Extrapolative Expectations and the Equity Premium *

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Abstract

Many stockholders irrationally believe that high recent market returns predict high future market returns. We argue that the presence of these extrapolative investors can help resolve the equity premium puzzle if the elasticity of intertemporal substitution (EIS) is greater than unity. Extrapolators’ overreaction to dividend news generates countercyclical expected returns. Rational investors respond by making their consumption growth more procyclical. The equity premium is high because extrapolators believe stocks are a bad hedge and rational investors have high consumption growth covariance with stocks. We estimate the model using the method of simulated moments. The resulting parameter combination matches the U.S. data with both a relative risk aversion and EIS of about 1.5.

JEL classification: D51, E44, G12

Keywords: equity premium puzzle, risk-free rate puzzle, equity volatility puzzle, irrational expectations, adaptive expectations, heterogeneous agents, elasticity of intertemporal substitution

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1 Introduction

A growing body of evidence suggests that the typical investor does not seem to have rational expectations about the premium. Survey and experimental studies show that the mean subjective forecast of the aggregate stock market return is positively correlated with recent returns (Andreassen and Kraus (1990), DeBondt (1993), Durell (2001), Fisher and Statman (2000), Qiu and Welch (2004), Vissing-Jørgensen (2003)), even though actual market returns exhibit no positive serial correlation (Fama and French (1988b)). In fact, Durell (2001) finds that average investor optimism about the stock market negatively predicts future returns. Further evidence that these extrapolative beliefs are mistaken comes from the return forecasts of more sophisticated market observers—such as professional economists, institutional investors, and investment newsletter editors—which are contrarian (DeBondt (1991), Shiller (2000), Clarke and Statman (1998)).

Even if extrapolative beliefs were present only among poor to moderately wealthy stockholders, they would play an important role in the response of aggregate consumption to stock returns because these households’ share of aggregate consumption is large. The bottom two stockholder consumption deciles accounted for 10.6% of total stockholder nondurable and services consumption in 1998. The next three deciles accounted for 23.3%.

1It is true that there is a small amount of positive serial correlation in index returns over a horizon of a few months (Lo and MacKinlay (1988)). However, investor expectation surveys typically ask about longer horizons, where return autocorrelations are negative.

2We use the classification of stockholders used by Vissing-Jørgensen (2002) to compute nondurable and services consumption from the Consumer Expenditure Survey, where observations are dropped for households reporting wages less than the statutory minimum wage. Nondurable and services consumption is the ND measure used in Krueger and Perri (2005).
However, the incidence of extrapolative beliefs does not diminish swiftly with wealth. Qiu and Welch (2004) report a 97% correlation between the returns expectations of the wealthy and poor. Graham and Harvey (2001) find that U.S. chief financial officers have extrapolative returns forecasts. Therefore, extrapolative beliefs are likely to have a significant price impact as well.

This paper argues that the presence of return extrapolators can help resolve the equity premium puzzle. We present a parsimonious general equilibrium model of an endowment economy where aggregate consumption growth has low covariance with market returns and relative risk aversion is low. Nonetheless, the average equity premium is high. Furthermore, stock returns are much more volatile than dividend growth and do not predict future dividend growth, which implies that stock returns are mean-reverting. The risk-free rate is low, stable, and does not predict aggregate consumption growth.

We estimate our model using the method of simulated moments. Therefore, we pick a set of asset pricing moments that our estimation targets. These moments are the expected excess return on equity, the volatility of equity, its Sharpe ratio, the risk-free rate as well as the autocorrelation of the price-dividend ratio. We find that we can match the target moments extremely well if both relative risk aversion coefficient and elasticity of intertemporal substitution (EIS) are around 1.5 and extrapolators constitute 75% of the population.

The equity premium is high when extrapolators are present. First, stocks

become riskier for rational agents due to extrapolators’ high demand following good news and low demand following bad news. Second, extrapolators themselves demand a high risk premium because they stocks have bad hedging properties for them. They think that low stock returns today predict low future returns and consumption and therefore demand a high risk premium.

The risk-free rate is low because sophisticates’ consumption growth is very volatile, generating a strong precautionary savings motive, while extrapolators are irrationally afraid that stocks are bad hedges. The risk-free rate is stable because the procyclicality of extrapolator demand for the risk-free asset is offset by the countercyclicality of sophisticate demand. Furthermore, both extrapolator and sophisticate demands are individually stable because their high EIS implies a weak consumption smoothing motive.

The model’s results rest crucially on the EIS being greater than unity. Our estimation procedure consequently points at an estimate of about 1.5. Although the empirical literature has not come to a consensus on the size of the EIS, a number of studies using micro data have found the EIS to be large. Gourinchas and Parker (2002) estimate an EIS of 2, Attanasio, Banks and Tanner (2002) estimate an EIS of 1.54, and Vissing-Jørgensen (2003) obtain estimates between 1.03 and 2.34. Thus asset pricing phenomena are consistent with the empirical literature if investors with extrapolative expectations are present.

It is commonly argued that even if individuals’ beliefs are irrational, their choices are rational because they are delegated to financial institutions. However, it is doubtful that financial institutions completely neutralize the effect of extrapolative beliefs because institutions’ role is typically restricted to al-
locating money within an asset class. They usually do not control the household’s consumption-savings decision and the fraction of household wealth allocated to each asset class.\textsuperscript{4} If extrapolators decide to invest more in equities, it is the money manager’s job to allocate that additional money optimally among stocks, even if she thinks that equities overall are overvalued. The model’s results hold even if every stock is correctly valued relative to every other stock.

Previous theoretical work has also shown that the presence of irrational noise traders can affect stock prices despite the presence of rational arbitrageurs (De Long, Shleifer, Summers and Waldmann (1990), Campbell and Kyle (1993), Shleifer and Vishny (1997), Barberis and Shleifer (2002)). Empirical studies that find important limits to the extent arbitrageurs can move prices towards fundamental values include Lee, Shleifer and Thaler (1991), Froot and Dabora (1999), Mitchell, Pulvino and Stafford (2002), Wurgler and Zhuravskaya (2002), and Lamont and Thaler (2003). To date, however, there has been little exploration of the effect noise traders have on the joint behavior of consumption and asset returns.\textsuperscript{5}

A recent strand of literature explores the asset pricing implications of an extrapolation bias in expectations. Hirshleifer and Yu (2013) study a full production economy governed by a representative agent with extrapolative

\textsuperscript{4}Financial institutions may not be given such a role because they cannot credibly communicate optimal savings rates and asset allocations when their compensation is based on total assets under management and varies by asset class.

\textsuperscript{5}An exception is Ingram (1990), who models an economy populated by sophisticates and noise traders who follow rule-of-thumb consumption and portfolio rules. She obtains an equity premium and volatility that are both an order of magnitude below the values observed in the data. A related line of research investigates the effect of near-rational expectations (see Hassan and Mertens (2011)). De Long, Shleifer, Summers and Waldmann (1990) suggest, as we do, that the riskiness of stocks may be both caused and hidden by noise trader consumption. However, they do not formally model this effect.
expectations. They find that, apart from improvements for pricing assets, the model also explains movements in macroeconomic variables such as high investment variability relative to consumption. Greenwood and Shleifer (2013) provide evidence for extrapolative expectations and demonstrate in a stylized model that these forecasts have implications for expected returns. In closely related work to ours, Barberis, Greenwood, Jin and Shleifer (2013) investigate the pricing implications when both rational traders and extrapolators are present. Compared to their work, we allow for non-tradable income, recursive preferences, and portfolio constraints which makes the model more realistic but loses some of its tractability. As a result of non-tradable income, our investors survive even in the long-run.

Section 2 presents the model. Section 3 describes the estimation of parameter values that produce the results in Section 4. Section 5 concludes. An appendix describes the computational procedures used to numerically solve and estimate the model.

2 Model description

Following Lucas (1978) and Mehra and Prescott (1985), we consider an infinite-horizon endowment economy where there is one share of equity that pays one unit of the aggregate consumption endowment as a stochastic perishable dividend $C_t$ to a risky asset and two units as labor income each period. There is also a risk-free asset present in zero net supply. The ex-dividend price of the risky asset at time $t$ is $P_t^S$, its return $R_{t,t+1}$, the price of a bond by $P_t^B$, and the risk-free rate is $R_{f,t}$.
2.1 Agents

There are two types of infinitely-lived agents, extrapolators, present in measure $Q$, and sophisticates, present in measure $1 - Q$ who disagree about the future path of the dividend process. Log dividend growth is independently and identically normally distributed with mean $\bar{g}$ and variance $\sigma_g^2$. Sophisticates know the true dividend growth process, but extrapolators believe next period’s log dividend growth, $\log(C_{t+1}/C_t)$, is normally distributed with mean $\hat{g}_t$ and variance $\sigma_g^2$, where they update $\hat{g}_t$ each period to get\(^6\)

$$\hat{g}_{t+1} = (1 - \phi) \hat{g}_t + \phi (\bar{c}_{t+1} - \bar{c}_t). \quad (2)$$

This inference is rational if the expected log dividend growth rate is a random walk (see Barsky and Long (1993)).

Extrapolators and sophisticates have recursive utility of the form derived by Epstein and Zin (1989) and Weil (1989). To indicate choices and value functions for each type, we denote extrapolators by an index $\times$ and sophisticates by $. Whenever an equation holds for either type, the index $\circ$ serves as a placeholder for sophisticates and extrapolators. The value function for both types is defined recursively via

$$U_{t,o} = \left[(1 - \beta) C_{t,o}^{(1 - \gamma)/\theta} + \beta E_{t,o} U_{t+1,o}^{1 - \gamma} \right]^{\theta/(1 - \gamma)}, \quad (3)$$

\(^6\)As a result, $\hat{g}_t$ is the geometrically weighted mean of present and past log dividend growth realizations:

$$\hat{g}_t = (1 - \phi) \sum_{\tau=0}^{\infty} \phi^\tau \log \left(\frac{C_{t-\tau}}{\bar{C}_{t-\tau}}\right), \quad 0 < \phi < 1. \quad (1)$$
where $C_{t,o}$ is the investor’s consumption at time $t$, $\gamma$ is the coefficient of relative risk aversion, $\psi$ is the EIS, $\theta \equiv (1 - \gamma) / (1 - 1/\psi)$, and $\beta < 1$ is the time discount factor.\footnote{When $\gamma = 1/\psi$, Epstein-Zin-Weil utility is equivalent to constant relative risk aversion utility.} Note that the expectations operator for the extrapolator is different from that for the sophisticate who holds rational expectations. Each agent maximizes utility, taking prices as given, subject to the budget constraint

$$C_{t,o} = Y_{t,o} + R_{r,t}P_{t-1}^{S}S_{t,o} + R_{f,t}P_{t-1}^{B}B_{t,o} - P_{t}^{S}S_{t+1,o} - P_{t}^{B}B_{t+1,o}$$

(4)

where $P_{t}^{S}$ denotes the price of a risky claim to one unit of consumption each period, $S_{t,o}$ asset holdings of the respective investor, $P_{t}^{B}$ the price of a bond, and $B_{t,o}$ the corresponding holdings. Labor income is given by $Y_{t,x} = 2QC_{t}$ for the representative extrapolator and by $Y_{t,*} = 2(1 - Q)\tilde{C}_{t}$ for the representative sophisticate. Agents face restrictions for short-selling assets. In particular, agents are restricted from short-selling bonds exceeding a limit of $\kappa_{B}$ and face an upper limit $\kappa$ for short-selling stocks given by the two constraints

$$S_{t,o} \geq -\kappa_{S}$$

(5)

and

$$B_{t,o} \geq -\kappa_{B}.$$  

(6)

Asset returns are defined by

$$R_{r,t} = \frac{P_{t}^{S} + \tilde{C}_{t}}{P_{t-1}^{S}} \quad R_{f,t} = \frac{\tilde{C}_{t-1}}{P_{t-1}^{B}}.$$  

(7)
The risk-free rate pays off a conditionally certain payoff $\bar{C}_{t-1}$.

### 2.2 Market clearing and equilibrium

In equilibrium, all agents optimize and assets market clear. To characterize the optimal decisions agents can take, we recognize that the recursively defined value function can be written in terms of the stock and bond holdings as arguments. We derive agents’ optimization by solving the budget constraint (4) for consumption, plugging it into the definition of utility and differentiate with respect to asset holdings. Optimality is satisfied when agents obey their first-order conditions for stocks

$$(1 - \beta)C_{t,o}^{1-\gamma}(-P_t^S) + \beta E_{t,o}[U_{t+1,o}^{1-\gamma}]^{1-1}E_{t,o}[U_{t+1,o}^{-\gamma}U_{S,t+1,o}] = 0 \quad (8)$$

and bonds

$$(1 - \beta)C_{t,o}^{1-\gamma}(-P_t^B) + \beta E_{t,o}[U_{t+1,o}^{1-\gamma}]^{1-1}E_{t,o}[U_{t+1,o}^{-\gamma}U_{B,t+1,o}] = 0 \quad (9)$$

whenever the constraints on asset holdings are not binding. The derivatives of the value functions appear in the agents’ first-order conditions. To compute their value, we include the envelope conditions with respect to stocks

$$U_{S,t,o} = U_{t,o}^{1-\gamma} \left( 1 - \beta \right) C_{t,o}^{1-\gamma} \left( P_t^S + \bar{C}_t \right) \quad (10)$$

and bonds

$$U_{B,t,o} = U_{t,o}^{1-\gamma} \left( 1 - \beta \right) C_{t,o}^{1-\gamma} \bar{C}_{t-1}. \quad (11)$$
As before and in what follows, these equations hold for sophisticates and extrapolators under their respective expectations. Equating the resulting aggregate demand with the supply of the assets allows us to solve for market clearing prices. Thus the market clearing condition for stocks is given by

\[ S_x + S_* = 1 \] (12)

and its counterpart for bonds by

\[ B_x + B_* = 0. \] (13)

All endogenous quantities, choices and prices, are functions of the underlying state space. The complete set of optimality conditions consists of the Bellman equations (3) for each agent, the first-order conditions (8) and (9), the market clearing conditions (12) and (13), the budget constraint (4), envelope conditions (10) and (11), as well as constraints for asset positions (5) and (6). Considering this system of equations, a sufficient set of state variables consists of the vector \((C_t, S_t, B_t, \hat{g}_t)\).

This system of equation will produce solutions that are instationary due to growth in dividends. To solve this problem, we exploit homogeneity of the value function to detrend all variables. We therefore define the value function \(V_{t,0} = U_{t,0}/\tilde{C}_t\). Dividing by the current dividend eliminates any trend from the value function. With this definition, we take the above system of equations, divide each equation by the current dividend \(\tilde{C}_t\) and slightly...
Each of these equations holds for extrapolators as well as sophisticates under their respective expectations.

3 Model estimation

The model produces time series on a quarterly frequency. To match fundamentals of the U.S. economy, we calibrate the parameters governing the process for consumption. Taking the stream of consumption as given, we estimate the remaining parameters from the data to see how well the model can match the pricing of U.S. securities.
3.1 Calibrated parameters

We take the process for fundamentals of the economy as given. These calibrated parameters closely reflect the U.S. experience for the stream of consumption. The moments for consumption should not be part of the estimation strategy since our exercise is to see what the pricing implications for a given stream of dividends might look like. We take the growth rate of consumption $\bar{g}$ to match the U.S. postwar aggregate per capita mean for non-durables and services which amounts to $4\bar{g} = 1.964\%$. The annual standard deviation of aggregate postwar per capita consumption growth is $1.07\%$ but empirical studies have found that stockholder consumption is significantly more volatile than nonstockholder consumption. This paper is concerned with stockholder behavior, since one does not expect the Euler equation for equities to hold for agents whose portfolios are at a corner solution. Therefore, we wish to choose consumption dividend volatility to match stockholder consumption volatility.\cite{Mimicking the more volatile stockholder consumption series makes the equity premium puzzle slightly easier to resolve because the puzzle is fundamentally about the smoothness of consumption growth.}

We use the midpoint of the Mankiw and Zeldes (1991) and Vissing-Jørgensen (2002) estimates for $2\sigma_g = 4.449\%$.

The crucial question concerns the relative size of the population of extrapolators versus sophisticates. Unfortunately, there is little guidance from the empirical literature on the exact proportion of extrapolators in the population. However, the evidence is strong that extrapolators outnumber contrarians. we therefore set $Q$, the proportion of extrapolators in the stock-holding population, to a value of 75\%.

\cite{Mimicking the more volatile stockholder consumption series makes the equity premium puzzle slightly easier to resolve because the puzzle is fundamentally about the smoothness of consumption growth.
3.2 Estimation

The remaining set of parameters are estimated when trying to match pricing moments from U.S. data. In particular, we aim for a good fit of moments in asset returns that have been hard to reconcile with standard asset pricing theorem. The most prominent among the moment is the extra return of equity earned on average over risk-free securities, the equity risk premium. Due to the movements in expectations by extrapolators, returns to equity are more risky and agents demand a high risk premium. Because the riskiness of equity plays an important role, we also include the standard deviation of returns. To emphasize the importance of the two moments, we additionally add the Sharpe ratio for equity, i.e. the excess expected return per unit of risk, as a target. The low level of the risk-free rate has been hard to reconcile with the return characteristics of equity and we let the estimation target this moment. Lastly, we include the autocorrelation of the price-dividend ratio. We list these target moments when we discuss the results in table 1.

To be able to match the data, our estimation strategy is allowed to choose the preference parameters, i.e. risk aversion \( \gamma \), the elasticity of intertemporal substitution \( \psi \), and time preference \( \delta \), the persistence of extrapolators’ beliefs \( \phi \), as well as the tightness of portfolio constraints for stocks \( \kappa_S \) and bonds \( \kappa_B \). Note that the standard model has problems in matching the data well. We allow for a single additional channel which is governed by the persistence of extrapolators’ beliefs.

For the estimation, we solve the model numerically. We choose perturbation methods over alternative methods since it is particularly good in handling the dimensionality of the state space. All endogenous variables are a function
of four state variables represented by the vector \( \{ \frac{C_{t+1}}{C_t}, S_{t,x}, B_{t,x}, \hat{g}_t \} \). The portfolio constraints are implemented via penalty functions on asset holdings. The numerical algorithm starts at the deterministic steady-state of the system in which, by definition, extrapolators and sophisticates are identical. In this special case, we can solve for choices in closed form. Then we build a higher-order approximation to all endogenous variables via a Taylor series as is standard in the literature. Given the solution of the system, we compute unconditional moments for variables of interest exactly. Appendix A lays out the details.

We use the Method of Simulated Moments to find a parameter combination that matches asset pricing moments moments most closely. The estimation is implemented via a Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. This numerical optimization algorithm minimizes the distance between our generated model moments and the moments in the data. The BFGS algorithm improves on a classical Newton algorithm by not having to invert the Hessian matrix at every step of the optimization. It allows us to update it from step to step and yet avoid costly inversions. We augment the BFGS algorithm along two dimensions. First, we implement a so-called Armijo rule to guarantee robust and fast convergence and second, we implement parameter restrictions to avoid infeasible parameter combinations such as a time preference factor greater than one. Appendix B contains details on the estimation procedure.
4 Results

4.1 Estimation results

Table 1 shows the target moments along with the model’s estimation results. The moments we include in the estimation are calculated for the postwar period. the expected logarithm of the return is 8.19% with a standard deviation of 15.74%. Together with an estimate of the risk-free rate of return of 0.9%, this yields a Sharpe ratio of the logarithm of equity of 0.457. The price-dividend ratio is persistent with an autocorrelation 0.784.

The model does extremely well in matching those moments as can be seen in table 1. The equity risk premium is slightly too high while its standard deviation slightly too low. Hence the Sharpe ratio is too high with a value of 0.486 relative to the estimate in the data of 0.457. Nevertheless, the results almost perfectly match the data. The risk-free rate is matched due to the flexibility in matching the risk-free rate and risk aversion separately. Although the model is rather parsimonious with a single source of uncertainty, it does a surprisingly good job at fitting the data. We discuss the mechanics of the model in detail below.

Our estimation produces the best fit for an elasticity of intertemporal sub-

<table>
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<th>Expected log return excess return</th>
<th>Log equity return standard deviation</th>
<th>Log equity Sharpe ratio</th>
<th>Mean risk-free rate</th>
<th>Autocorrelation P-D ratio</th>
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<td></td>
<td>7.19%</td>
<td>15.74%</td>
<td>0.457</td>
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<tr>
<td></td>
<td>7.63%</td>
<td>15.71%</td>
<td>0.486</td>
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<td>0.76</td>
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Table 1: List of target moments for the U.S (annualized) as well as the estimated moments from the model.
stitution of 1.50. With a simultaneous estimate for the coefficient of risk aversion of 1.63, this parameterization shows a clear departure from a utility function with constant relative risk aversion. Recursive utility with a high elasticity of intertemporal substitution is important in producing a low risk-free rate. Furthermore, the estimate is in line with the empirical literature using micro data. Often, these estimates range between 1 and 2.5. Most closely in line with our estimate is Attanasio, Banks and Tanner (2002) who obtains an estimate of 1.54 using consumption data. Note that our results do not result from the consumption process. Instead, the estimate shows that the pricing of securities points at a similar EIS. The low estimate of risk aversion of 1.5 shows that there is substantial risk due to the presence of agents with heterogeneous expectations. This pricing risk is sufficiently large that a small risk aversion coefficient, together with trading restrictions, suffices to match the moments in the data.

An important estimate for our procedure is the persistence of extrapolative expectations \( \phi \) which is estimated to be 0.80. The persistence of beliefs has an impact on both the riskiness of equity as well as the persistence of the price-dividend ratio. Lastly, we estimate the implied constraint on asset allocations. The short-selling constraints on securities is at \( \kappa_S = 0.17 \) which seems reasonable. It means agents could take short positions up to the size of 22% of their initial stock holdings. The borrowing constraint on risk-free securities is at \( \kappa_B = 2 \) which is a value of 5.2% of their financial wealth. Given that this constraint imposes restrictions on buying stocks on margin, a relatively tight constraint does not seem unrealistic. All parameter estimates are listed in table 2.
Elasticity of intertemporal substitution ($\psi$) | 1.63  
Risk aversion ($\gamma$) | 1.50  
Time preference ($\delta$) | 0.991  
Belief persistence ($\phi$) | 0.80  
Short-sales constraint equity ($\kappa_S$) | 0.17  
Short-sales constraint bonds ($\kappa_B$) | 2.0  

Table 2: List of estimated parameters which produce the best fit to match asset pricing moments.

4.2 Investor actions

Extrapolators generate a high equity premium through three channels. First, they make stocks highly risky, even though stocks are claims to a fundamentally low-risk technology. Extrapolators are more willing to buy stocks when there has been recent good news, and they are less willing to buy stocks when there has been recent bad news. In equilibrium, this generates procyclical price-dividend ratios, which lead to countercyclical expected stock returns because dividend growth is independently and identically distributed over time. Sophisticated investors with rational expectations respond by making their consumption more procyclical, since an EIS greater than one causes lower expected returns to increase consumption today. Sophisticates’ consumption growth therefore has a high covariance with equity returns, leading sophisticates to demand a high average equity return. This is consistent with the Aït-Sahalia, Parker and Yogo (2004) finding that the consumption of the very rich, who are likely to be disproportionately (although not universally) rational, can be better reconciled with the equity premium.

Second, extrapolators believe stocks to be a poor hedge, since they think that low stock returns today predict low future stock returns. Therefore, they too demand a high equity premium on average. In our model, extrapolators’
unconditional expectation of the equity premium is correct. It is only their conditional expectations that are incorrect.

Third, extrapolators obscure the risk they create in stocks. Extrapolators’ returns expectations move in the opposite direction of sophisticates’ contrarian expectations. Therefore, when sophisticates decrease their savings rate in response to a positive innovation in stock prices, extrapolators increase their savings rate because they think investment prospects have improved. Choi, Laibson, Madrian and Metrick (2009) find empirically that 401(k) investors cut their consumption growth in response to positive capital gains. Aggregate consumption statistics add together sophisticate and extrapolator consumption, creating an aggregate series whose growth has low covariance with stock returns.

Recall that we set $Q$, the proportion of extrapolators, in the economy, to 0.75. Because extrapolators hold incorrect conditional beliefs, extrapolators are on average poorer than sophisticates. However, due to asset market incompleteness, agents are unable to pledge their future labor income. As a result, extrapolators survive in the economy and make up a substantial fraction of investors’ wealth. Their presence leads equity to be risky to investors in the long-run and produces realistic asset pricing outcomes.

Figure 1 shows the price-dividend ratio for various expected growth rates where all other variables are at the deterministic steady-state. Extrapolators’ procyclical optimism about stocks generates a procyclical price-dividend ratio. Because the price-dividend ratio is stationary and the dividend growth process is independently and identically distributed, procyclical price-dividend ratios must lead to countercyclical expected stock returns. Consistent with
Figure 1: Price-dividend ratio as a function of extrapolators’ beliefs.

The data, the price-dividend ratio is persistent with an autocorrelation of 0.76. The main driver of this serial correlation is the persistence of beliefs that extrapolators hold. If, following high consumption growth, extrapolators continue to believe that consumption growth will remain high, prices will remain high along with the price-dividend ratio. At the same time, expected returns will be low under rational beliefs because extrapolators think the opposite. Consequently, sophisticate expected consumption growth is decreasing in $\hat{g}$ but extrapolator expected consumption growth is increasing. Extrapolators believe that the relationship between their consumption growth and $\hat{g}$ is steeper than it actually is. This is because when $\hat{g}$ is above its mean, extrapolators are not as rich next period as they had expected, whereas the reverse is true when $\hat{g}$ is below its mean.

Because investors have an EIS greater than unity, higher expected returns lead to a lower consumption-wealth ratio. Hence, sophisticates have procyclical consumption-wealth ratios and extrapolators have countercyclical ratios. Extrapolators’ consumption-wealth ratio is higher than those of sophisticates.
because extrapolators believe stocks have poor hedging properties. Hence they perceive the investment opportunity set to be less attractive than sophisticates do. Deteriorations in the investment opportunity set will decrease the savings rate when the elasticity of intertemporal substitution is greater than one. This result suggests a new mechanism that might explain some of the negative correlation between financial sophistication and savings rates, a phenomenon frequently attributed to differential self-control or foresight.

Although extrapolators are consistently mistaken about their conditional mean consumption growth, they have very accurate beliefs about their consumption growth volatility. Sophisticate consumption growth volatility is strongly procyclical, whereas extrapolator volatility is mildly countercyclical. Furthermore, the annualized standard deviation of log sophisticate consumption growth is substantially higher than that of extrapolators: 11.8% versus 1.8% in the median simulation run. The sophisticate consumption growth volatility is not far from the 15.8% average standard deviation Aït-Sahalia, Parker and Yogo (2004) estimate for the extremely wealthy, who are likely to be disproportionately sophisticated.

The accuracy about consumption growth volatility also extends to accurate predictions about asset market volatility on the part of extrapolators. In equilibrium, it turns out that extrapolators’ procyclical volatility expectations are very accurate. The procycliality of stock volatility is consistent with the findings of Campbell (1987) and Glosten, Jagannathan and Runkle (1993) that stock volatility is negatively related to expected returns.\footnote{On the other hand, Bollerslev, Engle and Wooldridge (1988) and French, Schwert and Stambaugh (1987) find a weak positive correlation between stock volatility and returns. Also, the model is not consistent with the negative correlation between stock return innovations and volatility innovations in the data.}
5 Conclusion

This paper argues that the equity premium and volatility puzzles can be explained with the presence of extrapolative investors who believe that high past stock market returns predict high future returns. When the elasticity of intertemporal substitution is greater than unity, this belief causes extrapolator savings rates to vary procyclically, pushing up price-dividend ratios during consumption booms and pushing down price-dividend ratios during consumption busts. The resulting predictability of stock returns causes sophisticated investors with rational expectations to make their savings rates countercyclical, increasing the covariance between their consumption growth and stock returns. Sophisticates therefore demand a high unconditional equity premium because of this high covariance. Extrapolators do not have high consumption growth covariance with stock returns, but they demand a high unconditional equity premium because they believe stocks are a poor hedge due to their perceived positive return serial correlation. Aggregate consumption statistics include both extrapolator and sophisticate consumption, hiding from the econometrician the stock market’s consumption risk to sophisticates.

We numerically solve a general equilibrium model of an endowment economy that is populated by sophisticated investors and extrapolators. Our estimation of the parameters suggests that we can generate a low, stable risk-free rate and a high equity Sharpe ratio consistent with the postwar U.S. data with both a relative risk aversion coefficient and an elasticity of intertemporal substitution (EIS) of about 1.5 when extrapolators constitute 75% of the population.
Numerous empirical studies have documented that extrapolative beliefs about stock returns are commonly held in the population, even among the rich. Nonetheless, it seems plausible that the rich are more likely to hold rational beliefs than the poor. Consistent with this notion, empirical studies have shown that the covariance of consumption growth with stock returns is higher for the rich. The model in this paper predicts that identifying the subset of investors who understand that stock returns are mean-reverting and measuring their consumption growth will yield an even higher covariance. Furthermore, the unconditional average portfolio share in cash and Treasury bills will be increasing with the strength of a household’s extrapolative tendencies, since knowledge of mean reversion in stock returns generates a positive hedging demand for equities.
A Computational algorithm

We solve the economy using perturbation methods. This computational technique approximates policy and price functions by its Taylor series around the deterministic steady-state. We follow Mertens and Judd (2013) for the numerical implementation of perturbation methods with heterogeneous agents and incomplete markets. Note that agents are identical in the deterministic version of the economy and hence their solution method applies directly.

To deal with portfolio constraints, we add a penalty function $\pi(S, B)$ defined over stocks and bonds. The optimality conditions are given by

$$V_t = \left\{ (1 - \delta) \left( \frac{C_t}{C_t^{1/\phi}} \right)^{1 - 1/\phi} - \pi(S, B) + \delta \left[ \hat{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{1 - \gamma} V_{t+1}^{1 - \gamma} \right] \right\}^{1/(1 - \gamma)} \tag{22}$$

$$\begin{align*}
(1 - \delta) \left( \frac{C_t}{C_t^{1/\phi}} \right)^{-\frac{1}{\phi}} \left( -\frac{P_t^S}{C_t} \right) &= \frac{1}{1 - \frac{1}{\phi}} \pi_S(S, B) + \delta \hat{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1 - \gamma} V_{t+1}^{1 - \gamma} \right]^{1 - \frac{1}{\phi}} \hat{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1 - \gamma} V_{t+1}^{1 - \gamma} \right] \tag{23}
\end{align*}$$

$$\begin{align*}
(1 - \delta) \left( \frac{C_t}{C_t^{1/\phi}} \right)^{-\frac{1}{\phi}} \left( -\frac{P_t^B}{C_t} \right) &= \frac{1}{1 - \frac{1}{\phi}} \pi_B(S, B) + \delta \hat{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1 - \gamma} V_{t+1}^{1 - \gamma} \right]^{1 - \frac{1}{\phi}} \hat{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1 - \gamma} V_{t+1}^{1 - \gamma} \right] \tag{24}
\end{align*}$$

$$\begin{align*}
V_{t, S} &= V_t^{\frac{1}{\phi}} \left[ (1 - \delta) \left( \frac{C_t}{C_t^{1/\phi}} \right)^{-\frac{1}{\phi}} \left( 1 + \frac{P_t^S}{C_t} \right) \right] \tag{25}
V_{t, B} &= V_t^{\frac{1}{\phi}} \left[ (1 - \delta) \left( \frac{C_t}{C_t^{1/\phi}} \right)^{-\frac{1}{\phi}} \left( \frac{C_{t+1}}{C_t} \right) \right] \tag{26}
\end{align*}$$

We take

$$\pi(S, B) = \pi^S(S) + \pi^B(B). \tag{27}$$

to be additively separable. We choose a particular form for $\pi^x$ where $x = S, B$ is either a stand-in for stocks or bonds, $\kappa_x$ their constraint, and $\bar{x}$ their value in the deterministic steady-state

$$\pi^x(x) = \frac{1}{10000 (x - \kappa_x)^2} + \frac{1}{5000 (\bar{x} - \kappa_x)^3} (x - \bar{x}) - \frac{1}{10000 (\bar{x} - \kappa_x)^2} \tag{28}$$
Figure 2: Penalty function for bond holdings using the estimated parameter combination.

Figure 2 plots the penalty function for bonds with the estimated parameter combination.

We compute unconditional moments for variables of interest. Therefore, we write the equations of motion for state variables in their linear form

$$x_{t+1} = \bar{x} + \rho_x x_t + \varepsilon_t$$  \hspace{1cm} (29)

for all state variables $x$ and compute unconditional expectations, covariances, and variances via the closed-form expression.

To compute moments, we list all state variables that influence endogenous variables in a vector. Returns to equity in period $t+1$, for example, depend on state variables both in period $t$ as well as $t+1$ because prices in those periods enter the return definition. We limit the moment computation to linear terms in state variables which includes higher-order terms mixed with the perturbation parameter for the standard deviation.
The law of motion for state variables follows

\[ s_{t+1} - \bar{s} = \bar{l} - \bar{s} + L(s_t - \bar{s}) + \sigma \epsilon_{t+1} \] (30)

where \( s_t = \left( \frac{C_t}{C_{t-1}}, S_{t-1}, B_{t-1}, \frac{\hat{C}_{t-1}}{C_{t-1}}, S_t, B_t, \hat{g}_t \right)' \), \( \epsilon \) a vector including the shock \( \epsilon \) for all stochastic variables, and \( \sigma \) the perturbation parameter with which we perturb the deterministic version of the economy.

Given this structure, we can directly compute first and second moments for the state variables. For the first moment, we take unconditional expectations on both sides of the equation to get

\[ E[s - \bar{s}] = \bar{l} - \bar{s} + L E[s - \bar{s}] \] (31)

which results in an easily solvable linear system of equations for the first moments. For the second moments, we compute unconditional expectations on both sides which results in the system of equations

\[ \text{Var}[s] = L \text{Var}[s] L' + \sigma^2 \Sigma \Sigma' \] (32)

where \( \Sigma \) denotes the variance covariance matrix of shocks \( \epsilon \).

Once we know the moments of the state variables, we can transform them into moments of the variables of interest which are functions of state variables via \( v = \bar{v} + V(s - \bar{s}) \). The first moments of \( v \) are then given by \( E[v] = \bar{v} + V(E[s] - \bar{s}) \), second moments by \( V \text{Var}[s] V' \).
B Estimation

To implement the method of simulated moments, we take the moments \( m \) from the numerical solution of the model (see appendix A). The moments depend on the parameterization \( \theta \) of the model and can thus be written as \( m(\theta) \). We measure the same moments in the data and arrive at the estimate \( \hat{m} \). The goal of the estimation procedure is now to find the vector of parameters \( \theta \) such that the model’s moments match the data most closely. The best match can be found using a numerical optimization technique. We minimize the distance between the moments from the data and the model

\[
G(\theta) = (m(\theta) - \hat{m}) W (m(\theta) - \hat{m})
\]  

(33)

where \( W \) is a weighting matrix. We set this weighting matrix to a diagonal matrix where the diagonal elements \( w_{i,i} \) are given by the inverse of the squared moment in the data \( 1/\hat{m}_{i,i}^2 \).

To implement numerical optimization of the objective function (33), we choose a Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. BFGS works similarly to Newton’s method in that it walks down the objective function towards the minimum. Compared to Newton’s method, BFGS does not require to compute the Hessian matrix at every step which would computationally be very costly. Instead, it updates the Hessian matrix using the gradient at each step. For a given current parameter combination \( \theta_t \), BFGS suggests a search direction for lower objective function values. We chose BFGS because of its known advantages both in terms of speed as well as in handling non-smooth optimization problems.
An Armijo rule ensures that the BFGS search direction is implemented efficiently. When BFGS suggests a direction for the next iterate, we search in this direction starting with the step length suggested by BFGS. If the next iterate would not lead to a sufficient decrease in the objective function as prescribed by the Armijo rule, we cut the step length in half and check again. We reduce the step length until the Armijo rule is satisfied. This rule imposes the following condition given a search direction $s_k$ in iteration step $k$ and a step length $l_k$

$$G(\theta_k + l_k s_k) \leq G(\theta_k) + l_k s_k' \nabla G(\theta_k).$$

(34)

Furthermore, we make sure constraints that certain constraints are satisfied when choosing a new iterate $\theta_{k+1}$. In particular, we impose that the time preference factor is between zero and an upper bound below an upper bound of 1. The persistence of extrapolators’ beliefs $\phi$ is required to stay in the unit interval $0 \leq \phi \leq 1$. And lastly, we make sure that both constraints have positive coefficients. The optimization procedure then chooses the parameter combination that matches the moments in the data most closely.
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