Technical Appendix for “Incentive Problems in Performance-based Online Advertising Pricing: Cost per Click versus Cost per Action”

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Technical Appendix: Optimal Reserve Price

In our main paper, we assumed that the reserve price for the publisher is exogenously given as zero, which implies that the publisher always accepts a non-negative bid. In this appendix, we explicitly solve the optimal reserve price under both the CPC and CPA cases and provide a micro-foundation for this assumption by identifying the conditions under which this assumption holds.

In practice, a revenue maximizing publisher can set a small reserve price that can potentially exclude some bidders from auction and thus increase her expected revenue, which is sometimes referred to as the exclusion principle. Suppose that the publisher now imposes a small reserve price of $x \geq 0$. It is well-known that in a second-price auction, a reserve price makes no difference to the behavior of the bidder; it is still a weakly dominant strategy to bid one’s value (Krishna 2010, p.21). We now consider two cases – under the CPC and CPA.

1. CPC pricing model

Since it is optimal for the advertiser to bid one’s own value as before, under the CPC, the advertiser’s bid is $b(m_i, \rho_i) = \frac{1}{2} (1 - r\sigma^2) m_i^2$. Hence, advertiser’s bids are uniformly distributed on $[0, \frac{1}{2} (1 - r\sigma^2)]$ because $m_i \sim U[0, 1]$. We denote $F$ as the distribution of the advertiser’s bid, $F(z) = \frac{z}{Z(1 + r\sigma^2)}$, and $F'(z) = f(z) = \frac{1}{Z(1 + r\sigma^2)}$. When there exists the publisher’s reserve price $x \geq 0$, the expected payment of an advertiser with bid $x$ is just $xF(x)$ (note that any bid below $x$ results in losing the auction for sure), and the expected payment of an advertiser with bid $b(m_i, \rho_i) \geq x$ is now

$$t_c = xF(x) + \int_x^{\frac{1}{2} (1 - r\sigma^2)} zf(z)dz,$$

(1)
because the winning advertiser should pay the reserve price \( x \geq 0 \) whenever the other advertiser’s bid is below \( x \).

We now assess the effect of reserve price on the publisher’s revenue. The ex ante expected payment of an advertiser is now

\[
E[t_c] = x (1 - F(x)) F(x) + \int_x^{\frac{1}{2}(1-r\sigma^2)} z (1 - F(z)) f(z) dz.
\]  

(2)

Then, the overall expected payoff of the publisher from setting a reserve price \( x \) is

\[
E[y_p] = \left( \frac{2}{1} \right) \times E[t_c] - \{F(x)\}^2 c_0
\]

(3)

where \( c_0 \) is the opportunity cost of the publisher if the slot is left unsold. The publisher has incurred the fixed cost to set it up for advertising on its website and this can be the source of the opportunity cost.

Taking the first order condition with respect to \( x \), we obtain

\[
\frac{dE[y_p]}{dx} = 2 \left[ 1 - F(x) - xf(x) \right] F(x) - 2F(x)f(x)c_0 = 0.
\]

(4)

Now, we define the hazard rate function associated with the distribution \( F \) as \( h(x) = \frac{f(x)}{1-F(x)} \). Then, we can re-write the equation (4) as following:

\[
\frac{dE[y_p]}{dx} = 2 \left[ 1 - (x + c_0) h(x) \right] (1 - F(x)) F(x) = 0.
\]

(5)

The first order condition implies that the optimal reserve price \( x^* \) must satisfy

\[
(x^* + c_0) h(x^*) = 1 \\
\Leftrightarrow x^* = \frac{1}{h(x^*)} - c_0.
\]

(6)

Since \( F(\cdot) \) is uniform distribution on \([0, \frac{1}{2}(1-r\sigma^2)]\), \( h(x) \) is increasing and thus, this condition is also sufficient. Using the fact that \( \frac{1}{h(x)} = \frac{1-F(x)}{f(x)} = \frac{1+r\sigma^2-2x}{2} \), the equation (6) is

\[
x^* = \begin{cases} 
\frac{1+r\sigma^2-2c_0}{4} & \text{if } c_0 < \frac{1+r\sigma^2}{2}, \\
0 & \text{if } c_0 \geq \frac{1+r\sigma^2}{2}.
\end{cases}
\]

(7)

Hence, when \( c_0 \geq \frac{1+r\sigma^2}{2} \), the publisher’s optimal reserve price is \( x^* = 0 \). ■
2. CPA pricing model

Similar to the CPC case, the advertiser bids one’s own value, and thus the advertiser’s bid is 
\( b(m_i, \rho_i) = \frac{m_i}{\rho_i} \). Hence, advertiser’s bids are uniformly distributed on \([0, \frac{1}{\rho_i}]\). Then, the distribution of 
the advertiser \( i \)'s bid is now \( F_i(z) = z \rho_i \), and \( F_i'(z) = f_i(z) = \rho_i \), where \( i \in \{B, D\} \), \( \rho_D = \alpha \), \( \rho_B = \beta \), 
and \( \alpha \geq \beta \). When there exists the publisher’s reserve price \( x \geq 0 \), the expected payment of an 
advertiser \( i \) with bid \( b(m_i, \rho_i) \geq x \) is now

\[
    t_a = x F_{-i}(x) + \int_x^{\frac{1}{\rho_i}} z f_{-i}(z) dz, 
\]

where \(-i\) denotes the other advertiser. This is so because the advertiser \( i \) should pay the reserve price
\( x \geq 0 \) whenever the other advertiser’s (which is denoted by \(-i\) ) bid is below \( x \).

The ex ante expected payment of an advertiser \( i \) is now

\[
    E[t_a] = x (1 - F_i(x)) F_{-i}(x) + \int_x^{\frac{1}{\rho_i}} z (1 - F_i(z)) f_{-i}(z) dz. \tag{9}
\]

Then, the overall expected payoff of the publisher from setting a reserve price \( x \) is

\[
    E[y_p^{CPA}] = x (1 - F_D(x)) F_B(x) + \int_x^{\frac{1}{\rho_B}} z (1 - F_D(z)) f_D(z) dz 
                  + x (1 - F_B(x)) F_D(x) + \int_x^{\frac{1}{\rho_D}} z (1 - F_B(z)) f_B(z) dz 
                  - F_D(x) F_B(x) c_0 \tag{10}
\]

where \( c_0 \) is the opportunity cost of the publisher if the slot is left unsold.

Taking the first order condition with respect to \( x \), we obtain

\[
    \frac{dE[y_p^{CPA}]}{dx} = [1 - F_D(x) - xf_D(x)] F_B(x) + [1 - F_B(x) - xf_B(x)] F_D(x) 
                          - [F_D(x)f_B(x) + F_B(x)f_D(x)] c_0 
                          = [1 - 2x\alpha] x\beta + [1 - 2x\beta] x\alpha - 2x\alpha\beta c_0 = x (\alpha + \beta - 2\alpha\beta c_0 - 2x\alpha\beta) = 0 
\]

The optimal reserve price \( x^* \) is

\[
    x^* = \begin{cases} 
        \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) - c_0 & \text{if } c_0 < \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \\
        0 & \text{if } c_0 \geq \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) 
    \end{cases} \tag{11}
\]

Hence, if \( c_0 \geq \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \), then the publisher’s optimal reserve price is \( x^* = 0 \). \[ \blacksquare \]

In sum, if the opportunity cost for the unsold advertising slot is sufficiently large such that \( c_0 \leq \max \left\{ \frac{1 + \alpha^2}{2}, \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \right\} \), then the optimal reserve price for the publisher is zero under both the CPC
and CPA cases. This implies that the publisher always accepts a non-negative bid.

References