 38.** Suppose that:

1. \( \rho > 1; \)
2. \( G' \geq 0 \) on \((0, \overline{\theta})\) (i.e. \( \theta_M = \overline{\theta} \));
3. \( \theta_2 \in (0, \overline{\theta}) \).

Then \( -\int_{[\theta_2, \overline{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) \, dG(h) \geq 0 \). In other words, Contribution 3 is non-negative.

**Proof.** Note first that \( G' \geq 0 \) on \((\theta_2, \overline{\theta}) \subset (0, \overline{\theta})\). Second, \( \rho > 1 \) Hence \( \frac{\partial S}{\partial \pi}(\cdot, \pi) < 0 \) on \((\theta_2, \overline{\theta}) \subset [\theta_2, \overline{\theta}]\). (Cf. the proof of Proposition.) Third, \( \frac{\partial S}{\partial \pi}(\overline{\theta}, \pi) = 0 \). Putting these three observations together, we obtain

\[
\int_{[\theta_2, \overline{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) \, dG(h) = \int_{[\theta_2, \overline{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) \, dG(h) + \int_{[\overline{\theta}, \pi]} \frac{\partial S}{\partial \pi}(h, \pi) \, dG(h) = \int_{[\theta_2, \overline{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) \, dG(h) \leq 0,
\]

as required. ■

**Remark 39.** If \( G' \geq 0 \) on \((0, \overline{\theta})\) then necessarily \( \Delta G(\overline{\theta}) < 0 \). The fact that \( \frac{\partial S}{\partial \pi}(\overline{\theta}, \pi) = 0 \) therefore plays an essential role in the proof of Proposition 38.

**12.2. The formula for \( \frac{\partial \psi}{\partial \theta} \).** Let \( T(h; \pi) \) and \( S(h; \pi) = \exp(-T(h; \pi)) \) as in the preceding section. Then the formula for \( \frac{\partial \psi}{\partial \theta} \) is given by the following proposition.
Proposition 40. Suppose that $\theta_2 < \overline{\theta}$. Then

$$\frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) = \left( \frac{\beta b\left(\frac{\theta_2}{\theta_1}\right)}{\theta_1 \left(\theta_2 + \beta b\left(\frac{\theta_2}{\overline{\theta}}\right)\right)} (G(\theta_2) - \phi(\theta_1, \pi)) + \frac{1}{\theta_1} (G(\theta_2) - G(\theta_1)) \right) S(\theta_2, \overline{\theta}).$$

Proof. This is simply a restatement of Lemma 20.

In view of Proposition 40, there are two main contributions to $\frac{\partial \psi}{\partial \theta_1}$, namely

1. $G(\theta_2) - \phi(\theta_1, \pi)$;
2. $G(\theta_2) - G(\theta_1)$.

We have already given conditions under which the first is strictly positive (in Proposition 34). The second is strictly positive under the same conditions:

Proposition 41. Suppose that:

1. $\theta_2 \in (\theta, \overline{\theta})$;
2. $\psi(\theta_1) \leq F(\overline{\theta})$;
3. Assumption A4 holds.

Then $G(\theta_2) - G(\theta_1) > 0$.

Proof. We break the proof down into the cases $\theta_1 \in \left(\frac{1}{1+\pi} \theta, \frac{1}{1+\pi} \overline{\theta}_F\right)$ and $\theta_1 \in \left[\frac{1}{1+\pi} \theta_F, \frac{1}{1+\pi} \overline{\theta}\right)$. In the first case, the proof parallels that of Lemma 22. We have $[\theta_1, \theta_2] \subset \left(\frac{1}{1+\pi} \theta_F, \overline{\theta}_F\right) \subset (0, \theta_M)$, and moreover $\theta_2 > \overline{\theta}$. Assumption A4 therefore implies that $G(\theta_2) > G(\theta_1)$. In the second case, Lemma 24 implies directly that that $G(\theta_2) > G(\theta_1)$. ■
13. The Strictly Slack Case

In this section we focus on the set $W$ of $(\theta_1, \pi)$ such that

1. $\theta_1 \in (0, \overline{\theta})$,
2. $\pi \in (0, \infty)$ and
3. $\theta_2 = (1 + \pi) \theta_1 > \overline{\theta}$.

In other words, we do not impose the requirement that $\theta_1 = \tau_1(\pi)$ (i.e. that $\theta_1$ be optimal for the given $\pi$), but we do require that the maximum-penalty constraint is slack in the sense that no types are choosing to pay the penalty.

13.1. The formula for $\frac{\partial \psi}{\partial \pi}$. The formula for $\frac{\partial \psi}{\partial \pi}$ is given by the following proposition.

**Proposition 42.** Suppose that $\theta_2 > \overline{\theta}$. Then

$$\frac{\partial \psi}{\partial \pi}(\theta_1, \pi) = 0.$$  

**Proof.** As in the proof of Lemma 21, we have

$$\psi(\theta_1, \pi) = \chi(\theta_1) \text{ for } \theta_1 \in \left(\frac{1}{1+\pi}, \overline{\theta}\right),$$

where

$$\chi(\theta_1) = \frac{1}{\overline{\theta} - \theta_1} \int_{\theta_1}^{\overline{\theta}} G(\theta) \, d\theta.$$  

Hence $\psi$ is independent of $\pi$ for such $\theta_1$. □

13.2. The formula for $\frac{\partial \psi}{\partial \theta_1}$. The formula for $\frac{\partial \psi}{\partial \theta_1}$ is given by the following proposition.

**Proposition 43.** Suppose that $\theta_2 > \overline{\theta}$. Then

$$\frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) = \frac{\chi(\theta_1) - G(\theta_1)}{\theta - \theta_1}.$$  

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Proof. As already noted, we have $\psi(\theta_1, \pi) = \chi(\theta_1)$ for $\theta_1 \in \left(\frac{1}{1+\pi}, \bar{\theta}\right)$. Moreover

$$\chi'(\theta_1) = \frac{\chi(\theta_1) - G(\theta_1)}{\theta - \theta_1},$$

as in the proof of Lemma 21. ■

In view of Proposition 43, there is really only one contribution to $\frac{\partial \psi}{\partial \theta_1}$, namely $\chi(\theta_1) - G(\theta_1)$. It is not possible to sign $\chi(\theta_1) - G(\theta_1)$ for all $\theta_1$, but it is possible to sign it when $\theta_1 = \tau_1(\pi)$, and indeed much more generally when $\theta_1 \in (0, \bar{\theta}_F)$. As before, this is enough for the purpose of our comparative statics.

Proposition 44. Suppose that $\theta_1 \in (0, \bar{\theta}_F)$. Then $\chi(\theta_1) - G(\theta_1) > 0$.

Proof. This is simply a special case of Lemma 27. ■

14. THE INTERMEDIATE CASE

Up to now we have focussed on the comparative statics of $\tau_1$. For example, we have shown that if A4 is satisfied and $\rho = 1$ then: (i) $\tau'_1 < 0$ when $\tau_2(\pi) < \bar{\theta}$; and (ii) $\tau'_1 = 0$ when $\tau_2(\pi) > \bar{\theta}$. However, this leaves open the question of what happens at the transition between the two cases. For example, does $\tau_1$ jump up when $\tau_2(\pi) = \bar{\theta}$? Does it jump down? Or is there more than one value of $\pi$ for which $\tau_2(\pi) = \bar{\theta}$?

In order to address these questions, we need to understand the comparative statics of $\tau_2$. These comparative statics are quite complex in the binding case. However, they simplify as the borderline between the two cases is approached. Moreover they are simpler still in the slack case.

14.1. COMPARATIVE STATICS OF $\tau_2$ IN THE WEAKLY BINDING CASE. We begin this section by looking at the comparative statics of $\tau_2$ when the maximum-penalty constraint is strictly binding (in the sense that $\tau_2(\pi) < \bar{\theta}$). More precisely, we show that $\tau'_2(\pi)$ satisfies a simple linear equation. We then go on to check whether this equation remains valid when the maximum-penalty constraint is only weakly binding (in the sense that $\tau_2(\pi) \uparrow \bar{\theta}$).

Proposition 45. Suppose that $\tau_2(\pi) < \bar{\theta}$. Then

$$D(\tau_1(\pi), \pi) \tau'_2(\pi) = N(\tau_1(\pi), \pi),$$

(31)
where

\[ D(\theta_1, \pi) = \frac{\theta_1 + \beta b \left( \frac{\theta_2}{\pi} \right)}{\theta_2 + \beta b \left( \frac{\theta_2}{\pi} \right)} (G(\theta_2) - \phi(\theta_1, \pi)) + (\phi(\theta_1, \pi) - G(\theta_1)), \]

\[ N(\theta_1, \pi) = \left( (\phi(\theta_1, \pi) - G(\theta_1)) + \frac{\pi (1 + \pi)}{S(\theta_2, \pi)} \int_{[\theta_2, \tilde{\theta}]} \frac{\partial S}{\partial \pi} (h; \pi) d\tilde{G}(h) \right) \theta_1 \]

and \( \tilde{G} \in BV([\theta_2, \tilde{\theta}], \mathbb{R}) \) is given by the formulae \( \tilde{G}_L(\theta_2) = \phi(\theta_1, \pi) \) and \( \tilde{G} = G \) on \((\theta_2, \tilde{\theta}]\).

**Proof.** We have

\[ \tau_2(\pi) = (1 + \pi) \tau_1(\pi) \]

and therefore

\[ \tau'_2(\pi) = \tau_1(\pi) + (1 + \pi) \tau'_1(\pi). \quad (32) \]

Now,

\[ \frac{\partial \psi}{\partial \theta_1}(\tau_1(\pi), \pi) \tau'_1(\pi) + \frac{\partial \psi}{\partial \pi}(\tau_1(\pi), \pi) = 0. \]

Hence, multiplying (32) through by \( \frac{\partial \psi}{\partial \theta_1}(\tau_1(\pi), \pi) \), we obtain

\[ \frac{\partial \psi}{\partial \theta_1} \tau'_2 = \frac{\partial \psi}{\partial \theta_1} \tau_1 + (1 + \pi) \frac{\partial \psi}{\partial \theta_1} \tau'_1 \]

\[ = \frac{\partial \psi}{\partial \theta_1} \tau_1 - (1 + \pi) \frac{\partial \psi}{\partial \pi}, \]

where we have suppressed the dependence of \( \frac{\partial \psi}{\partial \theta_1} \) and \( \frac{\partial \psi}{\partial \pi} \) on \( \tau_1(\pi) \) and \( \pi \), and the dependence of \( \tau_1 \) and \( \tau_2 \) on \( \pi \). We may therefore put

\[ D(\theta_1, \pi) = \frac{\pi \theta_1}{S(\theta_2, \tilde{\theta})} \frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) \]

and

\[ N(\theta_1, \pi) = \frac{\pi \theta_1}{S(\theta_2, \tilde{\theta})} \left( \frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) \theta_1 - (1 + \pi) \frac{\partial \psi}{\partial \pi}(\theta_1, \pi) \right). \]
Equation (31) now follows on applying the formulae for $\frac{\partial \psi}{\partial \pi}(\theta_1, \pi)$ and $\frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi)$ given in Propositions 32 and 40.

Equation (31) can be solved for $\tau^0_2(\pi)$ under the conditions of Proposition 34, namely that: (i) $\tau_2(\pi) \in (\theta, \bar{\theta})$; (ii) $\psi(\tau_1(\pi)) \leq F(\bar{\theta})$; and (iii) Assumption A4 holds. This is not, however, enough for our current purposes: we need to make sure that it can still be solved for $\tau^0_2(\pi)$ in the limiting case $\tau_2(\pi) \uparrow \theta$. To this end, recall that

$$V = \{(\theta_1, \pi) \mid \theta_1 \in (0, \bar{\theta}), \pi \in (0, \infty), \theta_2 < \bar{\theta}\},$$

and put

$$\partial V = \{(\theta_1, \pi) \mid \theta_1 \in (0, \bar{\theta}), \pi \in (0, \infty), \theta_2 = \bar{\theta}\}.$$

Furthermore, for all $(\theta_1, \pi) \in V \cup \partial V$, put

$$D(\theta_1, \pi) = (1 + \pi) \frac{\theta_1 + \beta b \left(\frac{\theta_1}{\beta}\right)}{\theta_2 + \beta b \left(\frac{\theta_1}{\beta}\right)} (G_L(\theta_2) - \phi(\theta_1, \pi))$$

$$+ (\phi(\theta_1, \pi) - \max \{G(\theta_1), G_L(\theta_1)\})$$

and

$$\overline{D}(\theta_1, \pi) = (1 + \pi) \frac{\theta_1 + \beta b \left(\frac{\theta_1}{\beta}\right)}{\theta_2 + \beta b \left(\frac{\theta_1}{\beta}\right)} (G_L(\theta_2) - \phi(\theta_1, \pi))$$

$$+ (\phi(\theta_1, \pi) - \min \{G(\theta_1), G_L(\theta_1)\}).$$

Then we have:

**Lemma 46.** Suppose that $(\tilde{\theta}_1, \tilde{\pi}) \in V \rightarrow (\theta_1, \pi) \in \partial V$. Then

$$D(\theta_1, \pi) \leq \lim \inf D(\tilde{\theta}_1, \tilde{\pi}) \leq \lim \sup D(\tilde{\theta}_1, \tilde{\pi}) \leq \overline{D}(\theta_1, \pi).$$

**Proof.** Note first that $b$ and $\phi$ are both continuous. Hence $b \left(\frac{\theta_1}{\beta}\right) \rightarrow b \left(\frac{\theta_1}{\beta}\right)$ and $\phi(\tilde{\theta}_1, \tilde{\pi}) \rightarrow \phi(\theta_1, \pi)$. Next, put $\tilde{\theta}_2 = (1 + \pi) \tilde{\theta}_1$ and $\theta_2 = (1 + \pi) \theta_1$. Then $\tilde{\theta}_2 \uparrow \theta_2$, and
therefore $G(\tilde{\theta}_2) \rightarrow G_L(\theta_2)$. Finally,
\[
\min \{G(\theta_1), G_L(\theta_1)\} \leq \lim \inf G(\tilde{\theta}_1) \\
\leq \lim \sup G(\tilde{\theta}_1) \\
\leq \max \{G(\theta_1), G_L(\theta_1)\}.
\]

The result follows. ■

The next step is to sign $D$. This cannot be done everywhere on $V \cup \partial V$. But it can be done when $\theta_2 = \overline{\theta}$ and $\theta_1 = \tau_1(\pi)$. Indeed, it is enough to require that $\theta_2 \in (\overline{\theta}_F, \overline{\theta}]$ (i.e. we do not actually have to be on the boundary) and that $\psi(\theta_1, \pi) \leq F(\overline{\theta})$ (i.e. we do not actually have to be at an optimum). We begin with a lemma.

**Lemma 47.** Suppose that:

1. $\theta_2 \in (\overline{\theta}_F, \overline{\theta}]$;
2. $\psi(\theta_1) \leq F(\overline{\theta})$.

Then $G_L(\theta_2) > G(\theta_1) \geq G_L(\theta_1)$.

**Proof.** The proof is similar to that of Lemma 24. Note first that
\[
G_L(\theta_2) \geq F(\overline{\theta})
\]
(with strict inequality if $\theta_2 < \overline{\theta}$)
\[
\geq \psi(\theta_1) = \Psi(\overline{\theta}; \theta_1) > \Psi(\theta_1; \theta_1) = G(\theta_1)
\]
(by assumption, by definition of $\psi$, because $\theta_1 < \theta_2 \leq \overline{\theta}$ and by construction of $\Psi$ respectively). Second, Lemma 10 tells us that $\psi > F(\overline{\theta})$ on $[\overline{\theta}_F, \overline{\theta})$. But we have $\psi(\theta_1) \leq F(\overline{\theta})$. Hence $\theta_1 < \overline{\theta}_F$ and therefore $G' \geq 0$ at $\theta_1 \in (0, \overline{\theta}_F) \subset (0, \theta_M)$. That is, $G(\theta_1) - G_L(\theta_1) = \Delta G(\theta_1) \geq 0$. ■

We can now sign $D$.

**Proposition 48.** Suppose that:
Online Appendix C to “Which Early Withdrawal Penalty Attracts the Most Deposits to a Commitment Savings Account?”

1. \( \theta_2 \in (\theta_F, \bar{\theta}] \);

2. \( \psi(\theta_1) \leq F(\bar{\theta}) \).

Then \( \mathcal{D}(\theta_1, \pi) > 0 \).

**Proof.** Two things follow from Lemma 47. First, \( G(\theta_1) \geq G_L(\theta_1) \). Hence the formula for \( \mathcal{D}(\theta_1, \pi) \) simplifies to

\[
\mathcal{D}(\theta_1, \pi) = (1 + \pi) \frac{\theta_1 + \beta b \left( \frac{\theta_1}{\bar{\theta}} \right)}{\bar{\theta} + \beta b \left( \frac{\theta_2}{\bar{\theta}} \right)} (G_L(\theta_2) - \phi(\theta_1, \pi)) + (\phi(\theta_1, \pi) - G(\theta_1)).
\]

In particular, \( \mathcal{D}(\theta_1, \pi) \) is a strictly positive linear combination of the two terms \( G_L(\theta_2) - \phi(\theta_1, \pi) \) and \( \phi(\theta_1, \pi) - G(\theta_1) \). Second, \( G_L(\theta_2) - G(\theta_1) > 0 \). Hence the sum of the two terms \( G_L(\theta_2) - \phi(\theta_1, \pi) \) and \( \phi(\theta_1, \pi) - G(\theta_1) \) is strictly positive. It therefore suffices to show that each of these two terms is non-negative. We have

\[
G_L(\theta_2) \geq F(\bar{\theta}) \geq \psi(\theta_1) = \Psi(\bar{\theta}; \theta_1)
\]

(as in the proof of Lemma 47)

\[
\geq \Psi(\theta_2; \theta_1) > \Psi(\theta_1; \theta_1)
\]

(since \( \Psi' \geq 0 \) on \( (\theta_1, \bar{\theta}_F) \) (by Proposition 18) and \( \Psi' > 0 \) on \( [\theta_F, \bar{\theta}] \) (by Proposition 19))

\[
= G(\theta_1)
\]

(again as in the proof of Lemma 47). In particular, since \( \Psi(\theta_2; \theta_1) = \phi(\theta_1, \pi) \), we have \( G_L(\theta_2) \geq \phi(\theta_1, \pi) \) and \( \phi(\theta_1, \pi) > G(\theta_1) \). \( \blacksquare \)

Since \( \mathcal{D} > 0 \), finding the sign of \( N \) and finding the sign of \( \tau_2'(\pi) \) amount to the same thing. Note first that

\[
\tau_2(\pi) = (1 + \pi) \tau_1(\pi)
\]

and hence

\[
\tau_2'(\pi) = (1 + \pi) \tau_1'(\pi) + \tau_1(\pi).
\]
We therefore face a tension. On the one hand, we are mainly interested in the case in which \( \tau_0'(\pi) < 0 \). For our purposes, then, the first contribution to \( \tau_2'(\pi) \) (namely \((1 + \pi) \tau_1'(\pi)\)) is negative. However, the second contribution (namely \( \tau_1(\pi) \)) is necessarily positive. The net effect is therefore ambiguous. Worse still, what we really need to show for the purposes of comparative statics is that \( \tau_2'(\pi) > 0 \) (so that the curve \((\tau_1(\pi), \pi)\) crosses the boundary \( \theta_2 = \bar{\theta} \) in a simple way). This is directly at odds with our interest in the case in which \( \tau_0'(\pi) < 0 \).

Fortunately, the problem of signing \( \tau_2'(\pi) \) at the boundary is much simpler than the problem of signing \( \tau_0'(\pi) \) in \( V \). With this end in mind, for all \((\theta_1, \pi) \in \partial V\), put

\[
N(\theta_1, \pi) = (\phi(\theta_1, \pi) - \max \{G(\theta_1), G_L(\theta_1)\}) \theta_1
\]

and

\[
\overline{N}(\theta_1, \pi) = (\phi(\theta_1, \pi) - \min \{G(\theta_1), G_L(\theta_1)\}) \theta_1.
\]

Then we have the following lemma.

**Lemma 49.** Suppose that \((\tilde{\theta}_1, \tilde{\pi}) \in V \rightarrow (\theta_1, \pi) \in \partial V\). Then

\[
N(\theta_1, \pi) \leq \lim \inf N(\tilde{\theta}_1, \tilde{\pi}) \leq \lim \sup N(\tilde{\theta}_1, \tilde{\pi}) \leq \overline{N}(\theta_1, \pi).
\]

**Proof.** The proof is similar to that of Lemma 46. Put \( \tilde{\theta}_2 = (1 + \pi) \tilde{\theta}_1 \) and \( \theta_2 = (1 + \pi) \theta_1 \). Then \( \tilde{\theta}_2 \uparrow \theta_2 = \bar{\theta} \), and therefore \( \frac{\partial S}{\partial \pi}(\tilde{\theta}_2, \pi) \rightarrow 0 \) and \( \int_{(\tilde{\theta}_2, \tilde{\pi})} \frac{\partial S}{\partial \pi}(h; \pi) \ dG(h) \rightarrow 0 \). Furthermore \( \phi(\tilde{\theta}_1, \tilde{\pi}) \rightarrow \phi(\theta_1, \pi) \) and

\[
\min \{G(\theta_1), G_L(\theta_1)\} \leq \lim \inf G(\tilde{\theta}_1)
\]

\[
\leq \lim \sup G(\tilde{\theta}_1)
\]

\[
\leq \max \{G(\theta_1), G_L(\theta_1)\}.
\]

Passing to the limit in the formula given for \( N \) in the statement of Proposition 45, we therefore obtain the required result.  

Combining Lemma 49 with the earlier Lemma 47, we obtain:

**Proposition 50.** Suppose that:
1. $\theta_2 = \theta$;

2. $\psi(\theta_1) \leq F(\theta)$.

Then $N(\theta_1, \pi) > 0$.

**Proof.** The proof is similar to that of Proposition 48. First, because $\psi(\theta_1) \leq F(\theta)$ and therefore $\theta_1 < \theta_F$, the formula for $N(\theta_1, \pi)$ simplifies to

$$N(\theta_1, \pi) = (\phi(\theta_1, \pi) - G(\theta_1)) \theta_1.$$ 

Second, we have

$$\Psi(\theta_1; \theta_1) > \Psi(\theta_1; \theta_1) = G(\theta_1).$$

It remains only to note that, because $\theta_2 = \theta$, we have $\phi(\theta_1, \pi) = \Psi(\theta; \theta_1)$.

Combining Propositions 48 and 50, we see that $\tau_0^2(\pi) > 0$ on $\partial V$. In other words, whatever the behaviour of the curve $(\tau_1(\pi), \pi)$ in $V$, it points out of $V$ at $\partial V$. I.e. it can exit, but not enter, $V$ at $\partial V$. In particular, there exists $\pi_1 \in (0, \infty)$ such that $\tau_2(\pi) < \theta$ iff $\pi \in [0, \pi_1]$.

**14.2. Comparative Statics of $\tau_2$ in the Weakly Slack Case.** We begin this section by looking at the comparative statics of $\tau_2$ when the maximum-penalty constraint is strictly slack (in the sense that $\tau_2(\pi) > \theta$). More precisely, we show that $\tau_2'(\pi)$ satisfies a simple linear equation. We then go on to check whether this equation remains valid when the maximum-penalty constraint is only weakly slack (in the sense that $\tau_2(\pi) \downarrow \theta$). Our first proposition is analogous to Proposition 45.

**Proposition 51.** Suppose that $\tau_2(\pi) > \theta$. Then

$$D(\tau_1(\pi), \pi) \tau_2'(\pi) = N(\tau_1(\pi), \pi), \quad (33)$$

where

$$D(\theta_1, \pi) = \frac{\chi(\theta_1) - G(\theta_1)}{\theta - \theta_1}$$

and

$$N(\theta_1, \pi) = \frac{\chi(\theta_1) - G(\theta_1)}{\theta - \theta_1} \theta_1.$$
Notice that, if \( \chi(\theta_1) - G(\theta_1) > 0 \), then we can divide through by \( D(\tau_1(\pi), \pi) \) to conclude that \( \tau'_2(\pi) = \theta_1 \). Furthermore \( \chi(\theta_1) - G(\theta_1) > 0 \) if \( \theta_1 = \tau_1(\pi) \), and indeed much more generally if \( \theta_1 \in (0, \theta_F) \). Cf. Proposition 44. But it does not hold for all \( (\theta_1, \pi) \in W \).

**Proof.** As in the proof of Proposition 45, we have

\[
\frac{\partial \psi}{\partial \theta_1} \tau'_2 = \frac{\partial \psi}{\partial \theta_1} \tau_1 - (1 + \pi) \frac{\partial \psi}{\partial \pi}.
\]

We may therefore put

\[
D(\theta_1, \pi) = \frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi)
\]

and

\[
N(\theta_1, \pi) = \frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) \theta_1 - (1 + \pi) \frac{\partial \psi}{\partial \pi}(\theta_1, \pi).
\]

Equation (33) now follows on applying the formulae for \( \frac{\partial \psi}{\partial \pi}(\theta_1, \pi) \) and \( \frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) \) given in Propositions 42 and 43.

The next step is to ensure that equation (33) can still be solved for \( \tau'_2(\pi) \) in the limiting case \( \tau_2(\pi) \downarrow \theta_F \). To this end, recall that

\[
W = \{ (\theta_1, \pi) \mid \theta_1 \in (0, \theta_F), \pi \in (0, \infty), \theta_2 > \theta_F \},
\]

and put

\[
\partial W = \{ (\theta_1, \pi) \mid \theta_1 \in (0, \theta_F), \pi \in (0, \infty), \theta_2 = \theta_F \}.
\]

Furthermore, for all \( (\theta_1, \pi) \in W \cup \partial W \), put

\[
\underline{D}(\theta_1, \pi) = \frac{\chi(\theta_1) - \max \{ G(\theta_1), G_L(\theta_1) \}}{\theta_F - \theta_1},
\]

\[
\overline{D}(\theta_1, \pi) = \frac{\chi(\theta_1) - \min \{ G(\theta_1), G_L(\theta_1) \}}{\theta_F - \theta_1}.
\]
and

\[ N(\theta_1, \pi) = \frac{\chi(\theta_1) - \max \{G(\theta_1), G_L(\theta_1)\}}{\bar{\theta} - \theta_1} \theta_1, \]
\[ \overline{N}(\theta_1, \pi) = \frac{\chi(\theta_1) - \min \{G(\theta_1), G_L(\theta_1)\}}{\bar{\theta} - \theta_1} \theta_1. \]

Then we have:

**Lemma 52.** Suppose that \((\tilde{\theta}_1, \tilde{\pi}) \in W \to (\theta_1, \pi) \in \partial W\). Then

\[ D(\theta_1, \pi) \leq \lim \inf D(\tilde{\theta}_1, \tilde{\pi}) \leq \lim \sup D(\tilde{\theta}_1, \tilde{\pi}) \leq \overline{D}(\theta_1, \pi) \]

and

\[ N(\theta_1, \pi) \leq \lim \inf N(\tilde{\theta}_1, \tilde{\pi}) \leq \lim \sup N(\tilde{\theta}_1, \tilde{\pi}) \leq \overline{N}(\theta_1, \pi). \]

**Proof.** Note first that \(\chi\) is continuous. Hence \(\chi(\tilde{\theta}_1) \to \chi(\theta_1)\). On the other hand, as in the proof of Lemma 46,

\[ \min \{G(\theta_1), G_L(\theta_1)\} \leq \lim \inf G(\tilde{\theta}_1) \]
\[ \leq \lim \sup G(\tilde{\theta}_1) \]
\[ \leq \max \{G(\theta_1), G_L(\theta_1)\}. \]

The result follows. □

The next step is to sign \(D\). This cannot be done everywhere on \(W \cup \partial W\). But it can be done when \(\theta_2 = \bar{\theta}\) and \(\theta_1 = \tau_1(\pi)\). Indeed, it is enough to require that \(\theta_2 \in [\bar{\theta}, \infty)\) (i.e. we do not actually have to be on the boundary) and that \(\psi(\theta_1, \pi) \leq F(\bar{\theta})\) (i.e. we do not actually have to be at an optimum). We begin with a lemma.

**Lemma 53.** Suppose that:

1. \(\theta_2 \in [\bar{\theta}, \infty)\);
2. \(\psi(\theta_1) \leq F(\bar{\theta})\).

Then \(G(\theta_1) \geq G_L(\theta_1)\).
Proof. The proof is identical to the relevant part of that of Lemma 47. Since \( \psi(\theta_1) \leq F(\bar{\theta}) \), we must have \( \theta_1 < \bar{\theta}_F \). Hence \( G' \geq 0 \) at \( \theta_1 \in (0, \bar{\theta}_F) \subset (0, \theta_M) \). ■

Proposition 54. Suppose that:

1. \( \theta_2 \in [\bar{\theta}, \infty) \);
2. \( \psi(\theta_1) \leq F(\bar{\theta}) \).

Then \( D(\theta_1, \pi), N(\theta_1, \pi) > 0 \).

Proof. Note first that, in view of Lemma 53, we have

\[
D(\theta_1, \pi) = \frac{\chi(\theta_1) - G(\theta_1)}{\bar{\theta} - \theta_1}
\]

and

\[
N(\theta_1, \pi) = \frac{\chi(\theta_1) - G(\theta_1)}{\bar{\theta} - \theta_1} \theta_1.
\]

Second, since \( \psi(\theta_1) \leq F(\bar{\theta}) \), we have \( \theta_1 < \bar{\theta}_F \). Finally, Proposition 44 tells us that \( \chi(\theta_1) - G(\theta_1) > 0 \) for \( \theta_1 \in (0, \bar{\theta}_F) \). ■

It follows from Proposition 54 that \( \tau_2'(\pi) > 0 \) on \( \partial W \). In other words, whatever the behaviour of the curve \((\tau_1(\pi), \pi)\) in \( W \), it points into \( W \) at \( \partial W \). I.e. it can enter, but not exit, \( W \) at \( \partial W \). In particular, there exists \( \pi_2 \in (0, \infty) \) such that \( \tau_2(\pi) > \bar{\theta} \) iff \( \pi \in (\pi_2, \infty) \).

14.3. Comparative Statics of \( \tau_2 \) in the Remaining Case. At this point we have established that there exist \( 0 < \pi_1 \leq \pi_2 < \infty \) such that \( \tau_2(\pi) < \bar{\theta} \) iff \( \pi \in [0, \pi_1) \) and \( \tau_2(\pi) > \bar{\theta} \) iff \( \pi \in (\pi_2, \infty) \).14 The remaining question is therefore whether it is possible that \( \pi_1 < \pi_2 \), in other words that there is a non-trivial interval \((\pi_1, \pi_2)\) over which \( \tau_2(\pi) = \bar{\theta} \).

Suppose for a contradiction that there is such an interval. Then, over this interval, we must have both

\[
\psi(\tau_1(\pi), \pi) = F(\bar{\theta}) \quad (34)
\]

14That \( \pi_1 \leq \pi_2 \) follows at once from the fact that we cannot have \( \tau_2(\pi) < \bar{\theta} \) and \( \tau_2(\pi) > \bar{\theta} \) simultaneously.
(because $\tau_1(\pi)$ is the optimal $\theta_1$) and

$$\tau_2(\pi) = \bar{\tau}.$$  \hfill (35)

Hence

$$F(\bar{\tau}) = \psi(\tau_1(\pi), \pi) = \Psi(\bar{\tau}; \tau_1(\pi), \pi) = \Psi(\tau_2(\pi); \tau_1(\pi), \pi)$$

(by equation (34), by definition of $\psi$ and by equation (35))

$$= \phi(\tau_1(\pi); \pi) = \frac{1}{\tau_2(\pi) - \tau_1(\pi)} \int_{\tau_1(\pi)}^{\tau_2(\pi)} G(\theta) \, d\theta = \frac{1}{\bar{\tau} - \tau_1(\pi)} \int_{\tau_1(\pi)}^{\bar{\tau}} G(\theta) \, d\theta$$

(by construction of $\Psi$, by definition of $\phi$, by equation (35) again). Multiplying through by $\bar{\tau} - \tau_1(\pi)$, we therefore obtain

$$\int_{\tau_1(\pi)}^{\bar{\tau}} G(\theta) \, d\theta = (\bar{\tau} - \tau_1(\pi)) F(\bar{\tau}).$$

Differentiating with respect to $\pi$, we then obtain

$$-G(\tau_1(\pi)) \tau'_1(\pi) = -\tau'_1(\pi) F(\bar{\tau})$$

or

$$(G(\tau_1(\pi)) - F(\bar{\tau})) \tau'_1(\pi) = 0.$$ 

But equation (35) implies that $(1 + \pi) \tau_1(\pi) = \bar{\tau}$ and therefore

$$\tau'_1(\pi) = -\frac{\theta_1}{1 + \pi} \neq 0.$$ 

We conclude that $G(\tau_1(\pi)) - F(\bar{\tau}) = 0$. This, however, is impossible. For we have

$$G(\tau_1(\pi)) = \Psi(\tau_1(\pi); \tau_1(\pi), \pi) \leq \Psi(\bar{\tau}; \tau_1(\pi), \pi) < \Psi(\bar{\tau}; \tau_1(\pi), \pi)$$

(by construction of $\Psi$, by Proposition 18 and by Proposition 19)

$$= \psi(\tau_1(\pi), \pi) = F(\bar{\tau})$$

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(as above). The only possible conclusion is therefore that $\pi_1 = \pi_2$.

15. Comparative Statics without A4

We divide our discussion into the same three cases that we considered in Section 12.1, namely:

1. $\rho < 1$ and $G' \leq 0$ on $(\theta, \infty)$;
2. $\rho = 1$;
3. $\rho > 1$ and $G' \geq 0$ on $(0, \theta)$.

Of these, the first is by far the simplest.

**Proposition 55.** Suppose that $\rho < 1$ and $G' \leq 0$ on $(\theta, \infty)$. Then $\underline{\tau} = \tau$ for all $\pi \in (0, \infty)$. Furthermore there exists $\pi_1 \in (0, \infty)$ such that: the maximum-penalty constraint is strictly binding for all $\pi \in (0, \pi_1)$; and the maximum-penalty constraint is strictly slack for all $\pi \in (\pi_1, \infty)$. Finally:

1. $\underline{\tau} = \tau$ is strictly decreasing on $(0, \pi_1)$; and
2. $\underline{\tau} = \tau$ is constant on $(\pi_1, \infty)$.

In other words, for all values of the maximum penalty $\pi \in [0, \infty)$, there is a unique optimum within our one-parameter family. Furthermore there exists a critical level $\pi_1$ of $\pi$. Below $\pi_1$, the maximum-penalty constraint is strictly binding and the optimal savings target is strictly increasing in $\pi$. Above $\pi_1$, the maximum-penalty constraint is strictly slack and the optimal savings target is independent of $\pi$.

**Proof.** Since $G' \geq 0$ on $(0, \theta)$ and $G' \leq 0$ on $(\theta, \infty)$, we can put $\theta_M = \theta$. For this choice of $\theta_M$, A4 holds. Indeed: the interval $[\theta_M, \theta]$ is empty, and therefore $G$ is certainly strictly increasing on $[\theta, \theta_M)!$ We may therefore apply the analysis of Sections 11-14 to conclude that there is a unique $\theta_1 = \tau_1(\pi)$ such that $\psi(\theta_1, \pi) = F(\theta)$, and that $\tau_1'(\pi) < 0$. ■
Remark 56. There is also a direct proof of Theorem 55. A sketch of this proof runs as follows. Since $G' \leq 0$ on $(\theta, \infty)$, we must have $\theta_F = \theta$. Furthermore we always have $\theta_1 < \theta_F$ and $\theta_2 > \theta$; and in the strictly binding case we also have $\theta_2 < \theta$. Hence, in the strictly binding case, we have

$$G(\theta_2) > F(\theta) = \psi(\theta_1, \pi) = \Psi(\theta_2; \theta_1, \pi) > \Psi(\theta_2; \theta_1, \pi) = \phi(\theta_1, \pi) > G(\theta_1).$$

In particular,

$$G(\theta_2) - \phi(\theta_1, \pi) > 0.$$

It then follows from the formulae for $\frac{\partial \psi}{\partial \pi}$ and $\frac{\partial \psi}{\partial \theta_1}$ given in Propositions 32 and 40 – both of which feature the term $G(\theta_2) - \phi(\theta_1, \pi)$ – that

$$\frac{\partial \psi}{\partial \pi}, \frac{\partial \psi}{\partial \theta_1} > 0.$$

That is, there is a unique $\theta_1 = \tau_1(\pi)$ such that $\psi(\theta_1, \pi) = F(\theta)$, and $\tau_1'(\pi) < 0$. (The important point here is the fact that our assumption on $G$ allows us to sign the core term $G(\theta_2) - \phi(\theta_1, \pi)$, and thereby the derivatives $\frac{\partial \psi}{\partial \pi}$ and $\frac{\partial \psi}{\partial \theta_1}$, directly.)

Proposition 57. Suppose that $\rho = 1$. Then there exists $\pi_0 \in [0, \infty)$ such that: $\tau < \tau$ for all $\pi \in (0, \pi_0)$; and $\tau = \tau$ for all $\pi \in (\pi_0, \infty)$. Furthermore there exists $\pi_1 \in (\pi_0, \infty)$ such that: the maximum-penalty constraint is strictly binding for all $\pi \in (0, \pi_1)$; and the maximum-penalty constraint is strictly slack for all $\pi \in (\pi_1, \infty)$. Finally:

1. $\tau$ is constant, and $\tau$ is strictly decreasing, on $(0, \pi_0)$;

2. $\tau = \tau$, is strictly decreasing on $(\pi_0, \pi_1)$; and

3. $\tau = \tau$ is constant on $(\pi_1, \infty)$.

In other words, there are two critical levels of $\pi$, namely $\pi_0$ and $\pi_1$. Below $\pi_0$, there is a continuum of optima from within our one-parameter family; and, above $\pi_0$, there is a unique optimum from within our one-parameter family. Below $\pi_1$, the maximum-penalty constraint is strictly binding; and, above $\pi_1$, the maximum-penalty constraint
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is strictly slack. Furthermore, below $\pi_0$: the smallest of the possible optimal savings targets is strictly increasing in $\pi$; and the largest of the possible optimal savings targets is independent of $\pi$. Between $\pi_0$ and $\pi_1$: there is only one optimal savings target, and this is strictly increasing in $\pi$. And, above $\pi_1$: there is again only one optimal savings target, and this is independent of $\pi$.

In order to prove Proposition 57, we shall need three Lemmas.

**Lemma 58.** Fix $\pi > 0$, and suppose that there exist $\theta_3, \theta_4 \in \left(\frac{1}{1+\pi} \bar{\theta}, \bar{\theta}\right)$ such that $\theta_3 < \theta_4$ and $\{\theta_1 | \psi(\theta_1, \pi) = F(\bar{\theta})\} = [\theta_3, \theta_4]$. Then:

1. $G < G(\theta_3)$ on $(0, \theta_3]$;
2. $G = G(\theta_3)$ on $[\theta_3, (1 + \pi) \theta_4)$;
3. $G > G(\theta_3)$ on $( (1 + \pi) \theta_4, \bar{\theta})$.

Furthermore $G(\theta_3) < F(\bar{\theta})$.

In other words, if there is a multiplicity of optimal savings targets, then $G$ must have a flat. Moreover the domain of this flat consists precisely of the half-open interval $[\theta_3, (1 + \pi) \theta_4)$, where $\theta_3$ is the smallest possible choice of $\theta_1$ and $\theta_4$ is the highest possible choice of $\theta_1$.

\{  
\text{i.e. the set of } \theta \text{ for which } G(\theta) = G(\theta_3),  
\text{union of the intervals } [\theta_1, (1 + \pi) \theta_1) \text{ associated with the optimal choices of } \theta_1.  
\}

**Proof.** Since $\psi(\theta_1, \pi) = F(\bar{\theta})$ for all $\theta_1 \in [\theta_3, \theta_4]$, we can take derivatives on the right to conclude that

$$\frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) = 0$$
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for all \( \theta_1 \in [\theta_3, \theta_4] \). But

\[
\frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) = \left( \frac{1 + \pi}{\pi} \frac{\theta_1 + \beta b}{\theta_2 + \beta b} \right) (G(\theta_2) - \phi(\theta_1, \pi))
\]

\[
+ \frac{1}{\pi \theta_1} (\phi(\theta_1, \pi) - G(\theta_1)) S(\theta_2, \theta).
\]

Hence

\[
G(\theta_2) - \phi(\theta_1, \pi) = \phi(\theta_1, \pi) - G(\theta_1) = 0 \quad (36)
\]

for all \( \theta_1 \in [\theta_3, \theta_4] \), where \( \theta_2 = (1 + \pi) \theta_1 \). But \( G' \geq 0 \) on \((0, \theta_F] \). Equation (36) therefore implies that \( G = G(\theta_3) \) on \([\theta_3, (1 + \pi) \theta_4] \), and that \( G_L((1 + \pi) \theta_4) = G(\theta_3) \). This establishes part 2 of the Lemma.

Now put \( \tilde{\theta}_3 \)

**Lemma 59.** Fix \( \pi > 0 \), suppose that there exist \( \theta_3, \theta_4 \in \left( \frac{1}{1+\pi} \theta, \theta_F \right) \) such that \( \theta_3 < \theta_4 \) and \( \{ \theta_1 \mid \psi(\theta_1, \pi) = F(\theta) \} = [\theta_3, \theta_4] \), and put \( \theta_5 = (1+\pi) \theta_4 \) and \( \pi_0 = \frac{\theta_5 - \theta_3}{\theta_3} \). Then, for all \( \hat{\pi} \in (0, \pi_0) \),

\[
\{ \theta_1 \mid \psi(\theta_1, \hat{\pi}) = F(\theta) \} = [\theta_3, \frac{1}{1+\pi} \theta_5] .
\]

In other words, if there is some \( \pi > 0 \) for which there is a multiplicity of optimal savings targets, then there is a whole range of \( \pi \) for which there is a multiplicity of optimal savings targets. Furthermore both of these multiplicities are associated with the same flat of \( G \).

**Proof.** ...  ■

**Lemma 60.** Fix \( \hat{\pi}, \pi > 0 \) and suppose that:

1. \( \hat{\pi} < \pi \);

2. there exist \( \hat{\theta}_3, \hat{\theta}_4 \in \left( \frac{1}{1+\pi} \theta, \theta_F \right) \) such that \( \hat{\theta}_3 < \hat{\theta}_4 \) and \( \{ \theta_1 \mid \psi(\theta_1, \hat{\pi}) = F(\theta) \} = [\hat{\theta}_3, \hat{\theta}_4] \);
3. There exist $\theta_3, \theta_4 \in \left( \frac{1}{1+\pi} \theta, \theta_F \right)$ such that $\theta_3 < \theta_4$ and $\{ \theta_1 \mid \psi(\theta_1, \pi) = F(\theta) \} = [\theta_3, \theta_4]$.

Then: $\hat{\theta}_3 = \theta_3; \hat{\theta}_4 = \frac{1+\pi}{1+\pi} \theta_4; \text{and } \pi < \pi_0 = \frac{(1+\pi)\theta_4 - \theta_3}{\theta_3}$.

In other words, if there is a multiplicity of optimal savings targets associated with both $\pi_0$ and $\pi$, then both multiplicities derive from the same flat of $G$.

**Proof.** \( \square \)

We are now in a position to prove Proposition 57.

**Proof.** [Proof of Proposition 57] \( \square \)

**Proposition 61.** Suppose that $\rho > 1$ and $G' \geq 0$ on $(0, \theta)$. Then there exists $\pi_1 \in (0, \infty)$ such that: the maximum-penalty constraint is strictly binding for all $\pi \in (0, \pi_1)$; and the maximum-penalty constraint is strictly slack for all $\pi \in (\pi_1, \infty)$. Furthermore:

1. $\tau$ and $\tau$ are both strictly decreasing on $(0, \pi_1)$;
2. $\tau = \pi$ is constant on $(\pi_1, \infty)$.

In other words, there exists a critical level $\pi_1$ of $\pi$. Below $\pi_1$: the maximum-penalty constraint is strictly binding; and the set of optimal savings targets is strictly increasing in $\pi$. Above $\pi_1$: the maximum-penalty constraint is strictly slack; and the optimal savings target is independent of $\pi$.

**Proof.** Let $L$ be the locus of all those $(\theta_1, \pi) \in V$ such that $\psi(\theta_1, \pi) = F(\theta)$, let $L_\theta$ be the projection of $L$ onto the first coordinate, and let $L_\pi$ be the projection of $L$ onto the second coordinate. Then, in order to prove the result, it suffices to show that there is a non-increasing function $\varpi: L_\theta \rightarrow L_\pi$, the graph of which is $L$. For then the inverse $\tau: L_\pi \rightarrow L_\theta$ of $\varpi$ is a strictly increasing correspondence.

Note first that, for all $(\theta_1, \pi) \in V$, we have

$$\frac{\partial \psi}{\partial \theta_1}(\theta_1, \pi) = \left( \frac{1+\pi}{\pi} \theta_1 + \beta b \left( \frac{\theta_1}{\theta} \right) \left( G(\theta_2) - \phi(\theta_1, \pi) \right) \right) + \frac{1}{\pi} \theta_1 \left( \phi(\theta_1, \pi) - G(\theta_1) \right) S(\theta_2, \theta)$$
and
\[
\frac{\partial \psi}{\partial \pi}(\theta_1, \pi) = \left( \frac{1}{\pi} S(\theta_2, \pi) - \theta_1 \frac{\partial S}{\partial h}(\theta_2, \pi) \right) (G(\theta_2) - \phi(\theta_1, \pi)) - \int_{[\theta_2, \overline{\theta}]} \frac{\partial S}{\partial \pi}(h, \pi) d\tilde{G}(h),
\]
where, as above, \( \tilde{G} \in BV([\theta_2, \overline{\theta}], \mathbb{R}) \) is given by the formulae \( \tilde{G}_L(\theta_2) = \phi(\theta_1, \pi) \) and \( \tilde{G} = G \) on \( (\theta_2, \overline{\theta}] \).\(^{15}\)

Next, since \( G' \geq 0 \) on \( (0, \overline{\theta}] \), we must have \( \phi(\theta_1, \pi) - G(\theta_1) \geq 0 \), \( G(\theta_2) - \phi(\theta_1, \pi) \geq 0 \) and \( \tilde{G}' \geq 0 \) on \( [\theta_2, \overline{\theta}] \). Hence \( \frac{\partial \psi}{\partial \pi} \geq 0 \).

Third, if in addition if \( \psi(\theta_1, \pi) = F(\overline{\theta}) \), then we must have \( \frac{\partial \psi}{\partial \pi} > 0 \). Indeed, it is always the case that \( \overline{\theta}_F < \overline{\theta} \) and \( G > F(\overline{\theta}) \) on \( (\overline{\theta}_F, \overline{\theta}) \). (See Lemma 9.) Moreover, if \( \psi(\theta_1, \pi) = F(\overline{\theta}) \), then we also have \( G(\theta_1) < F(\overline{\theta}) \). Overall, then, if \( \psi(\theta_1, \pi) = F(\overline{\theta}) \) then \( G \) is non-trivial on \( (\theta_1, \overline{\theta}) \). Now suppose for a contradiction that \( \frac{\partial \psi}{\partial \pi} = 0 \). Then we must have \( G(\theta_2) - \phi(\theta_1, \pi) = 0 \) (which is the same thing as saying that \( \tilde{G}' = 0 \) on \( \{\theta_2\} \)) and \( \tilde{G}' = 0 \) on \( (\theta_2, \overline{\theta}] \). Moreover the former implies that \( G(\theta_2) = G_L(\theta_2) = \phi(\theta_1, \pi) = G(\theta_1) \), and the latter implies that \( G_L(\overline{\theta}) = G(\theta_2) \). So \( G \) is trivial on \( (\theta_1, \overline{\theta}] \), which is the required contradiction.

Finally, since \( \frac{\partial \psi}{\partial \pi} > 0 \), there is a unique \( \pi = \varpi(\theta_1) \) such that \( \psi(\theta_1, \pi) = F(\overline{\theta}) \) and moreover
\[
\varpi'(\theta_1) = -\frac{\partial \psi}{\partial \pi} \frac{\partial \psi}{\partial \pi} \geq 0.
\]

This completes the proof. \( \blacksquare \)

**Remark 62.** It is interesting to compare the levels of uniqueness obtained in Propositions 55, 57 and 61. When \( \rho < 1 \), we have uniqueness for all \( \pi \in (0, \infty) \). When \( \rho = 1 \), a limited form of non-uniqueness can develop: there exists \( \pi_0 \in [0, \pi_1) \) such that there is non-uniqueness on \( (0, \pi_0) \) and uniqueness on \( (\pi_0, \infty) \). And, when \( \rho > 1 \), non-uniqueness takes the form that one might expect in a convex optimization problem. However, we do at least get strict monotonicity on the whole of \( (0, \pi_1) \).

\(^{15}\)For the definition of \( V \), see the beginning of Section 12.
16. **Existence of a Full Optimum**

Suppose that self $0$ is required to pick a $B$ satisfying Constraints 1 and 2. Then the utility curve $(u, w)$ that results will satisfy the following three conditions:

**I** $(u, w)$ is interior, in the sense that $u, w > U(0)$ on $\Theta$.

**M** $(u, w)$ is monotonic, in the sense that $u$ is non-decreasing and $w$ is non-increasing.

**DE** $(u, w)$ satisfies the differential equation $\theta du + \beta dw = 0$.

If $B$ is also convex, then $(u, w)$ will also satisfy:

**C** $(u, w)$ is continuous.

Now, the set $\Omega$ with which we have worked so far consists of utility curves $(u, w)$ that satisfy I, BV, DE and C, where BV is the condition:

**BV** $(u, w)$ is of bounded variation.

Since BV is weaker than M, this means that $\Omega$ contains all the utility curves that can result from convex $B$, and more besides. We have therefore solved a relaxed version of the convex-$B$ problem. Since the solution of this relaxed problem is feasible in the convex-$B$ problem, we have therefore also solved the convex-$B$ problem. The purpose of the present section is to solve the general problem in which $B$ is not required to be convex.

Suppose accordingly that $\Omega$ consists of all $(u, w) \in BV(\Theta, \text{ran}(U))^2$ such that $\theta du + \beta dw = 0$. In other words, let $\Omega$ consist of utility curves $(u, w)$ that satisfy I, BV, DE but not C. Put $X = BV(\Theta, \mathbb{R})^2, \Xi = BV(\Theta, \text{ran}(U))^2$ and $Z = BV(\Theta, \mathbb{R})$. Then the objective function $M$ and the constraint mappings $G_1$ and $G_2$ continue to make sense. The analysis of Luenberger (1969) therefore shows that $x_0 \in \Omega$ solves the problem

$$\text{maximize } M(x)$$

subject to

$$\begin{align*}
G_1(x) &\geq 0 \\
G_2(x) &\geq 0
\end{align*}$$

iff there exist $\lambda_1, \lambda_2 \in Z^*$ such that:
1. \( L(x_0, \lambda_1, \lambda_2) \geq L(x, \lambda_1, \lambda_2) \) for all \( x \in \Omega \), where

\[
L(x, \lambda_1, \lambda_2) = M(x) + \langle G_1(x), \lambda_1 \rangle + \langle G_2(x), \lambda_2 \rangle;
\]

2. \( G_1(x) \geq 0, \lambda_1 \geq 0 \) and \( \langle G_1(x), \lambda_1 \rangle = 0 \);

3. \( G_2(x) \geq 0, \lambda_2 \geq 0 \) and \( \langle G_2(x), \lambda_2 \rangle = 0 \).

In other words, there exists multipliers \( \lambda_1 \) and \( \lambda_2 \) such that: (1) \( x_0 \) maximizes \( L(\cdot, \lambda_1, \lambda_2) \) over \( \Omega \); (2) complementary slackness holds for the first constraint; and (3) complementary slackness holds for the second constraint.

At this point, however, we encounter an obstacle. While the dual space \( C(\Theta, \mathbb{R})^* \) of \( C(\Theta, \mathbb{R}) \) has a convenient representation as the space \( M(\Theta, \mathbb{R}) \), the dual space \( BV(\Theta, \mathbb{R})^* \) of \( BV(\Theta, \mathbb{R}) \) does not have a similarly convenient representation. This makes it difficult to use the necessity part of the Lagrangean characterization of the optimum. We can, however, still hope to use the sufficiency part.

The idea here is to note that the elements of \( M(\Theta, \mathbb{R}) \) can be used to induce continuous linear functionals on \( BV(\Theta, \mathbb{R}) \). For example, \( \mu \in M(\Theta, \mathbb{R}) \) induces \( \mu_R \in BV(\Theta, \mathbb{R})^* \) via the formula

\[
\langle z, \mu_R \rangle = \int z_R \, d\mu,
\]

where \( z_R \) denotes the right-continuous version of \( z \). However, in pursuing this idea, it is important to note that \( \mu \) also induces \( \mu_L \in BV(\Theta, \mathbb{R})^* \) via the formula

\[
\langle z, \mu_L \rangle = \int z_L \, d\mu,
\]

where \( z_L \) denotes the left-continuous version of \( z \). In other words, there is no canonical association between elements of \( M(\Theta, \mathbb{R}) \) and continuous linear functionals on \( BV(\Theta, \mathbb{R}) \).

Our plan is therefore to start from a \( \theta_1 \) such that \( \Psi(\theta; \theta_1) = F(\theta) \), in the hope that \( \Psi(\cdot; \theta_1) \) can be used to generate multipliers that can be used in the sufficiency part of the Lagrangean characterization of an optimum. Indeed, suppose that we are given such a \( \theta_1 \). Then, bearing in mind that \( \Delta \Psi(\theta_2; \theta_1) = 0 \), we may put \( d\tilde{\Lambda}_1 = d\Psi(\cdot; \theta_1) \) on \( [\theta, \theta_2] \), \( d\Lambda_2 = d\Psi(\cdot; \theta_1) \) on \( [\theta_2, \theta] \) and \( d\Lambda_1 = \frac{1}{K(w_0)} d\tilde{\Lambda}_1 \). Furthermore,
if we let $\lambda_1$ and $\lambda_2$ be the continuous linear functionals induced on $BV(\Theta, \mathbb{R})$ by $d\Lambda_1$ and $d\Lambda_2$ using integration with respect to the right-continuous versions of functions, then we have

\[
L(x, \lambda_1, \lambda_2) = \int \left( \theta u(\theta) + w(\theta) \right) dF(\theta) \\
+ \int \left( y - C(u(\theta)) - K(w(\theta)) \right) d\Lambda_1(\theta) \\
+ \int \left( b\left( \frac{\theta}{1+\pi} \right) u(\theta) - \frac{1}{\beta} a\left( \frac{\theta}{1+\pi} \right) - w(\theta) \right) d\Lambda_2(\theta)
\]

for all $x \in X$. Our objective is then to show that the utility curve $x_0 = (u_0, w_0)$ associated with $\theta_1$ maximizes $L(\cdot, \lambda_1, \lambda_2)$.

It suffices to show that, for all $x_1 \in \Omega$, the directional derivative $\nabla_x L(x_0, \lambda_1, \lambda_2)$ of $L$ at $x_0$ in the direction $x = x_1 - x_0$ is non-positive. As in Section 6, we have

\[
\nabla_x L(x_0, \lambda_1, \lambda_2) = \int \left( \theta u + w \right) dF - \int \left( C'(u_0) u + K'(w_0) w \right) d\Lambda_1 \\
+ \int \left( b\left( \frac{\theta}{1+\pi} \right) u - w \right) d\Lambda_2 \\
= \int \left( \theta u + w \right) dF - \int \left( \frac{C'(u_0)}{K'(w_0)} u + w \right) d\tilde{\Lambda}_1 \\
+ \int \left( b\left( \frac{\theta}{1+\pi} \right) u - w \right) d\tilde{\Lambda}_2.
\]

Furthermore, notwithstanding the fact that we are now working in a more general context, we can eliminate the terms $\int w dF$, $\int w d\tilde{\Lambda}_1$ and $\int w d\tilde{\Lambda}_2$ using integration by parts.

Indeed, the general formula for integration by parts tells us that

\[
\int_{[\underline{\theta}, \overline{\theta}]} w(\theta) dF(\theta) = [w F]_{\underline{\theta}}^{\overline{\theta}} - \int_{[\underline{\theta}, \overline{\theta}]} F(\theta) dw(\theta) + \sum_{\theta \in [\underline{\theta}, \overline{\theta}]} \Delta w(\theta) \Delta F(\theta),
\]

where

\[
[w F]_{\underline{\theta}}^{\overline{\theta}} = w(\overline{\theta}) F(\overline{\theta}) - w(\underline{\theta}) F(\underline{\theta}).
\]
We therefore have
\[ \int w \, dF = [w \, F']_{\underline{\theta}} - \int F \, dw + \sum w \, \Delta F \]
(where we have suppressed the dependence on $\theta$ and where the domains of all integrals and sums are understood to be the whole of $[\underline{\theta}, \overline{\theta}]$)
\[ = w(\overline{\theta}) \, F(\overline{\theta}) + \int F \, \frac{\theta}{\beta} \, du - \sum \frac{\theta}{\beta} \, \Delta u \, \Delta F \]
(because $F(\overline{\theta}) = 0$ and $dw = -\frac{\theta}{\beta} \, du$)
\[ = w(\overline{\theta}) \, F(\overline{\theta}) + \frac{1}{\beta} \int F \, \theta \, du - \frac{1}{\beta} \sum \theta \, \Delta u \, \Delta F. \]
Moreover
\[ \int F \, \theta \, du = [(F \, \theta) \, u]_{\underline{\theta}} - \int u \, d(F \, \theta) + \sum (F \, \theta) \, \Delta u \]
(applying the general formula for integration by parts to $\int F \, \theta \, du$)
\[ = \overline{\theta} \, u(\overline{\theta}) \, F(\overline{\theta}) - \int u \, (\theta \, dF + F \, d\theta) + \sum \theta \, \Delta F \, \Delta u \]
(since $F(\overline{\theta}) = 0$, $d(F \, \theta) = \theta \, dF + F \, d\theta$ and $\Delta(F \, \theta) = \theta \, \Delta F$). Overall, then,
\[ \int w \, dF = \left( \frac{\overline{\theta}}{\beta} \, u(\overline{\theta}) + w(\overline{\theta}) \right) \, F(\overline{\theta}) - \frac{1}{\beta} \int u \, (\theta \, dF + F \, d\theta). \]

By the same token, and bearing in mind that we did not use the fact that $\Delta F = 0$ in the derivation of the previous paragraph, we have
\[ \int w \, d\Lambda_1 = \left( \frac{\overline{\theta}}{\beta} \, u(\overline{\theta}) + w(\overline{\theta}) \right) \Lambda_1(\overline{\theta}) - \frac{1}{\beta} \int u \, (\theta \, d\Lambda_1 + \Lambda_1 \, d\theta) \]
and
\[ \int w \, d\Lambda_2 = \left( \frac{\overline{\theta}}{\beta} \, u(\overline{\theta}) + w(\overline{\theta}) \right) \Lambda_2(\overline{\theta}) - \frac{1}{\beta} \int u \, (\theta \, d\Lambda_2 + \Lambda_2 \, d\theta). \]
Online Appendix C to “Which Early Withdrawal Penalty Attracts the Most Deposits to a Commitment Savings Account?”

We therefore have

\[ \nabla_x L(x_0, \lambda_1, \lambda_2) = \int u\, du^\ast + w(\bar{\theta})\, r^\ast \]

where, as in section 6 above,

\[ du^\ast = -\frac{1}{\beta} \left( (1 - \beta) \theta \, dF + F \, d\theta \right) + \frac{1}{\beta} \left( \left( \theta - \beta \frac{C(u_0)}{K(u_0)} \right) \, d\Lambda_1 + \tilde{\Lambda}_1 \, d\theta \right) \]

\[ + \frac{1}{\beta} \left( \left( \theta + \beta b \left( \frac{\theta}{1 + \pi} \right) \right) \, d\Lambda_2 + \Lambda_2 \, d\theta \right) + \frac{\bar{\theta}}{\beta} \left( F(\bar{\theta}) - \tilde{\Lambda}_1(\bar{\theta}) - \Lambda_2(\bar{\theta}) \right) \, dI, \]

\[ r^\ast = F(\bar{\theta}) - \tilde{\Lambda}_1(\bar{\theta}) - \Lambda_2(\bar{\theta}) \]

and \( I \) is the distribution function of the unit mass at \( \bar{\theta} \). Finally, by construction of \( \tilde{\Lambda}_1 \) and \( \Lambda_2 \), we have \( u^\ast = 0 \) and \( r^\ast = 0 \). So in fact \( \nabla_x L(x_0, \lambda_1, \lambda_2) = 0 \). In particular, the utility curve \( x_0 = (u_0, w_0) \) associated with \( \theta_1 \) does indeed maximize \( L(\cdot, \lambda_1, \lambda_2) \).

Remark 63. Great care is needed in choosing the space \( Z \). One possible choice is \( \mathcal{C}(\Theta, \mathbb{R}) \), the space of all continuous functions on \( \Theta \) endowed with the sup norm. This choice has the advantage that there is a convenient representation for \( Z^\ast \). However, it also requires that \( \Omega \subset \mathcal{C}(\Theta, \mathbb{R}) \), and this is not an economically reasonable restriction. Another possible choice is \( \mathcal{B}(\Theta, \mathbb{R}) \), the space of all bounded functions on \( \Theta \) endowed with the sup norm. This choice has the advantage that it includes all economically relevant utility curves. Unfortunately, it leads to a different problem: the measures \( d\Lambda_1 \) and \( d\Lambda_2 \) associated with \( \Psi(\cdot ; \theta_1) \) do not induce continuous linear functionals on \( \mathcal{B}(\Theta, \mathbb{R}) \), since functions in \( \mathcal{B}(\Theta, \mathbb{R}) \) are not in general measurable. The results of Luenberger (1969) do not therefore apply. Our solution to this double problem is to use \( \mathcal{BV}(\Theta, \mathbb{R}) \). This space is big enough to include all economically relevant utility curves, but small enough that \( d\Lambda_1 \) and \( d\Lambda_2 \) can be used to induce continuous linear functionals on it (albeit not in a canonical way).

17. Distributions

17.1. Beta Distribution. The density of the generalization of the Beta that we consider is proportional to

\[ (x - a)^{\xi - 1}(b - x)^{\eta - 1} \]
on the interval $(a, b)$, where $0 < a < b$ and $\zeta, \eta > 0$. It is unbounded at $a$ if $\zeta < 1$, in which case we require that $\theta \in (a, b)$ in order to ensure that $A1$ is satisfied, and unbounded at $b$ if $\eta < 1$, in which case we require that $\overline{\theta} \in (a, b)$ in order to ensure that $A1$ is satisfied.

There are then four main cases. Three of the cases are easy to describe:

**Case 1** if $\zeta > 1$ and $\eta \geq 1$ then $A3$ is satisfied for all choices of $\theta, \overline{\theta} \in (a, b)$;

**Case 2** if $\zeta > 1$ and $\eta < 1$ then $A3$ is again satisfied for all choices of $\theta, \overline{\theta} \in (a, b)$, albeit for somewhat different reasons;

**Case 4** if $\zeta < 1$ and $\eta < 1$, then $A3$ is violated for some choices of $\theta, \overline{\theta} \in (a, b)$.

Case 3 is more involved. If $\zeta < 1$ and $\eta \geq 1$, then $A3$ is satisfied for all choices of $\theta, \overline{\theta} \in (a, b)$ iff

$$\frac{\left(\sqrt{1-\zeta} + \sqrt{\eta - 1} \sqrt{\frac{\overline{\theta}}{\theta}}\right)^2}{1 - \frac{a}{b}} \geq 1 + \frac{1}{1 - \beta}.$$  \hspace{1cm} (37)

As this inequality makes clear, $A3$ is more likely to be satisfied if: either (i) $\zeta$ is close to 0 (i.e. the spike at $a$ is very pronounced); or (ii) $\eta$ is large (i.e. the density decays very quickly towards $b$); or (iii) $\frac{a}{b}$ is close to 1 (i.e. the density is concentrated in a narrow band).\(^{16}\) It is also worth noting that, as $\frac{a}{b} \to 0$, the left-hand side of (37) converges to $1 - \zeta < 1$. Hence $A3$ is violated for some choices of $\theta, \overline{\theta} \in (a, b)$ when $\zeta < 1$ and $\frac{a}{b}$ is small. This is in striking contrast with the standard case studied in both Rice and Hogg et al. In that case $A3$ is satisfied for all $\theta, \overline{\theta} \in (a, b)$ when $\zeta < 1$ and $\frac{a}{b} = 0$.

Note finally that the right-hand side of (37) is strictly increasing in $\beta$. Hence, if we fix a distribution for which $\zeta < 1$ and $\eta \geq 1$, then the conclusion is that $A3$ will be satisfied provided that $\beta$ is far enough below 1. I.e., $A3$ is more likely to be satisfied when the decision maker is more time-inconsistent.

\(^{16}\)For the purposes of the present discussion, $\zeta \in (0, 1), \eta \in [1, \infty)$ and $\frac{a}{b} \in (0, 1)$. 


17.2. **Cauchy Distribution.** The density of the general form of the Cauchy distribution is proportional to

$$\left(1 + \left(\frac{x - \mu}{\sigma}\right)^2\right)^{-1}$$

on \(\mathbb{R}\), where \(\mu \in \mathbb{R}\) is a location parameter and \(\sigma > 0\) is a scale parameter. This distribution satisfies A3 for all \(\underline{\theta}, \overline{\theta} \in (0, \infty)\) iff

$$\frac{\mu}{\sigma} \leq \sqrt{\frac{1 - (1 - \beta)^2}{(1 - \beta)^2}}. \quad (38)$$

In other words, taking \(\beta\) as given, A3 is satisfied iff the distribution is not located too far to the right. If (38) does not hold then, for some choices of \(\underline{\theta}, \overline{\theta} \in (0, \infty)\), \(G\) is first increasing (at \(\underline{\theta}\)), then decreasing, then increasing again, then finally decreasing again (at \(\overline{\theta}\)).

We can also make \(1 - \beta\) the subject of the inequality (38). Doing so, we find that A3 is satisfied for all \(\underline{\theta}, \overline{\theta} \in (0, \infty)\) iff: either \(\mu \leq 0\); or \(\mu > 0\) and

$$1 - \beta \leq \left(1 + \frac{\mu^2}{\sigma^2}\right)^{-\frac{\beta}{2}}.$$

In other words, taking the parameters \(\mu\) and \(\sigma\) of the Cauchy distribution as given, A3 is satisfied iff: either \(\mu \leq 0\); or \(\mu > 0\) and \(\beta\) is sufficiently close to 1. I.e. A3 is more likely to be satisfied when the decision maker is less time-inconsistent.

17.3. **Log-Gamma Distribution.** The density of the Log-Gamma distribution is proportional to

$$x^{-\frac{\eta + 1}{\eta}}(\log(x))^{\zeta - 1}$$

on \((1, \infty)\), where \(\zeta, \eta > 0\). It is unbounded at 1 if \(\zeta < 1\), in which case we require that \(\underline{\theta} > 1\) in order to ensure that A1 is satisfied. It violates A3 for some choices of \(\underline{\theta}, \overline{\theta} \in (1, \infty)\) iff \(\zeta < 1\) and \(\eta > 1 - \beta\). In other words, taking \(\beta\) as given, A3 is violated iff there is a singularity at 1 and the rate of decay at \(\infty\) is sufficiently slow.

Note finally that, if we fix a distribution for which \(\zeta < 1\), then the conclusion is
that A3 will be satisfied provided that $\beta$ is far enough below 1. I.e. A3 is more likely to be satisfied when the decision maker is more time-inconsistent.

17.4. Pareto Distribution. The density of the Pareto type II distribution is proportional to

$$\left(1 + \frac{x - \mu}{\sigma}\right)^{-\zeta - 1}$$

on $(\mu, \infty)$, where $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter and $\zeta > 0$ is a shape parameter. It violates A3 for some choices of $\underline{\theta}, \overline{\theta} \in (\mu, \infty)$ iff

$$\zeta < \frac{1}{1 - \beta} \quad (39)$$

and

$$\frac{\mu}{\sigma} > \frac{1}{\zeta + 1} \left(1 + \frac{1}{1 - \beta}\right). \quad (40)$$

In other words, it violates A3 iff its right-hand tail is sufficiently fat and, taking the fatness of the tail as given, it is located sufficiently far to the right. In particular, if $\frac{\mu}{\sigma} \leq 1$, then the Pareto type II distribution satisfies A3 for all $\underline{\theta}, \overline{\theta} \in (\mu, \infty)$. For in that case: either (i) $\zeta \geq \frac{1}{1 - \beta}$ and therefore (39) is violated; or (ii) $\zeta < \frac{1}{1 - \beta}$, in which case $\frac{1}{\zeta + 1} \left(1 + \frac{1}{1 - \beta}\right) > \frac{1}{\zeta + 1} (1 + \zeta) = 1 \geq \frac{\mu}{\sigma}$ and therefore (40) is violated.

We can also make $1 - \beta$ the subject of these inequalities. Doing so, we find that A3 is violated for some $\underline{\theta}, \overline{\theta} \in (\mu, \infty)$ iff

$$\left(\frac{\mu}{\sigma} (\zeta + 1) - 1\right)^{-1} < 1 - \beta < \zeta^{-1}. \quad (41)$$

In particular, if $\frac{\mu}{\sigma} > 1$ and $\zeta > 1$ (so that $\left(\frac{\mu}{\sigma} (\zeta + 1) - 1\right)^{-1} < \zeta^{-1} < 1$), then A3 is satisfied iff $\beta$ is either close enough to 1 or far enough below 1.