A THEORETICAL AND EMPIRICAL INVESTIGATION OF THE DUAL PURPOSE FUNDS

An application of contingent-claims analysis

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Using the option pricing methods developed by Black and Scholes as a general technique for contingent claims analysis, this paper examines a class of mutual funds known as Dual Purpose Funds. By constructing a simplified model for these funds under the 'perfect hedge' conditions of Black and Scholes, it is demonstrated that the asset value of the fund will always exceed the market value and that it is not inconsistent with market equilibrium or efficiency for the capital shares to sell at a discount. The simple model predicts price fluctuations in the seven dual funds studied quite well; however, there is a persistent downward bias in the predicted price level. Finally, refinements to the model are examined to determine the nature of the misspecification causing this bias.

1. Introduction

Dual purpose funds are a special type of closed end investment company. Their purpose is to provide for investors with the diverse objectives of long-term capital gains and present income. This is accomplished through the formation of two classes of shares with claim on the same underlying assets. The two classes are Capital Shares which pay no dividends and are redeemable at net asset value at the maturity of the fund and Income Shares which have the rights to any dividends or income that the fund may earn, subject to a stated minimum cumulative dividend and are to be redeemed at a set price at the maturity of the fund.

Seven funds were formed in 1967, American Dual Vest, Gemini, Income and Capital, Leverage Fund, Hemisphere, Putnam Dou-Fund, and Scudder Duo-Vest. In addition to these seven there are two dual funds which are also tax-free exchange funds. These will not be considered here.

The capital shares of dual funds are entitled to no dividends until maturity and then receive the entire fund less repayment of the income shares. In this

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respect they are analogous to options. In this paper, the Black–Scholes (1972, 1973) option pricing equation is applied to the capital shares in an attempt to explain their values and discounts.

Sections 2 and 3 develop the pricing equations for the capital and income shares through option pricing techniques. Section 4 examines the characteristics of the two claims implied by the models. The later sections apply the developed model to the seven dual funds to empirically evaluate the model.

2. The capital shares

To derive the dual fund pricing equations the following assumptions are required:

(A-1) The capital markets are perfect in that: There are no differential taxes or transactions costs. Assets are perfectly divisible. Investors act as price-takers. There is unlimited borrowing or lending at the risk-free rate of interest. There are no restrictions on short sales.

(A-2) Trading in assets takes place continuously in time.

(A-3) The asset value of the dual fund, \( V \), satisfies the stochastic differential equation

\[
\frac{dV}{V} = (\alpha - C)dt + \sigma \, dz,
\]

where \( C \) is the total cash outflow paid out per unit time, the payout policy is not stochastic. Future values of \( C \) need not be certain; however, they must be known conditionally on knowing \( V \). \( \alpha \) and \( \sigma^2 \) are the instantaneous expected rate of return and variance of return on the underlying assets. The variance rate is non-stochastic and at most a known function of time. \( dz \) is a Gauss–Wiener process.

(A-4) The term structure is flat and non-stochastic.\(^1\) \( r \) is the instantaneous rate of interest.

Under these assumptions Merton (1974) has shown that any contingent claim, \( F(V, t) \), whose value can be written as a function of asset value and time must satisfy the partial differential equation

\[
\frac{1}{2} \sigma^2 V^2 F_{vv} + (rV - C)F_v - rF + F_t + c = 0,
\]  

(1)

where \( c \) is that portion of \( C \) payable to the contingent claim, and subscripts denote partial derivatives.

\(^1\)This assumption is later modified to allow for stochastic interest rates. However, this had minimal effect in the pricing of the dual funds, and the strict assumption is kept here for expositional purposes.
If the claim has a contractual expiration or maturity date $\tau$ periods in the future, then it may be written as $F(V, \tau)$, and (1) becomes

$$\frac{1}{2}\sigma^2 V^2 F_{vv} + (rV - C)F_v - rF - F_\tau + c = 0. \quad (2)$$

Eq. (2), two boundary conditions, and one initial condition are sufficient in principle to solve for $F$. In general the boundary conditions are $F(0, \tau) = 0$ and $F(V, \tau) \leq V$ which are derived from the limited liability of the claim and of the remaining claim(s) represented by $V - F$. The initial condition is contained in the indenture agreement. The initial condition and the functional form of the cash outflows, $C$ and $c$, completely determine the value of any contingent boundary claim with the above simple boundary conditions.

Solving (2) analytically is difficult with no further assumptions. The following simplifying assumptions are made here:

(A-5) $\sigma^2$ is a constant.

(A-6) Dividends are paid continuously to the income shares and are proportional to asset value, $D = \delta V$.\(^2\)

(A-7) Management fees are paid continuously and are proportional to asset value, $M = \mu V$.\(^3\)

For $f(\cdot)$, the value of the capital shares of the dual fund, $C = (\delta + \mu)V = \gamma V$ and $c = 0$. Then,

$$\frac{1}{2}\sigma^2 V^2 f_{vv} + (r - \gamma) V f_v - rf - f_\tau = 0, \quad (3)$$

\(^2\)Some of these funds have requirements that the fund’s assets be invested only in securities that currently earn at least some minimum fixed proportion in dividends or interest. If the managers, the large majority of which are elected by the capital share-owners, act to maximize the capital share market value subject to this constraint, they will insure that it is just met. That managers act in the best interests of a third party is a usual assumption in finance theory. Since many of these dual funds have penalties against the management fees if the minimum cumulative dividend is not met, there is some incentive for the managers to act in the income share-holders’ behalf and shift the portfolio into income producing securities when the asset value is low. On the other hand, once dividends are paid out, the management can no longer collect a percentage of that amount as a part of their fees. Hence for high asset values they have an incentive to shift the portfolio in favor of capital gains producing securities. (The minimum current yield restriction can be circumvented, for example, by purchasing convertible securities.) Together these two effects would tend to increase the intercept but decrease the ‘slope’ of the dividends as a function of asset value. Furthermore, to the extent that dividend yield on stocks is correlated with variance we will also tend to note that the variance is no longer a constant as assumed in (A-5) but a function of asset value as well. The empirical content of this assumption is examined in section 5. See also footnote 4.

\(^3\)This generally one-half percent per year. Some of the funds have provisions by which this fraction can be reduced; however, they have rarely if ever become effective. See also footnote 4.
subject to
\[ f(0, \tau) = 0, \quad f(V, \tau) \leq V, \quad f(V, 0) = \max(0, V - E), \quad (4) \]
where \( E \) is the redemption price of the income shares. To solve (3), let \( X = Ve^{-\tau^*} \) be the value of the assets net of all cash payments\(^4\) and \( g(X, \tau) = f(V, \tau) \).

Then \( f_0 = g_x e^{-\tau^*}, f_{\tau} = g_x e^{-\tau^*}, f_{\gamma} = g_x - \gamma Xg_x \).

Substituting these values into (3) we obtain
\[ \frac{1}{2}\sigma^2 X^2 g_{xx} + rXg_x - rg - g_\tau = 0. \quad (5) \]

This equation is isomorphic to that of an European call option on a stock paying no dividends. Thus, the solution is
\[ f(V, \tau; E, r, \sigma^2, \gamma) = W(e^{-\tau^*}V, \tau; E, r, \sigma^2), \quad (6) \]
where \( W(\cdot) \) is the option solution in Merton (1973). As demonstrated therein, the option function is homogeneous of degree one in stock price and the present value of the exercise price, hence the capital share price may be written
\[ f(V, \tau) = e^{-\tau^*}(ZN(x_1) - N(x_2)), \]
where
\[ Z \equiv Ve^{(r-\gamma^*)\tau}/E, \]
\[ x_1 \equiv (\log Z + \sigma^2 \tau/2)/\sigma \sqrt{\tau}, \quad (7) \]
\[ x_2 \equiv x_1 - \sigma \sqrt{\tau}, \]
\[ N(x) \equiv 1/\sqrt{(2\pi)} \int_{-\infty}^{x} e^{-s^2/2} ds, \quad \text{the cumulative normal.} \]

The behavior of the capital share price is shown in figs. 1 and 2. Two things should be noted. The capital share is always more valuable than \(\max(0, e^{-\tau^*}V - e^{-\tau^*}E)\). This is similar to the dominance constraint on European options. Nevertheless, the capital shares can sell for less than their net asset value, defined by \(V-E\) and indicated by the dashed line in fig. 1, if the asset value is sufficiently large. We note that for large asset values the capital shares behave asymptotically like \(e^{-\tau^*}V - e^{-\tau^*}E\); hence, the capital shares can sell at a discount.

\(^4\)If there is to be one lump sum disbursement in the future of a fraction \(s\) of the asset value (at that time), it is obvious that the current value of this payment is \(sV\), the same fraction of the present asset value. The value of the asset net of this payment is \((1-s)V\). Clearly if there are \(n\) such payments, the value of the asset net of payments is \((1-s)^nV\). If the disbursement fraction is alternatively stated as a rate \(\gamma\) and there are \(m\) payments per unit time, then each fractional payment will be \(\gamma V/m\) and there will be \(mt\) of them before maturity. The value net of payments in this case is \((1-\gamma/m)^mV\) which in the limit of continuous payout is \(e^{-\gamma}V\). Consequently the assumption of continuous payment in (A-6) and (A-7) is one of convenience only. Discrete payments could have been assumed in which case the factor \(e^{-\gamma}\) is everywhere replaced by \((1-\gamma/m)^m\).
Fig. 1. Capital share price as a function of asset value.

Fig. 2. Capital share price as a function of maturity. (1) $V > E$, $r > \gamma (V/E)$; (2) $V > E$, $\gamma (V/E) > r > \gamma \sqrt{V/E}$; (3) $V > E$, $\gamma \sqrt{V/E} > r$; (4) $V < E$. 
The comparative statics for the capital share price are

\[ f_v = e^{-\tau}N(x_1) \geq 0, \quad (8a) \]
\[ f_{vv} = (e^{-\tau}/V\sigma\sqrt{\tau})N'(x_1) \geq 0, \quad (8b) \]
\[ f_E = -e^{-\tau}N(x_2) \leq 0, \quad (8c) \]
\[ f_r = -\tau Ef_k \geq 0, \quad (8d) \]
\[ f_{o^2} = (Ee^{-\tau}/\sqrt{\tau/2}\sigma)N'(x_o) \geq 0, \quad (8e) \]
\[ f_\gamma = -\tau Vf_v \leq 0, \quad (8f) \]
\[ f_t = (\sigma^2 V^2/2)f_{vv} + (r - \gamma)Vf_v - rf \geq 0. \quad (8g) \]

Eqs. (8a) through (8e) give the expected results from option theory. \( f_v \) has the sign one would expect, namely the value of the capital shares is a decreasing function of the dividends and fees paid.

Examining \( f_r \) we find that it can be either positive or negative. For small asset values the first term dominates and it is positive. This result is reasonable since given any asset return dynamics a sufficiently small, though positive, value could be chosen for the asset value to virtually guarantee default. In this case the capital share holders would prefer a longer maturity to increase the probability that the asset would grow enough in value to redeem the income shares and have a residual for the capital shares. On the other hand for large asset values, \( f_r \sim -\gamma e^{-\tau} < 0 \). Again this is a sensible result since the ‘loan’ represented by the future payment of the redemption price is virtually riskless in terms of default while the compensation provided by the proportional dividends is in excess of that on a riskless loan.

3. The income shares

Under the same assumptions as in section 2 the income shares, \( F(\cdot) \), can be priced through the differential equation

\[ \frac{1}{2}\sigma^2 V^2 F_{vv} + (r - \gamma)VF_v - rF - F_r + \delta V = 0, \quad (9) \]

subject to

\[ F(0, \tau) = 0, \quad F(V, \tau) \leq V, \quad F(V, 0) = \min(V, E). \]

Making the substitutions \( X = Ve^{-\tau} \) and \( G = X - F \), eq. (9) becomes

\[ \frac{1}{2}\sigma^2 X^2 G_{xx} + rXG_x - rG - G_t = -\delta Xe^{\tau}, \quad (10) \]
subject to

\[ G(0, \tau) = 0, \quad G(X, \tau) \leq X, \quad G(X, 0) = \max (0, X - E). \]

Eq. (10) is again the European option pricing equation hence the two homogeneous solutions are \( G = W(X, \tau) \) and \( G = X \). The particular solution is \( -(\delta/\gamma)Xe^{\tau t} \). Using the initial condition to determine the appropriate constants and solving back for the income share price, we obtain

\[ F(V, \tau) = V\{e^{-\tau}(1-\delta/\gamma)+\delta/\gamma\} - f(V, \tau). \]  
\hspace{1cm} (11)

From (11) it is clear that the market value of the dual fund considered as a whole, \( \Phi = F + f \), is linear in asset value, \(^5\)

\[ \Phi(V, \tau) = V\{e^{-\tau}(1-\delta/\gamma)+\delta/\gamma\} = A(\tau)V. \]  
\hspace{1cm} (12)

The value of the future management fees is \( V - \Phi \) or \( (\mu/\gamma)(1-e^{-\tau})V \). Similarly the value of any proportional dividend claim (with no redemption at maturity) would be \( \omega(1-e^{-\tau}) \), where \( \omega \) is the fraction that these dividends are of the total cash outflow.

We can clarify the pricing of the income shares if we imagine them to be made up of two separable claims, one denoted \( h \) on the dividend stream and the other denoted \( H \) on the final redemption payment of \( E \). From above, the price of the pure dividend claim has the value

\[ h(V, \tau) = V(\delta/\gamma)(1-e^{-\tau}) = B(\tau). \]  
\hspace{1cm} (13)

The redemption claim is a 'discount bond' on the asset net of cash payments, and from Merton (1974) we can write

\[ H(V, \tau) = X - W(X, \tau) = e^{-\tau}V - f(V, \tau). \]  
\hspace{1cm} (14)

The income share price and the market value of the dual fund are plotted in fig. 3. The former is a monotonic increasing, concave function of asset value. For small asset value the 'discount bond' claim represents approximately the entire asset value net of future cash payments. For large asset values the 'discount bond' becomes essentially riskless and \( H \sim Ee^{-\tau} \).

\(^5\Phi \) would be the market value of a closed-end fund of the usual type (i.e., with only one class of shares) having the same asset, dividend, and fee characteristics. Since \( A(\tau) \) is less than one, we can see that all closed-end funds should sell at a discount from their asset values.
The comparative statics for the market value of the fund are

\[ \Phi_v = A(\tau) > 0, \]  
\[ \Phi_r = -\mu V e^{-\gamma r} \leq 0, \]  
\[ \Phi_\delta = V(\mu/\gamma^2)\{1 - e^{-\gamma r}(1 + \gamma \tau)\} \geq 0, \]  
\[ \Phi_\mu = -V(1/\gamma)\{(\delta/\gamma)(1 - e^{-\gamma r}) + \mu r e^{-\gamma r}\} \leq 0, \]  
\[ \Phi_\sigma = \Phi_E = \Phi_\sigma^2 = 0. \]  

The comparative statics for the income claim are then

\[ F_v = \Phi_v - f_v = (\delta/\gamma)(1 - e^{-\gamma r}) + e^{-\gamma r}\{1 - N(x_1)\} \geq 0, \]  
\[ F_{vv} = -f_{vv} \leq 0, \]  
\[ F_E = -f_E \geq 0, \]  
\[ F_r = -f_r \leq 0, \]  
\[ F_{\sigma^2} = -f_{\sigma^2} \leq 0. \]
\[ F_\delta = \Phi_\delta - f_\gamma \geq 0, \]  \hfill (16f)

\[ F_\mu = \Phi_\mu - f_\gamma = -V(\delta/\gamma^2)\{1-e^{-\gamma t}(1+\gamma t)\} - \{1-N(x_1)\}e^{-r\tau} \leq 0, \]  \hfill (16g)

\[ F_\tau = \Phi_\tau - f_\tau = H_\tau + h_\tau \geq 0, \]  \hfill (16h)

\[ h_\tau = \delta V e^{-r\tau} \geq 0, \]  \hfill (16i)

\[ H_\tau = -\gamma V e^{-r\tau}\{1-N(x_1)\} - rE e^{-r\tau}N(x_2) - \frac{1}{2}\sigma^2 V^2 f_{vv} \leq 0. \]  \hfill (16j)

The results in (16a) through (16e) are those expected for any debt-like claim. From (16f) and (16g) the income share is an increasing function of the dividends and a decreasing function of the management fees. Again these are expected. The ambiguous sign of \( F_\tau \) is inherent in the dual nature of the income share claim. When the asset value is low, the income shares behave like a discount bond since the dividends are small; hence, \( F_\tau < 0 \) and \( H_\tau \). If the asset value is large enough, then the value of the future dividends will dominate in the pricing of the income shares, and they will behave like a pure dividend stream; hence, \( F_\tau \sim h_\tau > 0 \) since the investor will receive the dividend stream for a longer time if the maturity is greater.

### 4. Fund characteristics

In this section we shall examine some of the characteristics of dual purpose funds that are of concern to investors. One item that appears to be of particular concern is discounts. Discounts for these funds are of two types. One, common to all closed-end funds, is reflected in the difference between the market value of the fund and the value of the fund's holdings. (I.e., the asset value of the fund.) The other, more commonly referred to discount, is the capital share discount. It is defined as that percentage by which the market price of the capital share falls below its ‘net asset value’, roughly \( V - E \). In the model presented in the previous sections these discounts would be

\[ \Delta \equiv 1 - \Phi/V, \quad \Delta_c \equiv 1 - f/(V - E), \quad V \geq E. \]  \hfill (17)

The income shares do not have an associated discount discussed in the literature; however, a definition consistent with the above would be

\[ \Delta_i \equiv 1 - F/E. \]  \hfill (18)

From section 3 we note that the fund will never sell at a premium and will sell at a discount except at the instant that it matures since \( 0 \leq \Delta = (1 - e^{-\gamma \tau}) (1 - \delta/\gamma) \), from (12). As noted in the previous section, this property is not
peculiar to the dual funds but is common to all closed-end funds. Furthermore, it is qualitatively independent of the proportionality assumptions previously placed on the dividends and fees. A closed-end fund will sell at a discount whenever: (1) there are payments from the fund's assets to which the owners of the fund are not entitled (e.g., management fees), and (2) the fund shares are not redeemable at asset value. This follows trivially from the realization that the future management fees must have a positive value to the management and a negative value to the fund owners regardless of their contractual structure or of dividend policy.6

The capital share discount appears to be of some importance both to investors and to the funds' managers. For example an American Dual Vest Fund report stated, '...the capital shares continue to sell at a substantial discount from net asset value. Your management believes that this disparity is not justified by the performance of the fund.' The belief that the capital shares should not sell at a discount appears quite widespread. It is perhaps associated with the true constraint that options must sell for more than their 'intrinsic value'.7 However, the latter is binding only from the arbitrage opportunity that would otherwise be present through the purchase and immediate exercise of the option. There is no apparent reason for a similar arbitrage condition to be applicable to the capital shares since they are not exerciseable except at maturity. Furthermore, the model presented in the previous sections exhibits a reasonable fund structure in which a capital discount is possible.

Before proceeding to examine the capital discount of this model in more detail it is worthwhile to point out that unlike the fund discount the behavior of this discount is qualitatively dependent upon the assumptions of section 2, in particular the proportionality assumptions (A-6) and (A-7).8 Uncertainty, however, is not a necessary condition for the discount although it does affect the magnitude. Appendix B demonstrates that the capital shares of a dual fund that holds only riskless securities will be subject to a discount.

6The existence of market imperfections may obviate the discount even in the presence of conditions (1) and (2) above. For example the benefits from economies of scale in transactions costs or information collecting and processing may outweigh the management fee costs to the investors. This might tend to explain the occasional premiums found on closed-end funds. Even in this case the management fees have negative value to the fund owners. The premium arises from the synergy that allows the economic value of the fund to exceed its asset value.

7The intrinsic value of a warrant or call option is the current stock price less the striking price.

8The matter of the capital discount is identical to the question of premature exercise of an option addressed by Merton (1973) and Samuelson and Merton (1969). They provide an example in which an early exercise (and, hence, in our model a discount) will never arise. The example is a constant dividend stream smaller than rE. This would be implausible for a dual fund since it would require that the income shares were issued to yield less than the riskless rate; nevertheless, it does demonstrate that capital discounts may never occur in some cases. It can be readily demonstrated that a necessary condition for discounts to be impossible on the capital shares is that there exist an upper bound on the size of the cash payouts to the other claims.
As shown in fig. 4, the capital discount is a monotone increasing, concave function of asset value with an upper bound of \(1 - e^{-\gamma}\). Below a critical asset value, \(V_c\), the discount is negative (i.e., a premium). The premium, expressed as it is in percentage terms, is unlimited in size. From (7) and (17) the critical asset value can be determined as the solution to

\[
V = E \frac{1 - N(x_2)e^{-\gamma t}}{1 - N(x_1)e^{-\gamma t}} = E \Gamma(V, \gamma).
\]

(19)

Examination of \(\Gamma\) reveals that it has an upper bound of \((1 - e^{-\gamma t})^{-1}\) hence discounts are possible for funds of any positive maturity.

![Fig. 4. Total discount from total market value and capital share discount from net asset value as a function of total asset value.](image)

When \(V > rE/\gamma\), the discount is an increasing function of maturity. Otherwise, the discount first decreases with maturity for short maturity dual funds and increases for long maturities. Since as a general rule the interest rate exceeds the payout parameter, \(\gamma\),\(^9\) either case above will be possible depending upon the current asset value of the fund.

In the perfect capital market of this model, concern over the capital discount is vacuous since it bears a strict functional relationship to the asset value and other parameters. Furthermore, since the fund as a whole always sells at a

\(^9\)Aside from the empirical resolution of this statement, this matter has a theoretical content developed later in this section.
discount (i.e., \( \Delta > 0 \)), and since

\[
\Delta = \{(V - E)/V\} \Delta_e + \{E/V\} \Delta,
\]

it is clear that either the capital shares or the income shares or both must always sell at a discount. If both discounts are perceived to be bad, then one group of shareholders must be disappointed at all times. If, on the other hand, the discounts are recognized to have no importance of their own, then the existence of a discount on the capital shares might be recognized as good since it corresponds to a high asset value.

Leaving discounts we turn our attention to the risk and return characteristics of the fund and its contingent claims. First we shall determine the equilibrium dividend policy. All of the funds promise to redeem the income shares at the initial offering price; furthermore, each was set up to have the capital and income shares offered at that same price.\(^{10}\) Ignoring the management fees then, the equilibrium dividend policy, \( \gamma^e \), is that value of the payout parameter for which the redemption price, \( E \), is the equilibrium price of both types of shares at a maturity equal to the original lifetime of the fund. If \( T \) is this lifetime then by setting the asset value to twice the redemption price and the capital share price to one-half the asset value in (7), the equilibrium dividend policy will be the solution for \( \gamma \) to

\[
1 = 2e^{-rT}N(x_1) - e^{-rT}N(x_2). \tag{21}
\]

Clearly the equilibrium policy is a function of only the lifetime, the riskless rate, and the variance of return on the assets. By the implicit function theorem

\[
0 < \partial \gamma / \partial r = -f_s / f_r < 1, \tag{22a}
\]

\[
0 < \partial \gamma / \partial \sigma^2 = -f_s / f_r, \tag{22b}
\]

\[
0 \leq \partial \gamma / \partial T = -f_T / f_r, \quad \text{as} \quad f_T \geq 0, \tag{22c}
\]

where \( f_T \equiv f_s |_{r = T} \). Also

\[
\gamma^e(T, r, \infty) = (\log 2)/T, \quad \gamma^e(T, r, 0) = r - (1/T) \log \{(1 + e^{rT})/2\}, \tag{23a}
\]

\[
\gamma^e(T, \infty, \sigma^2) = (\log 2)/T, \tag{23b}
\]

\[
\gamma^e(\infty, r, \sigma^2) = 0, \quad \gamma^e(0, r, \sigma^2) = \begin{cases} r/2, & \sigma^2 = 0, \\ 0, & \sigma^2 > 0. \end{cases} \tag{23c}
\]

\(^{10}\)For Putnam the capital share issue price was half that of the income shares; however, twice as many were offered. In all cases the initial leverage was fifty-fifty.
Since the capital share pricing function, \( f(\cdot) \), is homogeneous of degree zero in the interest rate, the payout parameter, the variance parameter, and the reciprocal of maturity, it follows that the equilibrium payout policy parameter is homogeneous of degree one in the other parameters. Hence for computational ease it may be written functionally as \( g(rT, \sigma^2T)/T \) and only the two variables 'lifetime uncertainty' and 'riskless return' are of concern. The partial derivatives in (22) have the expected signs. The income share owners must be promised a higher dividend rate when the fund is riskier, when the interest rate is higher, or when a change in maturity would hurt them. Note from (22a), however, that an increase in the interest rate is not fully reflected in an increase in the equilibrium payout policy.

The importance of the equilibrium payout policy should not be over-emphasized. It is an equilibrium only under the postulated offering and redemption format. While this is the format of all the existing dual funds and, given the legal restrictions of a maximum two to one leverage, it would be the likely format of any new fund, alternate offering and redemption structures would lead to other equilibrium dividend policies.

If the management fees are not neglected, then the determination of the equilibrium dividend policy, \( \delta^e \), and fee percentage, \( \mu^e \), becomes a simultaneous problem. The two equations to be satisfied are \( P_e = f(V, T) \) and \( P_I = F(V, T) \), where the \( P \)'s denote the offering prices of the shares. The first equation is derived from (7) and will be similar to (21). The latter is from (11). This procedure, however, leads only to the trivial solution \( \mu^e = 0, \delta^e = \gamma^e \), already determined. Since the management puts up no front money (i.e., it does not bid for the right to serve as the management and collect the fees), the sum of the offering prices must equal the initial asset value. From (12) the only solution set is clearly that given above since the fund would otherwise be subject to a discount. Under the perfect market assumptions in (A-1), no one would be willing to buy the initial offering of such a fund since he could costlessly perform the same services that the fund provides for himself and avoid the immediate loss of value that the discount would bring about. As discussed in footnote 5, the existence of market imperfections may allow a premium on the fund. In this case an equilibrium solution with a positive fee percentage would be possible; furthermore, the fee would be earned since the management would obviously be providing the service that allowed for the premium.

We have stated earlier that for the existing dual funds the interest rate is greater than the payout parameter. While this might be considered a purely empirical matter, in actuality it has more content. From (22) and (23) it is clear that the payout parameter is bounded by \( \log(2)/T \). Thus it can exceed the interest rate only if \( rT < \log(2) \). For an 'infinitely risky' dual fund with a lifetime of fifteen years, this would require an interest rate below 5 percent. For the small variance rates actually observed the true interest rate would have to be markedly smaller.
Once the equilibrium policies have been determined, the dividend structure of returns is established since the dividend policy, unlike the dividends themselves is nonstochastic by assumption. The structure of returns from capital gains remains to be determined.

From the derivation of the pricing equation\(^\text{11}\) it is clear that the no arbitrage condition

\[
\left(\alpha_f - r\right)/(\alpha - r) = \sigma_f/\sigma = Vf_c/f = q
\]

must hold at every instant of time since the capital shares are perfectly (instantaneously) correlated with the asset value. A similar relationship, of course, obtains for the income shares. We shall denote this ratio of relative excess (instantaneous) returns and relative (instantaneous) risk\(^\text{12}\) by \(q\) for the capital shares and \(Q\) for the income shares. \(Z\), the proxy for asset value, defined in (7), and \(U = \sigma^2\tau\), a term measuring the remaining uncertainty, are sufficient statistics for the former; the latter is described fully by asset value, maturity, and the instantaneous variance.

Fig. 5 depicts the behavior of these two measures. When the remaining uncertainty is small, the final resolution of the claims becomes nearly certain either because the variance is small and the future asset value is predictable or because the maturity is small and the resolution is imminent. In this case, if the asset value is large \((Z > 1)\), then the fund will almost certainly be able to repay the income share owners. Hence these shares become riskless \((Q = 0)\), and the capital shares take on the levered risk of the assets \((q = e^{-\tau}V/f)\). If the asset value is small \((Z < 1)\), then the fund will almost certainly default. The capital shares become infinitely risky, and the income shares acquire the entire asset risk \((Q = 1)\).

The response of the capital shares' risk to asset value depicted in fig. 5a is as expected; however, that of the income shares' risk deserves some explanation. If we consider the dual nature of the income shares as we did in section 3, then the relative risk measure of the redemption claim, \(Q^H\), is identical to that of a discount bond;\(^\text{13}\) thus, it is a decreasing function of asset value. The relative risk measure of the dividend claim, \(Q^h\), is identically one. The risk measure of the income shares is a weighted average of these two with weights proportional to the sub-claims' values,

\[
Q = (h/F)Q^h + (H/F)Q^H = (h/F) + (H/F)Q^H.
\]

\(^{11}\)See appendix A or Merton (1974).

\(^{12}\)Generally some measure of systematic variation such as covariance with the market is used as a risk measure. However, since the contingent claims are perfectly correlated, the ratio of their standard deviations of returns must equal the ratio of their covariances with the market return.

\(^{13}\)See Merton (1974).
Fig. 5(a–b). Relative risk of the capital shares as a function of asset value proxy and uncertainty.

Fig. 5(c–e). Relative risk of the income shares as a function of asset value, maturity, and variance rate.
Initially as the asset value increases, the relative risk associated with redemption decreases and $Q$ falls with it; however, as the asset value gets large the weight $H/F$ goes to zero and the risk measure, $Q$, rises to one again. The same process is at work in figs. 5d and 5e. Both figures are similar to Figure 8 in Merton (1974) although they are distorted in the same fashion as is fig. 5c since the importance of the dividends, through the weight $h/F$, approaches one as the variance or maturity becomes large. Summarizing the comparative statics,

\[
q_x < 0, \quad q_u < 0, \quad Q \geq 0, \\
Q_x > 0, \quad Z \geq 1; \quad Q_x \geq 0, \quad Z < 1; \\
Q_{\sigma^2} > 0, \quad Z \geq 1; \quad Q_{\sigma^2} \geq 0, \quad Z < 1. \quad (26)
\]

The relative risk ratio $g$ is a concept similar to what is popularly known as capital share leverage. The term leverage is used here in the same sense that it would be applied to the equity of any company with debt in its capital structure. For the dual purpose funds there are two common uses which we will distinguish as structural and effective leverage. The 'structural leverage' of the capital shares is defined as the ratio of asset value to net asset value of the capital shares, $L = V/(V - E)$. The effective leverage is the more common ratio, $\lambda = V/f$. Both of these are intended to measure the number of dollars working for the capital shares per dollar invested although clearly effective leverage being based on market values rather than book values does so in a more relevant manner. However, this does not imply that it is necessarily more meaningful as a measurement of risk and return. From its definition and that of the risk measure $g$, the effective leverage is always greater than $g$ since $\lambda = g/f > g$. Since the structural leverage is smaller than the effective leverage when the capital shares are selling at a discount, $L = (1 - \Delta c)\lambda$, the former might well be a better estimate of the risk-return ratio $g$.

5. The sample data

The model as developed in the previous sections was tested on the seven dual purpose funds – American Dual Vest, Gemini, Hemisphere, Income & Capital, Leverage Fund, Putnam Dou-Fund, and Scudder Dou-Vest. The time period examined extended from May 1967 (near that time all seven were first offered to the public) through December 1973. The data on capital share price and net asset value per share was taken from the weekly reports of the Lipper Analytical Division of Steiner Rouse and Co. The former was checked against the ISL Daily Price Index where the income share prices were also obtained. Dividend data was found in Moody’s Dividend Record. Expenses and fees were obtained from Moody’s Bank and Finance Manuals. The time series for asset value per
share was constructed as the sum of the listed net asset value, par value of the income share,\textsuperscript{14} cumulative amount by which promised dividends exceeded actual dividends (i.e., dividend arrearage), and accumulated (unpaid) dividends. Accumulated dividends were not directly observable hence they were approximated as follows. If the minimum dividend was met in each quarter of the year, gross accumulated dividends were assumed to have accrued over the year at a constant weekly rate equal to the total dividends divided by the number of weeks in the year.\textsuperscript{15} If less than the minimum dividend was paid in any quarter of the year, the accrual was assumed to have been at the rate paid quarter by quarter through the last quarter that the minimum was not met. For the remainder of the year the previous method was applied to the remaining dividends. Net accumulated dividends were figured as the maximum of zero and gross dividends less dividends actually paid.

This method should closely approximate the actual asset value of the dual fund; however, since dividends actually accrued sporadically the approximation may produce a time series whose growth rate displays an incorrect variance. Spurious autocorrelation may also be introduced into the time series.\textsuperscript{16}

‘Market returns’ were computed weekly as $R'_i = (V'_i + D'_i)/V'_{i-1}$, where $V'_i$ denotes market value (i.e., the sum of the prices of the income shares and capital shares) at time $t$, and $D'_i$ denotes the value of any dividend that went ‘ex’ during week $t$. ‘Value returns’ were computed weekly as $R_i = (V'_i + D_i)/V'_{i-1}$, where $V'_i$ denotes the constructed asset value at time $t$, and $D_i$ the value of any dividend paid during week $t$.

The data was first checked against the various assumptions in section 2. The first-order autocorrelation coefficient was examined to test for serial correlation in the Wiener process. The results are presented in table 1. Positive serial correlation does seem to be evident in the value return series although this result might be attributable to the construction process. The market series seem to be free from serial correlation; therefore, we will assume that the construction process is at fault.

Standard skewness and kurtosis tests were also performed on the log-returns series. These results are also presented in table 1. The market series seem to

\textsuperscript{14}In the case of American and Income & Capital the par value was adjusted annually to reflect the accounting procedure used to amortize the difference between the net money received by the dual fund per income share at offering and the redemption price.

\textsuperscript{15}The accumulated dividends had to be computed in this annual manner since the funds all pay only the minimum dividend (or less) for the first three quarters and then declare an extra dividend at year’s end.

\textsuperscript{16}Some negative autocorrelation might be expected in the true asset value series due to the time lag between the payment date of the dividends and the ‘ex’ date. If stock prices adjust downward by the amount of the dividend on the ‘ex’ date, then the asset value of the fund will do likewise. This will be followed by an increase in the asset value when payment is received. This occurrence is not found in the reported net asset value series, however, since the dividend receipts do not enter into the net asset value being reported rather as accumulated (unpaid) dividends. To the extent that dividends accrue to the fund regularly over the year this phenomenon will be reduced, but it might not be eliminated.
Table 1
Summary statistics of dual fund data; upper row – market series, lower row – constructed asset value series.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>Return$^a$</th>
<th>Variance$^a$</th>
<th>Auto-correlation</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>American</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>345</td>
<td>-0.027</td>
<td>0.037</td>
<td>-0.010</td>
<td>0.306$^b$</td>
<td>3.958$^b$</td>
</tr>
<tr>
<td>345</td>
<td>-0.034</td>
<td>0.016</td>
<td>0.310$^b$</td>
<td>-0.388$^b$</td>
<td>3.390</td>
</tr>
<tr>
<td>Gemini</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>345</td>
<td>0.036</td>
<td>0.034</td>
<td>0.032</td>
<td>0.163$^b$</td>
<td>3.617$^b$</td>
</tr>
<tr>
<td>337</td>
<td>0.009</td>
<td>0.024</td>
<td>-0.058</td>
<td>-0.252$^b$</td>
<td>1.582$^b$</td>
</tr>
<tr>
<td>Hemisphere</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>331</td>
<td>-0.120</td>
<td>0.053</td>
<td>-0.106</td>
<td>0.369$^b$</td>
<td>5.335$^b$</td>
</tr>
<tr>
<td>329</td>
<td>-0.084</td>
<td>0.024</td>
<td>0.097</td>
<td>-0.634$^b$</td>
<td>1.308$^b$</td>
</tr>
<tr>
<td>Inc–Cap</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>345</td>
<td>0.022</td>
<td>0.042</td>
<td>-0.046</td>
<td>0.086</td>
<td>3.314</td>
</tr>
<tr>
<td>333</td>
<td>0.008</td>
<td>0.022</td>
<td>0.222$^b$</td>
<td>-0.190</td>
<td>3.242</td>
</tr>
<tr>
<td>Leverage</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>345</td>
<td>-0.016</td>
<td>0.030</td>
<td>-0.028</td>
<td>0.216</td>
<td>3.120</td>
</tr>
<tr>
<td>343</td>
<td>-0.007</td>
<td>0.021</td>
<td>0.042</td>
<td>-0.191</td>
<td>7.464$^b$</td>
</tr>
<tr>
<td>Putnam</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>322</td>
<td>0.039</td>
<td>0.027</td>
<td>0.088</td>
<td>-0.130$^b$</td>
<td>5.675$^b$</td>
</tr>
<tr>
<td>313</td>
<td>0.017</td>
<td>0.015</td>
<td>0.208</td>
<td>-0.185</td>
<td>3.377</td>
</tr>
<tr>
<td>Scudder</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>345</td>
<td>-0.012</td>
<td>0.033</td>
<td>-0.058</td>
<td>0.041$^b$</td>
<td>4.037$^b$</td>
</tr>
<tr>
<td>335</td>
<td>-0.004</td>
<td>0.011</td>
<td>0.227</td>
<td>-0.246</td>
<td>2.901</td>
</tr>
</tbody>
</table>

$^a$Annualized.
$^b$Significant at the one-percent level.

demonstrate positive skewness while the value series show negative skewness. This effect is most pronounced for American and Hemisphere. Lepto-kurtosis, also present in many of the series, suggests that the log-returns series are not normally distributed. These results are not surprising since these tests are highly sensitive to even small deviations from normality in samples of this size. Two alternate explanations for lepto-kurtosis are a stable distribution with infinite variance or a normal distribution with variable variance.$^{17}$

In the former case the sample variance will not be a reliable estimate of the dispersion of the log-returns series, and our estimation procedure using it may give poor results. In the latter case the sample variance will be a good estimate of the contemporaneous population variance; however, it may not be the best estimate of the future prevailing variance. In either case the derivation in section 2 is no longer rigorously valid since we have assumed that the random component of the returns is a Gauss–Wiener process, (A-3), and that the variance is non-stochastic, (A-3), and constant, (A-5).

The proportionality of dividends and fees was checked by performing a time series regression of annual dividends and per share annual fees and expenses.$^{18}$

$^{17}$See Mandelbrot (1963) and Rosenberg (1972), respectively.
$^{18}$Expenses were included with management fees as a part of cash flow on the presumption that the shareholders expected to pay them in the future even though they were not contractually obligated to do so as with the management fees.
against average asset value during each year. These results are presented in table 2. The intercept terms were significantly positive for Hemisphere's, Putnam's, and Scudder's dividends and for American's and Hemisphere's

Table 2

<table>
<thead>
<tr>
<th>Dividends</th>
<th>Equation</th>
<th>Correlation</th>
<th>t-stat. intercept</th>
<th>δ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>American</td>
<td>$D = 0.57 + 0.015 V$</td>
<td>0.35</td>
<td>1.15</td>
<td>3.72</td>
</tr>
<tr>
<td>Gemini</td>
<td>$D = -0.27 + 0.046 V$</td>
<td>0.59</td>
<td>0.33</td>
<td>3.61</td>
</tr>
<tr>
<td>Hemisphere</td>
<td>$D = 0.51 + 0.006 V$</td>
<td>0.35</td>
<td>3.77$^a$</td>
<td>3.59</td>
</tr>
<tr>
<td>Inc-Cap</td>
<td>$D = 0.49 + 0.014 V$</td>
<td>0.26</td>
<td>1.13</td>
<td>3.62</td>
</tr>
<tr>
<td>Leverage</td>
<td>$D = 0.67 + 0.009 V$</td>
<td>0.32</td>
<td>1.57</td>
<td>3.30</td>
</tr>
<tr>
<td>Putnam</td>
<td>$D = 1.64 - 0.011 V$</td>
<td>-0.46</td>
<td>4.60$^a$</td>
<td>3.50</td>
</tr>
<tr>
<td>Scudder</td>
<td>$D = 0.70 - 0.005 V$</td>
<td>-0.38</td>
<td>5.72$^a$</td>
<td>3.31</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expenses</th>
<th>Equation</th>
<th>Correlation</th>
<th>t-stat. intercept</th>
<th>μ %</th>
<th>γ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>American</td>
<td>$E = 0.12 + 0.0002 V$</td>
<td>0.32</td>
<td>16.05$^d$</td>
<td>0.97</td>
<td>4.69</td>
</tr>
<tr>
<td>Gemini</td>
<td>$E = -0.02 + 0.0040 V$</td>
<td>0.71</td>
<td>0.36</td>
<td>0.85</td>
<td>4.46</td>
</tr>
<tr>
<td>Hemisphere</td>
<td>$E = 0.05 + 0.0010 V$</td>
<td>0.62</td>
<td>4.29$^d$</td>
<td>0.93</td>
<td>4.52</td>
</tr>
<tr>
<td>Inc-Cap</td>
<td>$E = 0.13 - 0.0020 V$</td>
<td>-0.30</td>
<td>1.81</td>
<td>0.75</td>
<td>4.37</td>
</tr>
<tr>
<td>Leverage</td>
<td>$E = 0.03 + 0.0009 V$</td>
<td>0.29</td>
<td>0.77</td>
<td>0.71</td>
<td>4.01</td>
</tr>
<tr>
<td>Putnam</td>
<td>$E = 0.09 + 0.0052 V$</td>
<td>0.66</td>
<td>0.97</td>
<td>0.76</td>
<td>4.26</td>
</tr>
<tr>
<td>Scudder</td>
<td>$E = 0.08 - 0.0021 V$</td>
<td>-0.43</td>
<td>1.88</td>
<td>0.72</td>
<td>4.03</td>
</tr>
</tbody>
</table>

$^a$D = annual dividends paid.
$^b$E = annual expenses + management fees.
$^c$V = average asset value during the year.
$^d$δ = estimated dividend percentage assuming strict proportionality.
$^e$μ = estimated expenses percentage assuming strict proportionality.
$^f$γ = estimated payout parameter assuming strict proportionality.
$^g$Significant at the one-percent level.

For the others the hypothesis of proportionality could not be rejected; however, the small sample size of seven years precluded a powerful test.$^{19}$ The correlations for all of the regressions were low, and in a few cases negative. The negative slope in the case of Putnam and Scudder is probably due to the

$^{19}$Square and higher-order terms could not be added to the regression without drastically reducing the degrees of freedom.
managements' reluctance to pay dividends below the minimum. Currently Putnam is faced with a rising minimum dividend and Scudder with meeting an already existing arrearage at a time when the asset values of all the funds are low. Consequently, we might expect the managers to shift the portfolios to income producing securities and therefore pay larger dividends than they did earlier when the asset value was higher.

6. The model

Using eq. (7), modeled values for the capital shares were estimated both from the constructed asset value series, and their associated parameters, and from the market value series, and their associated parameters. All parameters were estimated using only historic data except during the eight months of 1967 for which that whole year's data was used. The variance rates were estimated by the sample variances of the appropriate log-returns series. This estimate was updated weekly. The cash flow proportionality constant was approximated annually by the historic mean of the ratio of cash flow to average asset value or the fraction dividends divided by average market value. The risk-free rate chosen was the median yield to maturity on all taxable government bonds with maturities in excess of fifteen years.

Table 3 presents summary statistics for the two models. The square of the correlation coefficient, the root mean square error, and the mean error comprise the overall summary of the model. The last of these items indicates that the model is biased low in all cases save the asset based model for Scudder. The 'R-squared' indicate that the market based models capture the price fluctuations more realistically while the root mean square statistic at times favors the asset based model. This discrepancy is most likely due to the smaller mean error of prediction of the asset based model indicating that this model, while not performing as well on fluctuations, predicts price levels more accurately.

To obtain a more discriminating viewpoint on this matter, a simple linear regression of the actual prices upon the modeled prices was performed. The estimated slope of this regression, \( b \), is presented as the fourth item in the table. The intercept is not reported since it conveys the same message as the mean error. This regression enables us to decompose the mean square error into three components: the portion due to bias or difference of the intercept of the regression from zero, the portion due to over or under response or the difference of the estimated slope from unity, and the portion due to the remaining variation about the regression line. A comparison of the importance of each of these items

\[20\text{The pricing formulas developed earlier were based on asset value rather than market value; however, the derivation goes through exactly the same if market value is independent of the leverage on the dual fund. The market based model using the sum of the capital share price and income share price as input for } V \text{ may be better if the constructed asset values are in error or if expenses are not proportional to asset values.}\]
indicates that bias is the biggest handicap of the model based on market data while residual variance accounts for the majority of the problem of the asset based model.

The bias and incorrect response of the models are probably caused by misspecification of one form or another in the original model. Market imperfections and dividends that are not strictly proportional to asset (or market) value are but two possible examples. Errors in the estimated parameters represents another possibility. The residual error may also be due to the above

<table>
<thead>
<tr>
<th></th>
<th>American</th>
<th>Gemini</th>
<th>Hemisphere</th>
<th>Inc–Cap</th>
<th>Leverage</th>
<th>Putnam</th>
<th>Scudder</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Market based model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.981</td>
<td>0.695</td>
<td>0.987</td>
<td>0.927</td>
<td>0.889</td>
<td>0.967</td>
<td>0.821</td>
</tr>
<tr>
<td>RMSE</td>
<td>2.206</td>
<td>2.717</td>
<td>1.464</td>
<td>2.059</td>
<td>2.516</td>
<td>1.046</td>
<td>0.911</td>
</tr>
<tr>
<td>Mean error</td>
<td>2.107</td>
<td>2.424</td>
<td>1.309</td>
<td>1.788</td>
<td>2.450</td>
<td>0.949</td>
<td>0.790</td>
</tr>
<tr>
<td>Estimated slope</td>
<td>1.188</td>
<td>1.061</td>
<td>1.228</td>
<td>1.404</td>
<td>1.243</td>
<td>1.166</td>
<td>0.818</td>
</tr>
<tr>
<td>Fraction of error due to:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bias</td>
<td>0.912</td>
<td>0.796</td>
<td>0.799</td>
<td>0.754</td>
<td>0.948</td>
<td>0.822</td>
<td>0.751</td>
</tr>
<tr>
<td>( b \neq 1 )</td>
<td>0.049</td>
<td>0.001</td>
<td>0.146</td>
<td>0.126</td>
<td>0.012</td>
<td>0.066</td>
<td>0.046</td>
</tr>
<tr>
<td>Residual variance</td>
<td>0.039</td>
<td>0.203</td>
<td>0.055</td>
<td>0.120</td>
<td>0.040</td>
<td>0.112</td>
<td>0.203</td>
</tr>
<tr>
<td>Misspec. error</td>
<td>4.676</td>
<td>5.883</td>
<td>2.024</td>
<td>3.730</td>
<td>6.076</td>
<td>0.971</td>
<td>0.661</td>
</tr>
<tr>
<td>Residual error</td>
<td>0.189</td>
<td>1.498</td>
<td>0.117</td>
<td>0.508</td>
<td>0.253</td>
<td>0.122</td>
<td>0.168</td>
</tr>
</tbody>
</table>

|                      |          |        |            |         |          |        |         |
| **Asset based model** |          |        |            |         |          |        |         |
| \( R^2 \)            | 0.870    | 0.477  | 0.908      | 0.342   | 0.185    | 0.710  | 0.268   |
| RMSE                 | 1.802    | 3.617  | 1.511      | 2.731   | 1.845    | 1.132  | 1.039   |
| Mean error           | 1.337    | 3.203  | 0.542      | 1.693   | 0.254    | 0.199  | -0.173  |
| Estimated slope      | 1.173    | 0.750  | 1.587      | 1.012   | 0.346    | 1.350  | 0.456   |
| Fraction of error due to: |          |        |            |         |          |        |         |
| Bias                 | 0.551    | 0.784  | 0.129      | 0.384   | 0.019    | 0.031  | 0.028   |
| \( b \neq 1 \)       | 0.057    | 0.020  | 0.300      | 0.000   | 0.439    | 0.137  | 0.333   |
| Residual variance    | 0.392    | 0.196  | 0.371      | 0.616   | 0.542    | 0.832  | 0.639   |
| Misspec. error       | 1.974    | 10.517 | 1.435      | 2.863   | 1.558    | 0.214  | 0.389   |
| Residual error       | 0.185    | 2.564  | 0.847      | 4.594   | 1.844    | 1.066  | 0.440   |

or possibly to temporary demand disequilibria or related factors. The last two items in table 3 present the magnitude of these two types of errors so that the two models may be directly compared. Clearly the initial observation was correct. The model based upon market data is superior in capturing fluctuations while inferior at predicting levels.

Fig. 6 shows sample plots of the two models together with the true prices. The bias is clearly evident. Furthermore, this bias appears to be an increasing function of maturity. To verify this qualitative result the following regression
Fig. 6. Comparison of market prices and proportional dividend model estimates.
was performed:

\[
\log \left( \frac{\text{true price}}{\text{model price}} \right)_t = a + b \log \left( \frac{\text{true price}}{\text{model price}} \right)_{t-1}.
\]

The results shown in table 4 confirm the hypothesis that the error decreases with time since the estimated slope is significantly less than one for most cases.\(^{21}\) Two alternative explanations for this behavior in the discrepancy are: (1) The model is misspecified in a manner that introduces bias which is an increasing function of maturity, or (2) the market participants do not fully utilize the

<table>
<thead>
<tr>
<th>Market based model</th>
<th>American</th>
<th>Gemini</th>
<th>Hemisphere</th>
<th>Inc-Cap</th>
<th>Leverage</th>
<th>Putnam</th>
<th>Scudder</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.031</td>
<td>0.002</td>
<td>0.031</td>
<td>0.007</td>
<td>0.039</td>
<td>0.027</td>
<td>0.007</td>
</tr>
<tr>
<td>(b)</td>
<td>0.870</td>
<td>0.986</td>
<td>0.884</td>
<td>0.955</td>
<td>0.839</td>
<td>0.834</td>
<td>0.939</td>
</tr>
<tr>
<td>s.e. ((b))</td>
<td>0.028</td>
<td>0.011</td>
<td>0.026</td>
<td>0.015</td>
<td>0.030</td>
<td>0.030</td>
<td>0.020</td>
</tr>
<tr>
<td>t-stat. ((b = 0))</td>
<td>30.637</td>
<td>87.436</td>
<td>33.537</td>
<td>63.547</td>
<td>28.254</td>
<td>27.811</td>
<td>47.391</td>
</tr>
<tr>
<td>t-stat. ((b = 1))</td>
<td>4.597</td>
<td>1.257</td>
<td>4.403</td>
<td>2.971</td>
<td>5.421</td>
<td>5.548</td>
<td>3.104</td>
</tr>
<tr>
<td>D.W.</td>
<td>2.335</td>
<td>2.534</td>
<td>2.341</td>
<td>2.510</td>
<td>2.245</td>
<td>1.967</td>
<td>2.440</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset based model</th>
<th>American</th>
<th>Gemini</th>
<th>Hemisphere</th>
<th>Inc-Cap</th>
<th>Leverage</th>
<th>Putnam</th>
<th>Scudder</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.007</td>
<td>0.008</td>
<td>-0.004</td>
<td>0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.003</td>
</tr>
<tr>
<td>(b)</td>
<td>0.936</td>
<td>0.963</td>
<td>0.960</td>
<td>0.982</td>
<td>0.965</td>
<td>0.958</td>
<td>0.961</td>
</tr>
<tr>
<td>s.e. ((b))</td>
<td>0.019</td>
<td>0.015</td>
<td>0.015</td>
<td>0.010</td>
<td>0.012</td>
<td>0.015</td>
<td>0.015</td>
</tr>
<tr>
<td>t-stat. ((b = 0))</td>
<td>48.146</td>
<td>65.596</td>
<td>63.618</td>
<td>99.623</td>
<td>78.147</td>
<td>64.659</td>
<td>66.240</td>
</tr>
<tr>
<td>t-stat. ((b = 1))</td>
<td>3.319</td>
<td>2.498</td>
<td>2.672</td>
<td>1.832</td>
<td>2.863</td>
<td>2.822</td>
<td>2.701</td>
</tr>
<tr>
<td>D.W.</td>
<td>2.372</td>
<td>2.487</td>
<td>2.612</td>
<td>2.493</td>
<td>2.733</td>
<td>2.251</td>
<td>2.494</td>
</tr>
</tbody>
</table>

information contained in the past estimates of the variance or other parameters in forming their investment decisions; however, as the day of reckoning at maturity approaches, when the model must hold exactly, large deviations are not so likely as they are earlier.

If the market price is 'incorrect' [i.e., explanation (2) is at least partially valid], then the model will indicate opportunities for abnormal profits by buying the undervalued and selling the overvalued claims. Furthermore, this opportunity

\(^{21}\) Care must be taken in interpreting these results since the Durbin-Watson statistic indicates the presence of autocorrelation. If \(\rho\) is the autocorrelation coefficient then \(\hat{b}\) is an inconsistent estimate of \(b\) \([\lim \hat{b} = b + (1 - b^2)/(1 + b \rho)]\), and in our regression \(\hat{b}\) is biased away from one. The usual correction procedure leads to a regression of the form \(y_t = (\rho + b) y_{t-1} - \rho b y_{t-2}\) which is not identified for \(\rho\) and \(b\).
may have little associated risk since we know how to hedge the capital shares with positions in the fund’s assets and riskless bonds.

To test whether the market model discrepancy is due in part to the market’s inefficiency in utilizing any information contained in the model in pricing the capital shares and income shares, the following simulation test was performed. Each week the model value for the capital share was compared to the existing market price. If the former was larger (smaller), one dollar was invested in a long (short) position in the capital shares. This investment was financed by an opposite position in the assets of the dual fund and borrowing or lending at the riskless rate. The aggregate investment was constrained to be zero. Furthermore, the relative positions were taken so as to form a hedged portfolio that would minimize the risk. In the simple world of the model the risk can be eliminated entirely by choosing the proper investment weights. This is the principle upon which the Black–Scholes (1973) derivation hinges.\(^{22}\)

Under the idealized conditions of the model, the proper hedge is \(-V_{fo}/f\) ‘dollars’ in the fund’s assets for each ‘dollar’ in the capital shares. Care must be taken in interpreting this statement since ‘dollar’ refers to investment at the true price. If the model rather than the market reflects the correct price, then the proper amount to invest in the fund is \(-V_{fo}/P_c\) (where \(P_c\) is the market price of the capital shares).\(^{23}\)

In the test performed the returns computed were not riskless. Variations in returns would be expected from three sources: (1) weekly rather than continuous portfolio updating, (2) change in the market model deviation, and (3) improper hedges if the model price were not correct. The second source is that from which profits are to be made. The other two will introduce noise into the return series. To reduce the noise a combination portfolio was also formed. In this portfolio one-seventh dollar was invested in each of the capital shares and the seven hedges were also utilized. The combination portfolio will show improved results to the extent that the noise represents unsystematic risk.

Table 5 presents the results of this market simulation test. The top line for each period is the mean weekly return on the hedging portfolio in cents.\(^{24}\)

The lower lines give the standard deviation of return and the \(t\)-statistics. The mean weekly returns are positive in just over sixty percent of the cases and are positive for each fund over the six year period. However, the confidence level on the overall mean return of 0.102 cents is only nine percent. From these results we should be hesitant about rejecting the hypothesis that the market is efficient with regard to the model contained information. Hence our preliminary conclusion must be that any market model discrepancy must be due to errors in the model.

\(^{22}\)See Black and Scholes (1973), Merton (1973, 1974), or equation (A.4) in appendix A.

\(^{23}\)If \(V\) and \(f\) are the true (fair) prices, then \(-f_o\) ‘shares’ of the fund’s assets must be held to hedge one capital share; hence, \(-f_o V/P_c\) dollars must be invested in the fund per dollar in the capital shares.

\(^{24}\)Percentage returns cannot be computed since there is zero net investment in the portfolio.
7. Model error – Parameters

In this section we shall examine possible sources of model error. In particular inaccuracies that might produce an error which would be an increasing function of maturity will be sought. We shall confine our attention to the market based model since it has a smaller residual error. Changes examined here should mostly affect the misspecification error.

Table 5
Results of market simulation test (weekly).*

| Year | American | Gemini | Hemisphere | Inc–Cap | Leverage | Putnam | Scudder | Year avg.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1968</td>
<td>-0.164</td>
<td>-0.130</td>
<td>-0.012</td>
<td>-0.212</td>
<td>-0.014</td>
<td>-0.404</td>
<td>-0.052</td>
<td>-0.142</td>
</tr>
<tr>
<td></td>
<td>(2.00)</td>
<td>(1.96)</td>
<td>(2.32)</td>
<td>(1.94)</td>
<td>(2.47)</td>
<td>(3.07)</td>
<td>(1.83)</td>
<td>(2.25)</td>
</tr>
<tr>
<td></td>
<td>-0.59</td>
<td>-0.48</td>
<td>-0.04</td>
<td>-0.79</td>
<td>-0.04</td>
<td>-0.93</td>
<td>-0.20</td>
<td>-1.19</td>
</tr>
<tr>
<td></td>
<td>(0.354)</td>
<td>(0.021)</td>
<td>-0.047</td>
<td>0.151</td>
<td>0.048</td>
<td>0.459</td>
<td>0.115</td>
<td>0.148</td>
</tr>
<tr>
<td>1969</td>
<td>(2.28)</td>
<td>(1.62)</td>
<td>(3.27)</td>
<td>(1.65)</td>
<td>(2.22)</td>
<td>(3.69)</td>
<td>(2.15)</td>
<td>(2.50)</td>
</tr>
<tr>
<td></td>
<td>1.12</td>
<td>-0.09</td>
<td>-0.10</td>
<td>0.63</td>
<td>0.16</td>
<td>0.86</td>
<td>0.37</td>
<td>1.11</td>
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<tr>
<td></td>
<td>(0.135)</td>
<td>0.042</td>
<td>0.340</td>
<td>0.205</td>
<td>0.068</td>
<td>0.362</td>
<td>0.633</td>
<td>0.254</td>
</tr>
<tr>
<td>1970</td>
<td>(2.99)</td>
<td>(2.00)</td>
<td>(5.42)</td>
<td>(2.18)</td>
<td>(2.54)</td>
<td>(5.97)</td>
<td>(3.72)</td>
<td>(3.82)</td>
</tr>
<tr>
<td></td>
<td>0.33</td>
<td>0.15</td>
<td>0.45</td>
<td>0.67</td>
<td>0.19</td>
<td>0.44</td>
<td>1.19</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>-0.092</td>
<td>0.282</td>
<td>-0.161</td>
<td>0.006</td>
<td>-0.009</td>
<td>-0.021</td>
<td>-0.023</td>
<td>-0.003</td>
</tr>
<tr>
<td>1971</td>
<td>(2.85)</td>
<td>(1.20)</td>
<td>(4.20)</td>
<td>(1.98)</td>
<td>(1.96)</td>
<td>(3.93)</td>
<td>(2.14)</td>
<td>(2.80)</td>
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<tr>
<td></td>
<td>-0.23</td>
<td>1.60</td>
<td>-0.28</td>
<td>0.02</td>
<td>-0.03</td>
<td>-0.04</td>
<td>-0.08</td>
<td>-0.02</td>
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<tr>
<td></td>
<td>(0.045)</td>
<td>0.050</td>
<td>0.030</td>
<td>0.072</td>
<td>0.013</td>
<td>-0.253</td>
<td>-0.206</td>
<td>-0.035</td>
</tr>
<tr>
<td>1972</td>
<td>(2.25)</td>
<td>(1.19)</td>
<td>(3.78)</td>
<td>(1.82)</td>
<td>(1.93)</td>
<td>(3.33)</td>
<td>(1.58)</td>
<td>(2.41)</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.30</td>
<td>0.06</td>
<td>0.28</td>
<td>0.05</td>
<td>-0.54</td>
<td>-0.94</td>
<td>-0.27</td>
</tr>
<tr>
<td></td>
<td>0.012</td>
<td>0.259</td>
<td>0.966</td>
<td>0.583</td>
<td>0.242</td>
<td>0.278</td>
<td>0.134</td>
<td>0.354</td>
</tr>
<tr>
<td>1973</td>
<td>(2.78)</td>
<td>(2.52)</td>
<td>(3.99)</td>
<td>(2.38)</td>
<td>(2.42)</td>
<td>(3.50)</td>
<td>(1.73)</td>
<td>(2.84)</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0.73</td>
<td>1.75</td>
<td>1.75</td>
<td>0.72</td>
<td>0.55</td>
<td>0.55</td>
<td>2.35</td>
</tr>
<tr>
<td>Fund</td>
<td>(2.54)</td>
<td>(1.81)</td>
<td>(3.95)</td>
<td>(2.08)</td>
<td>(2.25)</td>
<td>(4.03)</td>
<td>(2.29)</td>
<td>(2.83)</td>
</tr>
<tr>
<td>avg.</td>
<td>0.40</td>
<td>0.68</td>
<td>0.75</td>
<td>1.38</td>
<td>0.45</td>
<td>0.36</td>
<td>0.70</td>
<td>1.67</td>
</tr>
</tbody>
</table>

*Top line – mean return on hedging portfolios in cents per dollar invested in capital shares, second line – standard deviation of returns on hedging portfolios, third line – t-statistic of mean returns.

Positive measurement error in the payout proportionality constant is one possible explanation. Since it appears only in the factor $e^{-\tau}$ in the first argument of the option function $W(\cdot)$, and since $W_1 > 0$, it is clear that overestimation of this parameter will lead to an underestimation of the capital share values; furthermore, this error will increase with maturity. Similarly a negative measurement error in the variance is a possible explanation. To see this we write $f(\cdot) = W(Z, U; 1, 0, 1) e^{-\tau}$, where $Z$ is the asset value proxy and $U = \sigma^2 \tau$.
is the uncertainty term introduced earlier, and note that \( W_2 \geq 0 \). However, either of these explanations would require a measurement error in the parameter of the same sign for all dual funds during each year.\(^{25}\)

An alternate explanation that appears more plausible is that the bias effect is due to ignoring the stochastic nature of the interest rate. Appendix A prices the capital shares under the assumption of a stochastic interest rate; the solution from (A-10) is

\[
\begin{align*}
  f^*(V, P, \tau) &= EP(\tau)W(Z^*, U^*; 1, 0, 1), \\
  Z^* &\equiv Ve^{-\gamma_t}/EP(\tau), \\
  U^* &\equiv \int_0^\tau (\sigma^2 + v^2 - 2r\sigma v) \, ds. \\
\end{align*}
\]

\( P(\tau) \) is the price today of a riskless dollar delivered at \( \tau \), \( v^2(\tau) \) is the instantaneous variance of return on the bond, \( \rho(\tau) \) is the instantaneous correlation coefficient of returns on the bond and the asset value of the fund, and \( W(\cdot) \) is the option solution defined earlier. Therefore, if the asset variance is assumed constant as before and the term \( v^2 - 2\rho\sigma v \) which is the addition to the uncertainty term \( U^* \) due to incorporating a stochastic interest rate, is positive then the earlier model will underestimate the capital share values by an error that increases with maturity.

The parameters of the bond's price dynamics, \( v \) and \( \rho \) cannot be easily determined from a time series of bond prices since they must be functions of maturity if the unanticipated returns to bonds are to be serially independent.\(^{26}\) If \( R(\tau) \) is the yield to maturity on a \( \tau \) period discount bond (i.e., \( P = e^{-R\tau} \)) and is assumed to be approximately equal to the rate on bonds with nearby maturities, then the dynamics for \( R \) are

\[
dR = a(R)dt + b(R)d\xi, 
\]

where \( a \) and \( b \) are independent of maturity and \( d\xi \) is the Gauss–Wiener process of the interest rate. Using Ito's Lemma the dynamics for the bond price, \( P \), can be determined

\[
dP/P = (R - a\tau + \frac{1}{2} b^2 \tau^2)dt - b\tau d\xi, 
\]

and

\[
\begin{align*}
  v^2 &= \tau^2 \text{Var}\{b(R)d\xi\} = \tau^2 \text{Var}\{dR\}, \\
  \rho\sigma v &= -\tau \text{Cov}\{b(R)d\xi, \sigma dz\} = -\tau \text{Cov}\{dR, dV/V\}. \\
\end{align*}
\]

\(^{25}\) Even if the sample variance is an unbiased estimate of the true variance, it does not follow that \( f \) is an unbiased estimate of the capital share price. \( f(\cdot) \) is neither a strictly concave nor convex function of variance, thus Jensen's inequality is inapplicable and the sign of the bias depends on the current value of the assets. To test for measurement error, the model was estimated using all the available data to estimate the payout parameters and the variances. No improvement, and little change at all, was found in these new modeled price series.

\(^{26}\) For example, we know that \( v \to 0 \) as \( \tau \to 0 \) since the return over the last instant before the bond matures must be at the riskless rate for certain.
The variance and covariance terms do not depend on maturity; therefore, they can be estimated from a time series of long interest rates. The time series of changes in the interest rate demonstrated heteroscedasticity that appeared linear [i.e., \( b(R) = kR \)]; therefore, the estimates of \( \nu \) and \( \rho \) employed were

\[
\hat{\nu}^2 = R^2 \tau^2 \text{Var} \{dR/R\} \equiv R^2 \tau^2 \hat{\Sigma}^2,
\]

\[
\hat{\rho} = -(1/\hat{\nu}) R\tau \text{Cov} \{dR/R, dV/V\} \equiv -R\tau \hat{\Gamma}/\hat{\nu}.
\]

Performing the integration for \( U^* \),

\[
U^* = \hat{\sigma}^2 \tau + \frac{1}{2} R^2 \tau^3 \hat{\Sigma}^2 + R\tau^2 \hat{\Gamma}.
\]

The sample variance over the seven year period for \( dR/R \) was \( 2.58 \times 10^{-4} \). The sample covariances ranged from \( -3.70 \times 10^{-5} \) for Leverage to \( -9.48 \times 10^{-5} \) for Hemisphere. Consequently the addition to the uncertainty term \( U^* \) due to the stochastic nature of the interest rate was very small, and the largest difference found in estimated model values was less than ten cents.

The stochastic nature of the interest rate is not the only misspecification in the model which could result in an error of the type found that would apply uniformly to all the dual funds. A personal income tax can be shown to have similar effects.

In a world in which a personal income tax is assessed, it is reasonable to assume that investors are concerned with their after-tax return on investment and that contingent claims should therefore be priced by their relative after-tax value. If the returns on two claims are subject to different taxes, then their relative prices will depend upon the taxes assessed.

As a simple model assume that ordinary income is taxed at the rate \( T \), the same for all income levels and constant over time. Capital gains are not taxed at all. If an investor forms a portfolio with \( W_1 \) dollars in the dual fund’s assets, \( W_2 \) dollars in \( F \), the contingent claim, and \( W_3 \) dollars in a riskless bond paying taxable interest continuously at the rate \( r \), then his after-tax return will be

\[
dx = \frac{dV + C(1-T)dt}{V} W_1 + \frac{dF + c(1-T)dt}{F} W_2 + r(1-T)W_3 dt.
\]

This equation is identical to Equation (4) in Merton (1974) where the cash payments, \( C \) and \( c \), and the interest rate are replaced by their ‘after-tax’ equivalents; hence, we can immediately write the implied differential equation for the contingent claim.

\(^{27}\)It is here that the assumption that \( b(R) \) is independent of maturity is required.
\[ \frac{1}{2} \alpha^2 V^2 F_v + (1 - T)(r - \gamma)VF_v - (1 - T)rf - F_r + (1 - T)c = 0. \quad (32) \]

Under the same assumptions as before, the value of the capital shares is

\[ f(V, \tau) = W(V e^{-\gamma(1 - T)}; \tau; E, (1 - T) r, \sigma^2). \quad (33) \]

The marginal impact of the tax is

\[ f_T = \tau q \{ \gamma - r(1 - 1/q) \}, \quad (34) \]

where \( q \) is the relative risk measure introduced in section 4. Ceteris paribus the impact will be larger in absolute value earlier in the fund's history; however, the sign of the effect is uncertain. \( f_T \geq 0 \) as \( \gamma \geq r(1 - 1/q) \). For the no dividend case the lower inequalities hold. If the payout parameter is at least as large as the interest rate, then the upper set will hold. Both cases follow immediately upon observing that \( q \) is greater than one. From section 4,

\[ \lim_{z \to 0} q = \infty, \quad f_T \sim \gamma - r Ve^{-\gamma N(x_1)} \geq 0, \]

\[ \lim_{z \to \infty} q = 1, \quad f_T \sim Ve^{-\gamma} > 0; \quad (35) \]

hence, in the usual intermediate case when the payout parameter is positive but less than the interest rate, \( f_T \) will take on first negative and then positive values as the asset value increases.

The uncertainty in the sign of \( f_T \) may seem counter-intuitive. One might expect the capital shares which escape taxation to benefit at the expense of the income shares. This effect is present; nevertheless, at small asset values it is smaller than the relative decrease in value of the capital shares due to the smaller effective (i.e., after-tax) interest rate.\(^{28}\) When dividends are large (i.e., asset value is large) the former effect will dominate and vice versa.

Ignoring for the moment the non-dividend cash payments of the dual funds, which are minor, we can see immediately that the capital shares must be an increasing function of the tax rate whenever the current yield on the income shares is greater than the riskless rate. I.e., if \( \gamma V/(V - f) > r \), then \( \gamma > r(1 - f/V) > r(1 - f/V) = r(1 - 1/q) \) which from above is a sufficient condition for \( f \) to be positive. In principle, current yields on the income shares need not exceed the interest rate. For example, if the income shares were selling at a discount and a capital gain were expected or if the growth in dividends expected through the capital appreciation of the assets were sufficiently large, the current yield might be smaller than the riskless rate. However, the dividend yields did exceed

\(^{28}\)The capital share price is an increasing function of the interest rate. See (8d).
the riskless rate in general for the sample period. Hence introducing taxes will increase the capital share prices and do so to a greater extent for the longer maturities.

Table 6 shows statistics for this model. A comparison with table 3 shows the expected decrease in mean square error and more particularly in the ‘misspecification error’ as we consider the tax rates 0 percent, 25 percent and 50 percent. Even in this last case, however, each model was on average too low. Although tax rates greater than 50 percent are possible in the United States and

<table>
<thead>
<tr>
<th></th>
<th>American</th>
<th>Gemini</th>
<th>Hemisphere</th>
<th>Inc-Cap</th>
<th>Leverage</th>
<th>Putnam</th>
<th>Scudder</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tax rate = 50%</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.985</td>
<td>0.767</td>
<td>0.985</td>
<td>0.963</td>
<td>0.904</td>
<td>0.971</td>
<td>0.851</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.396</td>
<td>1.096</td>
<td>0.656</td>
<td>0.864</td>
<td>1.406</td>
<td>0.490</td>
<td>0.581</td>
</tr>
<tr>
<td>Mean error</td>
<td>1.339</td>
<td>0.238</td>
<td>0.536</td>
<td>0.558</td>
<td>1.324</td>
<td>0.356</td>
<td>0.228</td>
</tr>
<tr>
<td>Estimated slope</td>
<td>1.027</td>
<td>0.996</td>
<td>0.977</td>
<td>1.192</td>
<td>0.943</td>
<td>0.964</td>
<td>0.701</td>
</tr>
<tr>
<td>Fraction of error due to:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bias</td>
<td>0.920</td>
<td>0.047</td>
<td>0.666</td>
<td>0.417</td>
<td>0.886</td>
<td>0.528</td>
<td>0.153</td>
</tr>
<tr>
<td>$b \neq 1$</td>
<td>0.004</td>
<td>0.000</td>
<td>0.012</td>
<td>0.234</td>
<td>0.004</td>
<td>0.021</td>
<td>0.430</td>
</tr>
<tr>
<td>Residual variance</td>
<td>0.076</td>
<td>0.953</td>
<td>0.322</td>
<td>0.350</td>
<td>0.110</td>
<td>0.451</td>
<td>0.417</td>
</tr>
<tr>
<td>Misspec. error</td>
<td>0.145</td>
<td>0.056</td>
<td>0.291</td>
<td>0.485</td>
<td>1.759</td>
<td>0.131</td>
<td>0.196</td>
</tr>
<tr>
<td>Residual error</td>
<td>0.011</td>
<td>1.144</td>
<td>0.138</td>
<td>0.261</td>
<td>0.217</td>
<td>0.108</td>
<td>0.140</td>
</tr>
<tr>
<td><strong>Tax rate = 25%</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.983</td>
<td>0.737</td>
<td>0.988</td>
<td>0.948</td>
<td>0.911</td>
<td>0.970</td>
<td>0.839</td>
</tr>
<tr>
<td>RMSE</td>
<td>1.793</td>
<td>1.796</td>
<td>1.021</td>
<td>1.447</td>
<td>1.947</td>
<td>0.735</td>
<td>0.692</td>
</tr>
<tr>
<td>Mean error</td>
<td>1.722</td>
<td>1.355</td>
<td>0.930</td>
<td>1.179</td>
<td>1.890</td>
<td>0.648</td>
<td>0.499</td>
</tr>
<tr>
<td>Estimated slope</td>
<td>1.104</td>
<td>1.035</td>
<td>1.095</td>
<td>1.294</td>
<td>1.098</td>
<td>1.060</td>
<td>0.760</td>
</tr>
<tr>
<td>Fraction of error due to:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bias</td>
<td>0.923</td>
<td>0.587</td>
<td>0.828</td>
<td>0.663</td>
<td>0.942</td>
<td>0.778</td>
<td>0.520</td>
</tr>
<tr>
<td>$b \neq 1$</td>
<td>0.026</td>
<td>0.001</td>
<td>0.066</td>
<td>0.163</td>
<td>0.004</td>
<td>0.021</td>
<td>0.164</td>
</tr>
<tr>
<td>Residual variance</td>
<td>0.051</td>
<td>0.412</td>
<td>0.105</td>
<td>0.174</td>
<td>0.053</td>
<td>0.201</td>
<td>0.316</td>
</tr>
<tr>
<td>Misspec. error</td>
<td>3.050</td>
<td>1.896</td>
<td>0.931</td>
<td>1.729</td>
<td>3.586</td>
<td>0.431</td>
<td>0.327</td>
</tr>
<tr>
<td>Residual error</td>
<td>0.163</td>
<td>1.328</td>
<td>0.109</td>
<td>0.364</td>
<td>0.200</td>
<td>0.108</td>
<td>0.151</td>
</tr>
</tbody>
</table>

such higher rates would probably reduce the market model discrepancy even more, 50 percent was the largest tax rate considered here for three reasons: (1) Those investors in the very high tax brackets generally have their investments personally managed rather than using mutual funds as investment vehicles. (2) Excluding the capital gains tax in the model formulation as we have done tends to overstate the impact of the tax.²⁹ (3) The income shares of these funds were

²⁹See next page.
largely purchased by tax-free endowment funds and corporations which are
allowed an 85 percent exclusion of dividends paid to them on their corporate
taxes. Consequently the effective tax rate on dividends is only 7.5 percent
for corporations with a tax rate of 50 percent.

Inclusion of a tax in the model has helped to reduce the discrepancy in the
manner sought; nevertheless, it appears that only a partial correction is possible
under the best of circumstances.

8. Model Error – Misspecification

The alterations to the basic dual fund model examined in the previous section
were accomplished through the change of one or more parameters. In this
section we turn our attention to model error caused by misspecification. In
particular assumption (A-3) about the asset return dynamics will be examined.

The derived pricing eq. (7) is based on log-normal return dynamics with a
constant variance. In section 5 evidence was presented suggesting that this simple
assumption is not strictly true, and two explanations were offered. To review,
the first postulated that the infinitesimal process driving the returns is not a
Gauss–Wiener process but rather drawn from a stable distribution with an
infinite second moment. The second argues that the process is as postulated in
(A-3); however, the variance is not a constant. The first hypothesis is probably
more damaging to the model presented here. In this case Ito’s Lemma may not
be used to deduce the contingent claims’ price dynamics. In addition the expected
change in asset value over any interval would be infinite as noted by Samuelson.\(^3\)

Under such conditions, Merton\(^3\) has conjectured that the only equilibrium
option price would be the stock price itself. In the model here this would imply
that the capital shares would sell for \(X\), the value of the assets net of cash
payments. Under the second explanation the exact nature of the process driving
the variance is important. If the variance is a known function of time, then the
entire derivation is valid and the term \(\sigma^2 \tau\) in (7) is replaced by a generalized
uncertainty term \(U = \int_0^\tau \sigma^2(s)ds\) much as in the case of a stochastic interest
rate. If the variance is stochastic but it is a known function of asset value (and
possibly time), then the derivation of (2) is valid; however, it does not have
the simple closed form solution (7). Finally, if the variance is stochastic even
conditionally on asset value, then the hedging derivation breaks down entirely
as no portfolio will be completely riskless. Even in this last case if it is not ‘too
stochastic’ in nature, then the model may closely approximate the true solution.

\(^2\)If a capital gains tax is also assessed and collected continuously then \((1 - T)\) in the model is
everywhere replaced by \((1 - T_i)/(1 - T_c)\), where \(T_i\) is the tax rate on ordinary income and \(T_c\)
is the tax rate on capital gains. For a capital gains tax of Max (0.25, \(T_c/2\)) our computed models
then give results for investors in the 40% and 72.5% tax brackets. While 25% and 50% are
underestimates of the true tax brackets modeled, 40% and 72.5% are overestimates since
capital gains taxes are due only when the gain is realized.

\(^3\)See Samuelson (1967, footnote 1) and Ohlson (1975) for a proof.

\(^3\) See Merton (1973, footnote 42).
Determining which explanation is more plausible is important if we are to
decide whether this model can be improved upon or should be abandoned. If
the log-price changes are drawn from a stable distribution other than the normal
(i.e., the characteristic exponent \( \alpha \) is less than two), then the population variance
is infinite. However, as long as it is greater than one, scale parameters of degree
one (e.g., mean absolute deviation, interquartile range, etc.) do have finite
expectations. If \( S_n \) is such a scale parameter based upon independent changes
over \( n \) periods, then Mandelbrot (1963) has shown that

\[
S_n = n^{1/\alpha} S_1. \tag{36}
\]

From this result we can deduce that \( S_1 \sqrt{n}/S_n \) should be a decreasing function
of \( n \) reaching zero in the limit for all stable distributions except the normal.

Table 7
Results of scale parameter test: \( S_1 \sqrt{n}/S_n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>American</th>
<th>Gemini</th>
<th>Hemisphere</th>
<th>Inc-Cap</th>
<th>Leverage</th>
<th>Scudder</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>1.019</td>
<td>0.958</td>
<td>1.047</td>
<td>1.101</td>
<td>1.120</td>
<td>0.989</td>
</tr>
<tr>
<td>3</td>
<td>0.983</td>
<td>0.949</td>
<td>1.131</td>
<td>0.984</td>
<td>0.987</td>
<td>1.019</td>
</tr>
<tr>
<td>4</td>
<td>1.135</td>
<td>1.053</td>
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<td>0.972</td>
<td>1.051</td>
<td>0.957</td>
</tr>
<tr>
<td>5</td>
<td>0.999</td>
<td>0.961</td>
<td>1.181</td>
<td>0.908</td>
<td>1.052</td>
<td>1.044</td>
</tr>
<tr>
<td>6</td>
<td>1.058</td>
<td>0.863</td>
<td>1.093</td>
<td>0.953</td>
<td>1.079</td>
<td>1.032</td>
</tr>
<tr>
<td>7</td>
<td>0.879</td>
<td>1.004</td>
<td>1.210</td>
<td>1.013</td>
<td>1.127</td>
<td>1.084</td>
</tr>
<tr>
<td>8</td>
<td>1.060</td>
<td>1.040</td>
<td>1.123</td>
<td>1.040</td>
<td>1.222</td>
<td>1.048</td>
</tr>
<tr>
<td>9</td>
<td>0.989</td>
<td>0.907</td>
<td>0.999</td>
<td>0.931</td>
<td>1.012</td>
<td>0.892</td>
</tr>
<tr>
<td>10</td>
<td>1.109</td>
<td>1.125</td>
<td>1.323</td>
<td>1.203</td>
<td>1.214</td>
<td>1.064</td>
</tr>
<tr>
<td>11</td>
<td>1.021</td>
<td>0.958</td>
<td>0.998</td>
<td>0.895</td>
<td>1.107</td>
<td>1.108</td>
</tr>
<tr>
<td>12</td>
<td>0.942</td>
<td>0.870</td>
<td>0.946</td>
<td>0.947</td>
<td>1.065</td>
<td>0.903</td>
</tr>
<tr>
<td>13</td>
<td>0.933</td>
<td>1.171</td>
<td>1.196</td>
<td>1.241</td>
<td>1.063</td>
<td>1.138</td>
</tr>
<tr>
<td>14</td>
<td>1.092</td>
<td>1.123</td>
<td>1.157</td>
<td>1.234</td>
<td>1.160</td>
<td>1.314</td>
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<tr>
<td>15</td>
<td>0.979</td>
<td>1.130</td>
<td>1.200</td>
<td>0.958</td>
<td>0.971</td>
<td>1.135</td>
</tr>
<tr>
<td>16</td>
<td>0.984</td>
<td>1.016</td>
<td>0.976</td>
<td>0.988</td>
<td>1.015</td>
<td>1.091</td>
</tr>
<tr>
<td>17</td>
<td>1.000</td>
<td>1.190</td>
<td>1.518</td>
<td>1.104</td>
<td>1.225</td>
<td>1.866</td>
</tr>
<tr>
<td>18</td>
<td>0.958</td>
<td>0.995</td>
<td>1.176</td>
<td>1.019</td>
<td>1.037</td>
<td>0.940</td>
</tr>
<tr>
<td>19</td>
<td>1.041</td>
<td>1.131</td>
<td>1.215</td>
<td>1.191</td>
<td>1.223</td>
<td>1.600</td>
</tr>
<tr>
<td>20</td>
<td>0.885</td>
<td>1.060</td>
<td>1.132</td>
<td>1.011</td>
<td>1.021</td>
<td>0.929</td>
</tr>
</tbody>
</table>

Table 7 presents this ratio based upon the mean absolute deviation for \( n = 1 \)
to 20 weeks. This evidence, although not highly discriminating against stable
distributions with characteristic exponents near two, offers little support to the
hypothesis that the distribution of log price changes is a non-normal member of
the Pareto–Lévy family.

The alternate hypothesis is somewhat more difficult to test with no further
knowledge of the process controlling the variance. Rosenberg (1972) has postu-
lated that this process is governed by general market forces. If this is the case,
then we might expect that the computed variances for the seven dual funds would tend to move together. To test this possibility the sample variances were computed every year for each fund and a two-way analysis of variance was performed. The F-statistic across years was significant at the one-percent level indicating that some market factors probably were affecting all the funds in a similar manner. It is not clear if the effect was directly caused by the co-movement of the underlying assets' variances or if it was due to similar reactions of the funds' managers to the market. This distinction, however, is immaterial to the pricing of the dual funds ex post.

Although the evidence presented here is more supportive of the Rosenberg hypothesis than of Mandelbrot's, we have no clue as to how to proceed to improve upon the model already presented so we turn our attention to another matter.

The dividend policy assumed in the model was a proportional one. While this appears to be a reasonable assumption, it was chosen primarily because an analytical solution is known for eq. (2) only in this case. A more realistic assumption would be that dividends are linear in asset value \( C = a + bV; a, b > 0 \).\(^\text{32}\) In this case (2) becomes

\[
\frac{1}{2} \sigma^2 V^2 f'_{w} + (rV - a - bV)f_{v} - r f - f_{v} = 0,
\]

where \( f(\cdot) \) represents the price of the capital shares as computed from this new model. While an analytic solution to (37) is not known, we can compute an asymptotic value for large asset values. First we transform the equation by making the substitutions \( Y = Ve^{-br} \) and \( \psi(Y, \tau) = f \). Then (37) becomes

\[
\frac{1}{2} \sigma^2 Y^2 \psi_{y} + (rY - ae^{-br})\psi_{y} - r\psi - \psi_{y} = 0.
\]

We can now consider \( \psi \) as the capital share on a fund with asset value \( Y \) and paying dividends, \( C = ae^{-br} \) which are a function of maturity only. Consequently as the 'pseudo asset value' \( Y \) becomes large, the income shares, \( \Psi \), will approach their limiting value equal to the future dividends and redemption value discounted at the riskless rate,

\[
\lim_{Y \to \infty} \Psi = Ee^{-rt} + [a/(r-b)]e^{-rt}(e^{(r-b)\tau} - 1) \equiv Ee^{-rt} + L(\tau).
\]

(39)

The asymptotic behavior of the capital share price is then \( \psi \sim Y - \Psi \), and

\[
f \sim Ve^{-br} - Ee^{-rt} - L(\tau).
\]

(40)

If we approximate the linear dividend solution, \( f \), by the proportional dividend

\(^{32}\)See footnote 2 for an explanation of why we might expect \( a > 0 \) for a linear dividend model.
solution, $f$, where the payout parameter is chosen such that $\gamma = C/V = b + a/V$ then expanding $f$

$$f \sim Ve^{-bt}\{1 - a\tau/V + \ldots\} - Ee^{-rt},$$

and

$$f - \hat{f} \sim \frac{ae^{-rt}}{r-b}\left[\frac{1-(r-b)\tau}{e^{(b-r)\tau}} - 1\right] \geq 0, \quad \text{as} \quad r \leq b. \tag{42}$$

If the linear dividend model is an improved description of reality, then (42) is consistent with the observed errors since $r > \gamma > b$. It is, however, only an asymptotic result. In general (37) must be solved by numerical integration. This technique can handle problems of a quite arbitrary form; however, a complete solution (i.e., for all asset values and maturities) must always be computed. Single values of $\hat{f}(-\cdot)$ cannot be obtained.

To isolate the effect of the linear dividends from those of the other factors still not explained by the model, the following procedure was employed. The numerical solution was computed for a fictitious dual fund which completely satisfied the assumptions (A-1) to (A-5) and had a linear dividend policy.\textsuperscript{33} Each value of the computed capital share price was then compared to that obtained from the approximation $f(-\cdot)$ with $\gamma = C/V = b + a/V$.

Table 8 presents a representative sample of values for the proportional approximation and the true price from the numerical solution. The latter is uniformly greater than the former (except at maturity when they are equal), and the difference increases with maturity for any asset value. Since this is the same pattern as found in the market model discrepancy, there is reason to believe that the linear dividend solution for each fund would show improved predictive ability.

With this encouragement, a numerical solution for each fund could have been computed every week. There are, however, two drawbacks to this idea. First, the dividend parameters, $a$ and $b$, can be computed only with the knowledge of the entire time series of values and dividends, and even then only seven data points exist for the linear regression due to the manner in which dividends are paid.\textsuperscript{34} Under this circumstance the linear dividend model might have an unfair advantage in a comparison with the original model which uses only past data to estimate its parameters. On the other hand, the proportionality estimation scheme imposes a structure which must produce a reasonable ‘expected’ dividend policy while the linear estimation need not. In particular, the estimated policies for Putnam and Scudder which depend on value in a negative fashion\textsuperscript{35} are certainly not good estimations of the investors’ ex ante expected dividend

\textsuperscript{33}The values chosen for the parameters were: $\sigma^2 = 0.035$, $r = 0.08$, $a = 0.50$, $b = 0.01$, and $E = 10$. The first four were chosen as representative of the values actually found for the seven funds. The last is merely a scaling factor.

\textsuperscript{34}See footnote 14.

\textsuperscript{35}See table 2.
Table 8
Capital share prices computed from linear dividend model (upper) and proportional approximation.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
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</thead>
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<td>0.00</td>
<td>5.00</td>
<td>10.00</td>
<td>15.00</td>
<td>20.00</td>
<td>25.00</td>
<td>30.00</td>
<td>35.00</td>
<td>40.00</td>
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<tr>
<td></td>
<td>0.00</td>
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<td>20.00</td>
<td>25.00</td>
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<td>35.00</td>
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<td>5.31</td>
<td>10.17</td>
<td>15.07</td>
<td>19.97</td>
<td>24.86</td>
<td>29.76</td>
<td>34.66</td>
<td>39.55</td>
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<tr>
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<td>5.27</td>
<td>10.13</td>
<td>15.02</td>
<td>19.92</td>
<td>24.82</td>
<td>29.72</td>
<td>34.62</td>
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<td>5.62</td>
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<td>19.88</td>
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<td>5.46</td>
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<td>29.30</td>
<td>34.09</td>
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<td>5.84</td>
<td>10.39</td>
<td>15.06</td>
<td>19.75</td>
<td>24.45</td>
<td>29.16</td>
<td>33.87</td>
<td>38.58</td>
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<tr>
<td></td>
<td>0.02</td>
<td>1.64</td>
<td>5.53</td>
<td>10.05</td>
<td>14.70</td>
<td>19.38</td>
<td>24.07</td>
<td>28.76</td>
<td>33.46</td>
<td>38.16</td>
</tr>
<tr>
<td>8</td>
<td>0.10</td>
<td>2.12</td>
<td>6.00</td>
<td>10.43</td>
<td>14.99</td>
<td>19.59</td>
<td>24.20</td>
<td>28.81</td>
<td>33.43</td>
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<td>1.71</td>
<td>5.50</td>
<td>9.86</td>
<td>14.40</td>
<td>18.97</td>
<td>23.55</td>
<td>28.14</td>
<td>32.73</td>
<td>37.33</td>
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<td>6.10</td>
<td>10.44</td>
<td>14.90</td>
<td>19.40</td>
<td>23.92</td>
<td>28.44</td>
<td>32.97</td>
<td>37.51</td>
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<td>9.65</td>
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<td>10.41</td>
<td>14.78</td>
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<td>28.05</td>
<td>32.49</td>
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<td>13.63</td>
<td>17.96</td>
<td>22.33</td>
<td>26.71</td>
<td>31.10</td>
<td>35.50</td>
</tr>
<tr>
<td>14</td>
<td>0.19</td>
<td>2.45</td>
<td>6.19</td>
<td>10.35</td>
<td>14.64</td>
<td>18.96</td>
<td>23.30</td>
<td>27.64</td>
<td>31.99</td>
<td>36.35</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>1.65</td>
<td>5.07</td>
<td>9.04</td>
<td>13.18</td>
<td>17.40</td>
<td>21.65</td>
<td>25.93</td>
<td>30.22</td>
<td>34.53</td>
</tr>
</tbody>
</table>

policies. Second, the original model was computed using values for the interest rate and variance which were updated weekly. To follow a similar procedure for the new model would present a formidable task since, as mentioned before, a complete solution for the capital share model prices would have to be computed each week requiring that some 2400 numerical integrations be performed.

Nevertheless, it is desirable to have a comparison between the linear dividend model and the market price that is more revealing than simply stating that the discrepancy between each one and the original model seems to behave in the same manner. Consequently the following compromise scheme was employed. Each week the market value of the fund was observed, and an estimate for the capital shares was computed by the original proportional dividend model (as described in section 6). The ratio of the exact price (as previously computed by the numerical integration) of the fictitious fund to the approximate price at this market value\(^{36}\) and maturity was found next. This ratio was then used as a multiplicative correction factor for the prices from the original estimation. This compromise method uses the individual funds’ parameters, variance and payout, in the estimation; nevertheless, it hopefully corrects the major deficiencies of the misspecification due to the strict proportionality assumption.

\(^{36}\)The comparison was actually made at the same value of the fraction market value over redemption price for scaling purposes.
Although this method is heuristic, it should give us a rough evaluation of the linear dividend model since the interest rate and variance estimates did not change greatly over time. The $a$ and $b$ parameters of the fictitious linear model are close to the estimated values for American, Hemisphere, Income and Capital, and Leverage. More importantly, they should be reasonable approximations to the ex ante expected values for all the funds.

Fig. 7 shows the new model as well as the proportional market based model and the market price for the capital shares, both from fig. 6. The linear dividend model seems to be an improvement. Table 9 confirms this evaluation as the misspecification error has been markedly decreased.

### Table 9
Statistics for linear dividend model.

<table>
<thead>
<tr>
<th></th>
<th>American</th>
<th>Gemini</th>
<th>Hemisphere</th>
<th>Inc-Cap</th>
<th>Leverage</th>
<th>Putnam</th>
<th>Scudder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>0.969</td>
<td>0.810</td>
<td>0.965</td>
<td>0.963</td>
<td>0.910</td>
<td>0.950</td>
<td>0.716</td>
</tr>
<tr>
<td>RMSE</td>
<td>1.346</td>
<td>1.396</td>
<td>0.765</td>
<td>1.162</td>
<td>1.382</td>
<td>0.434</td>
<td>0.704</td>
</tr>
<tr>
<td>Mean error</td>
<td>1.209</td>
<td>0.940</td>
<td>-0.377</td>
<td>0.818</td>
<td>1.275</td>
<td>-0.046</td>
<td>-0.113</td>
</tr>
<tr>
<td>Estimated slope</td>
<td>1.078</td>
<td>1.226</td>
<td>1.133</td>
<td>1.334</td>
<td>1.244</td>
<td>1.008</td>
<td>0.639</td>
</tr>
<tr>
<td>Fraction of error due to:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bias</td>
<td>0.808</td>
<td>0.453</td>
<td>0.244</td>
<td>0.495</td>
<td>0.852</td>
<td>0.011</td>
<td>0.026</td>
</tr>
<tr>
<td>$b \neq 1$</td>
<td>0.027</td>
<td>0.069</td>
<td>0.209</td>
<td>0.312</td>
<td>0.041</td>
<td>0.001</td>
<td>0.435</td>
</tr>
<tr>
<td>Residual variance</td>
<td>0.165</td>
<td>0.477</td>
<td>0.547</td>
<td>0.193</td>
<td>0.107</td>
<td>0.987</td>
<td>0.539</td>
</tr>
<tr>
<td>Misspec. error</td>
<td>1.511</td>
<td>1.018</td>
<td>0.262</td>
<td>1.087</td>
<td>1.705</td>
<td>0.002</td>
<td>0.227</td>
</tr>
<tr>
<td>Residual error</td>
<td>0.298</td>
<td>0.929</td>
<td>0.320</td>
<td>0.261</td>
<td>0.203</td>
<td>0.185</td>
<td>0.267</td>
</tr>
</tbody>
</table>

### 9. Conclusion

The first four sections of this paper formulated a dual purpose fund pricing function based on the option studies of Black–Scholes and Merton. The characteristics of this function were examined in order to explain the behavior of the share prices of the dual funds' claims. In particular it was found that under the idealized conditions within the model any closed-end fund should sell at a discount from its asset value. It is also not inconsistent to find the capital shares selling at a price below their net asset value. This latter fact, in particular, has not heretofore been fully appreciated.

The formulated model was tested in the later sections. There it was found that the model predicted price fluctuations in the capital shares quite well. There was, however, evidence of misspecification in the model. Several methods of eliminating the misspecification were examined. The two most promising alterations...
Fig. 7. Comparison of market prices and linear dividend model estimates.
were the inclusion of a simplified income tax in the model and a more realistic dividend policy. We can conclude that analysis of this type can be quite useful in the pricing of contingent claims other than the simple option contracts previously examined by Black and Scholes.

Further work is required to ascertain that these results were not influenced by the bias inherent in the ex-post specification process (i.e., in the correction for observed errors rather than a better ex-ante specification). Also study into the improved incorporation of asset value, rather than market value, in the pricing function would be desirable as this could be applied to all the closed end investment companies to explain the encountered discounts and premiums. In this context alternate payout policies and return dynamics that are other than log-normal could be tested by the numerical techniques used in this paper.

Appendix A

This appendix presents a formal derivation of the contingent claims pricing equation used in the text along the lines of the option pricing method of Merton (1973). The assumptions stated in section 2 are assumed throughout with the substitution of

(A-4) The price of a riskless in terms of default discount bond promising one dollar, $\tau$ periods from now is $P(\tau)$. The dynamics of the returns on this bond are described by

$$dP/P = \pi dt + \nu dz,$$

where $\pi$ and $\nu^2$ are the instantaneous expected return and variance of return on the bond and $dz$ is a Gauss–Wiener process.

If $F(V, P, \tau)$ is the contingent claim to be priced, then by Ito's lemma

$$dF = \beta Fdt + \lambda Fdz + \eta Fdz,$$  \hspace{1cm} (A.1)

$3^7$By writing $F(\cdot)$ as a function of only the $\tau$ period bond price, we are implicitly assuming that only the $\tau$ period interest rate is of concern in pricing this claim. In the case of an option on a stock paying no dividends this is a reasonable assumption since we know that the option will never be prematurely exercised and there are no interim considerations. In the general contingent claims case with cash disbursements, it is perhaps more reasonable to assume that the claim is a function of all the bonds whose maturities match payment dates. For the continuous payment case studied in this paper, we would then be presented a formidable problem. This problem can be surmounted by assuming that interest rates of all maturities are perfectly functionally related; in which case the inclusion of any one will give all the required information. A sufficient condition for bond prices to be perfectly functionally related is that the instantaneous compounding interest rate follow a Markov process and that the expectations hypothesis hold. See also the discussions in footnotes 4 and 38.
where

\[
\beta F \equiv \frac{1}{2} \sigma^2 V^2 F_{11} + \rho \sigma v V P F_{12} + \frac{1}{2} \nu^2 P^2 F_{22} + (\alpha V - C) F_1 + \pi P F_2 - F_3,
\]

\[
\lambda \equiv \sigma V F_1 / F,
\]

\[
\eta \equiv \nu P F_2 / F,
\]

\[
\rho \equiv \text{Cov} (dz, d\zeta).
\]

We now form a portfolio with \(W_1\) dollars invested in the asset, \(W_2\) dollars invested in the contingent claim, and \(W_3\) dollars invested in the bond. The dollar return on this portfolio will be

\[
\text{dx} = W_1 \frac{dV + Cdt}{V} + W_2 \frac{dF + cdt}{F} + W_3 \frac{dP}{P}.
\]  \(\text{(A.2)}\)

Constraining the portfolio to have zero net investment (i.e., \(W_1 + W_2 + W_3 = 0\)) and substituting for \(dV\), \(dF\), and \(dP\), we have

\[
\text{dx} = \{W_1 (\alpha - \pi) + W_2 (\beta - \pi + c/F)\} dt + \{W_1 \sigma + W_2 \lambda\} dz - \{W_1 \nu + W_2 (\nu - \eta)\} d\zeta.
\]  \(\text{(A.3)}\)

If we now choose \(W_1\) and \(W_2\) to eliminate the uncertainty in \(dx\), then the portfolio will be riskless and since it requires no investment, its expected return must be zero. These conditions can be expressed as the system of equations,

\[
(\alpha - \pi) W_1 + (\beta - \pi + c/F) W_2 = 0,
\]

\[
\sigma \quad W_1 + \lambda \quad W_2 = 0,
\]

\[
\nu \quad W_1 + (\nu - \eta) \quad W_2 = 0.
\]  \(\text{(A.4)}\)

A non-trivial solution set to \(\text{(A.4)}\) will exist only if

\[
(\beta - \pi + c/F)/(\alpha - \pi) = \lambda/\sigma = (\nu - \eta)/\nu.
\]  \(\text{(A.5)}\)

The first condition in \(\text{(A.5)}\) together with the definitions of \(\beta\) and \(\lambda\) implies that

\[
\frac{1}{2} \sigma^2 V^2 F_{11} + \rho \sigma v V P F_{12} + \frac{1}{2} \nu^2 P^2 F_{22} - CF_1 - F_3 + c
\]

\[
-\pi (F - VF_1 - PF_2) = 0.
\]  \(\text{(A.6)}\)
If the bond price dynamics are not stochastic then \( v = 0 \) and \( F_t = F_3 - rPF_2 \). If the term structure is flat then \( \pi = r \). In this case (A.6) reduces to

\[
\frac{1}{2} \sigma^2 V^2 F_{vv} + (rV - C)F_v - rF - F_t + c = 0,
\]

(A.6')

which is eq. (2) in the text. In the general case, (A.6) can be simplified since the term in parenthesis is zero from the second condition in (A.5) and the definitions of \( \lambda \) and \( \eta \),

\[
VF_t/F = \lambda/\sigma = (v - \eta)/\nu = (F - PF_2)/F.
\]

(A.7)

Therefore,

\[
\frac{1}{2} \sigma^2 V^2 F_{11} + \rho \nu \sigma VPF_{12} + \frac{1}{2} \nu^2 P^2 F_{22} - CF_1 - F_3 + c = 0.
\]

(A.8)

For a capital share \( c = 0 \) and making the substitutions\(^{38}\) \( X = e^{-\nu t} V \) and \( G(X, P, \tau) = F(V, P, \tau) \), we derive

\[
\frac{1}{2} \sigma^2 X^2 G_{11} + \rho \nu \sigma XPG_{12} + \frac{1}{2} \nu^2 P^2 G_{22} - G_3,
\]

(A.9)

which is identical to Equation (34) in Merton (1973); therefore, we can immediately write its solution,\(^{39}\)

\[
f^*(V, P, \tau) = EP(\tau) W(Z^*, U^*, 1, 0, 1),
\]

where

\[
Z^* = V e^{-\nu t}/EP(\tau),
\]

\[
U^* = \int_0^t \{ \sigma^2(s) - 2\rho(s)\nu(s)\sigma(s) + \nu^2(s) \} ds.
\]

(A.10)

Appendix B

In this appendix we shall demonstrate that even the capital shares of a dual fund holding only riskless assets and liable for no management fees may still be subject to a discount.

The dividend policy of a dual fund can be expressed in general as \( C = C(V, t) + \epsilon \). If the dividend policy is certain conditionally on \( V \) and \( t \) (i.e., the error is zero), then in the case of a riskless fund, the explicit value dependence may be

\(^{38}\)In the case of proportional payments assumed here, we have a stronger reason to ignore the shorter maturity bond prices in \( F(\cdot) \) since we may choose to work with the asset value net of payments, \( X \), rather than the raw asset value. For a further discussion on this point see footnote 4.

\(^{39}\)If the variance on the bond's return is a function of time only as assumed here and by Merton (1973) there is no guarantee that the \( \tau \) period interest rate will be positive.
suppressed since \( V \) itself is a deterministic function of time. If the dividend policy \( C(t) \) is an equilibrium policy that allows the income share offering price to be equal to \( E \), the redemption price, then the capital shares will sell at a discount at time \( t \) if

\[
\int_0^t c(s)e^{-rs}ds < E(1-e^{-r}) \tag{B.1}
\]

The left-hand side of (B.1) is the initial present value (i.e., at the offering time) of the dividends that will be paid through time \( t \). The right-hand side is the initial present value of a stream of dividends paid continuously at a constant rate of \( rE \) per unit time over the same time period. The two sides of eq. (B.1) must be equal at the offering date of the fund; hence, the capital shares will sell at a discount (premium) if the dividends on the income shares are in arrears (ahead) of the constant rate policy in a present value sense.

The proof of (B.1) comes directly from \( f = V - F, F(0) = E, f(0) = E \), and

\[
F(t) = Ee^{-r(t-t)}+\int_t^T c(s)e^{-r(s-t)}ds, \tag{B.2}
\]

from which it will be observed that \( f < V - E \) whenever (B.1) holds.

If the dividend payments have been small enough in the past, (B.1) is clearly a possible condition. The relevant question here is under what conditions it will

---

**Fig. 8.** Asset value of one capital plus one income share \((V)\), net asset value of capital share \((V-E)\), and price of capital share \((f)\) as a function of time.
hold for a fund paying proportional dividends at the equilibrium rate. Since the fund is riskless $V(t) = 2e^{(r-\gamma)T}$, $f(t) = e^{rt}$, and from (23a) $\gamma = r - \log [(1+e^T)/2]/T$. The capital shares will sell at a discount, whenever

$$[(1+e^T)/2]^{1/T} > [(1+e^t)/2]^{1/t},$$

but this is always the case since $T > t$ and $[(1+e^T)/2]^{1/T}$ is an increasing function of $t$.\(^{40}\) We have proved then that the capital shares of a riskless dual fund paying proportional dividends at the equilibrium rate will always sell at a discount from net asset value. The behavior of the discount over time can be seen in fig. 8. Early in the fund’s history it increases in size reaching a maximum, and then decreasing until it just disappears at the maturity date.

\(^{40}\)I thank D. Fehr for a proof of this.

References


Rosenberg, B., 1972, The behavior of random variables with non-stationary variance and the distribution of security prices, Graduate School of Business Working Paper no. 11 (University of California, Berkeley, Calif.).

