Optimal consumption and portfolio rules with intertemporally dependent utility of consumption*

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In the usual consumption portfolio problem, lifetime utility is assumed to be time-additive. This assumption has been criticized for failing to capture important intertemporal dependencies in utility such as intertemporal risk aversion and habit formation. This paper studies the consumption portfolio problem for a class of intertemporally dependent utility functions.

1. Introduction

In the usual consumption portfolio problem, lifetime utility is assumed to be time-additive. The time-additive representation has been criticized as not capturing intertemporal risk aversion or intertemporal complementarity and substitution effects.

This paper examines the continuous-time consumption portfolio problem from the viewpoint of an investor whose utility function is not time-additive. By allowing current felicity to be affected by past consumption these effects can be captured in an expanded representation of lifetime utility which can still be solved by standard stochastic dynamic programming methods.

Section 2 discusses the nature of intertemporal dependencies in utility. Section 3 introduces a linear felicity function that captures these dependencies. Sections 4 and 5 solve the consumption portfolio problem for an infinite and finite horizon, respectively, and discuss the differences with the standard model. Section 6 examines a special case of this utility representation which is consistent with the Hindy–Huang–Kreps desirata.

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The criticism on the absence of 'complementarity' between consumption at different points in time has been less precisely focused. For example, Hicks (1965, p. 261) states: 'It is nonsense to assume that successive consumptions are independent; the normal condition is that there is strong complementarity between them.'

Clearly, Hicks and others who have criticized the lack of complementarity have not used this term in the same sense commonly employed in consumer demand theory. For example, consider the time-additive isoelastic family $U[C, t] = e^{-\rho t}C^\gamma / \gamma$. The marginal rate of substitution between consumption at two dates, $t$ and $s$, is

$$\frac{dC(s)}{dC(t)} = -\frac{\partial T / \partial C(t)}{\partial T / \partial C(s)} = -e^{-\rho(t-s)} \left( \frac{C(t)}{C(s)} \right)^{\gamma-1}. \quad (3)$$

For $\gamma = 1$ (risk neutrality), the marginal rate of substitution is constant, and, apart from the time discounting, consumption at any date is a perfect substitute for that at any other. At the extreme of risk aversion (as $\gamma \to -\infty$), the marginal rate of substitution goes to 0 or $-\infty$ depending on whether $C(t) \geq C(s)$, and consumption at any date is a strict complement to that at any other. Thus, time-additive utility can admit to any degree of complementarity or substitution across time. This is also true of the multiplicatively separable utility representation.\(^4\)

The concern over the lack of intertemporal complementarity is rather an objection to the independence of the marginal rate of substitution between consumption at two dates to the level of consumption at other dates. Hicks (1965, ibid.) states this as: 'The sacrifice [in current consumption] which one would be willing to make to fill a gap [in the consumption stream at some later time] must normally be much greater than what it would be worth while to incur for a mere extra.' Because of the independence properties inherent in the additive and multiplicative representations, it is clear that the marginal rate of substitution between consumption at any two dates does not depend on the level of consumption at any other dates and, in this sense the preferences do not display intertemporal complementarity.\(^5\)

One way to introduce intertemporal dependence is to admit the possibility that the contribution to lifetime utility by a single period's felicity depends on

\(^4\)For example, take $T[\ln(C(t))] = \prod_i \exp(C_i / \gamma)$. Then the marginal rate of substitution is $-\{C(t)/C(s)\}^{\gamma-1}$ with the same interpretation as (3).

\(^5\)This point has been emphasized by some authors who choose to use the expression 'habit formation' in place of complementarity. We shall continue to use 'complementarity' as no similar expression for its counterpart 'substitution' has arisen in the literature.
past as well as current consumption, i.e.,

$$T \left[ \langle C(t) \rangle^T_0 \right] = \sum_{t=0}^{T} U \left[ \langle C(s) \rangle^T_0, t \right].$$

(4)

We wish time $t$ felicity, $U[\langle C(s) \rangle^t_0, t]$, to measure the contribution to lifetime utility as assessed at time $t$; hence it cannot depend on consumption after that date. This feature as well as the assumed additivity in (4) entails no loss of generality as lifetime utility could always be expressed by identifying the final felicity function with lifetime utility; i.e.,

$$U \left[ \langle C(s) \rangle^t_0, t \right] = \begin{cases} U \left[ \langle T(t) \rangle^T_0 \right], & t = T, \\ 0, & t \neq T. \end{cases}$$

(5)

The representation in (4) is obviously not unique because canceling terms can be added to two felicity functions. For example, time-additive utility can be represented as in (5) or in the usual fashion as $U[\langle C(s) \rangle^t_0, t] = U[C(t), t]$.

It might also be desirable to impose additional structure by requiring that the importance of past consumption in the felicity function be stationary, e.g.,

$$U \left[ \langle C(s) \rangle^t_0, t \right] = e^{-\rho \tau} u[C(t), x(t)],$$

(6)

where

$$x_i(t) = \sum_{s=1}^{t} \theta_i(s) C(t-s) \quad \text{with} \quad \theta_i(s) \geq 0.$$

The vector $x$ consists of a set of positively weighted averages of past levels of consumption with weighting functions $\theta_i(\cdot)$.\(^6\) Note that the weighting function for $x_i(t)$ depends only on lags and not on calendar time, $t$. This and the constant rate of time preference is how stationarity has been introduced. With stationarity the marginal contribution of time $t$ consumption to time $t + \tau$ felicity is the same for all $t$ up to a proportional discount factor.

\(^6\)See Ryder and Heal (1973) for an introduction to the formulation used here in the context of optimal growth under certainty.

\(^7\)In a discrete time problem the use of the state variables $x$ is simply one of notational convenience. In a continuous time problem, it entails the implicit assumption that what is of importance in the past history of consumption $\langle C(s) \rangle^T_0$ can be summarized by a finite number of pieces of data. With no loss of generality we assume that each weighting function $\theta_i(\cdot)$ is everywhere nonnegative, $\theta_i(\cdot) \geq 0$. This can always be assured by separating a component of $x$ with any negative weights into two parts: $x^{+}$ and $x^{-}$ with $\theta^{+}(s) = \max[\theta(s), 0]$, $\theta^{-}(s) = \max[-\theta(s), 0]$, respectively. We will also assume that each $\theta_i(\cdot)$ has total measure of unity so that each $x_i(t)$ is a weighted average. This is merely a normalization.
For the representation in (6) the marginal contribution of time $t$ consumption to lifetime utility is

$$\frac{\partial T}{\partial C(t)} = e^{-\rho t} u_C[C(t), x(t)]$$

$$+ \sum_{t} \sum_{s=t+1}^{T} e^{-\rho s} u_{x_s}[C(s), x(s)] \theta_t(s-t),$$  \hspace{1cm} (7)$$

where subscripts on $u$ indicate partial derivatives. Clearly intertemporal dependence has been introduced into the utility representation. Under what conditions this dependence takes the form of intertemporal substitution, complementarity, or risk aversion remains to be shown.

The definition of intertemporal risk aversion was given previously. One natural way to define intertemporal complementarity and substitution is to say that state variable $x_{i}$ contributes to intertemporal complementarity (substitution) at time $t$ if the marginal rate of substitution between $C(t)$ and $x_{i}(t)$ in the felicity function $u[C(t), x(t)]$ is negative (positive). By assumption $u_{C(\cdot)} > 0$, so $x_{i}$ contributes to intertemporal complementarity (substitution) if $u_{x_{i}}$ is negative (positive).

Under this definition intertemporal complementarity means that a high level of consumption at time $s$ creates subjective anticipations about the future so that an even higher level of consumption is required at time $t (s)$ to achieve the same felicity. In other words, a decrease in consumption causes a form of felicity regret in addition to the direct felicity consequences of the lower consumption. With intertemporal substitution, past consumption creates a partial 'satiation' that keeps felicity in the future high even with lowered current consumption.

Because we are defining intertemporal substitution and complementarity with respect to the felicity function rather than lifetime utility, the definitions are not symmetric in time. That is, for $s < t$, $C(s)$ can substitute (or be a complement) for $C(t)$ but not vice versa, because $C(t)$ does not appear in the time $s$ felicity function. This is another distinction between our use of the terms complement and substitute and their standard usage.

If a given consumer's choices are characterized by both intertemporal substitution and complementarity, we would typically think of substitution as being the shorter-run phenomenon. That is, abnormally high consumption in the recent past would satiate the consumer and increase current felicity, while sustained high past consumption (or even momentary high consumption in the more distant past) would lower current felicity by creating habits or raising anticipations.
Mehra and Prescott (1985), Merton (1980), Shiller (1981), and others find results which are puzzling or seem incompatible when viewed in the context of models based on additive utility structures.

In a study of portfolio holdings Blume and Friend concluded that the relative risk aversion of the market is around 2. Merton's study using the distribution of returns on the market portfolio reached a similar conclusion. On the other hand, Hall found that the intertemporal elasticity of substitution for consumption at two dates is around 0.1. For the time-additive utility model the marginal rate of substitution across periods and the relative risk aversion are closely related as shown in (3). Unless the rate of time preference is huge these two numbers are inconsistent.

Using a consumption model, Mehra and Prescott determined that the excess expected rate of return on equity over the period 1889–1978 of 6.18% is similarly inconsistent with the growth of per capita consumption over the same period. In particular they conclude that this premium should be no larger than 0.35%. As a result of his work Shiller (1981) claims that the stock market is too variable.

3. Linear felicity

To explore the differences in these utility representations we examine a special class of felicity functions. We assume felicity is a function of some linear combination of consumption and the taste variables in \( x \). For notational simplicity, we assume that the state variable vector \( x \) has only two components, one capturing the effects of intertemporal substitution and the other capturing the effects of complementarity,

\[
u(C, x) = \nu(ac + b_1x_1 - b_2x_2),\]

with \( a, b_1, b_2 \geq 0 \), \( \nu'(\cdot) > 0 \), and \( \nu''(\cdot) < 0 \). For the standard case, \( a > 0 \) and \( b_i = 0 \). For the Hindy–Huang–Kreps representation, \( a = 0 \).

The marginal rates of substitution between \( C(t) \) and \( x_i(t) \) holding felicity constant are

\[
\frac{\partial C}{\partial x_1} = \frac{b_1}{a} < 0, \quad \frac{\partial C}{\partial x_2} = \frac{b_2}{a} > 0.
\]

The first component of \( x \) thus represents the intertemporal substitution effect. The second embodies intertemporal complementarity. Note that for the linear representation the marginal rates of substitution are constant. In general \( b_1 \) measures the tolerance to intertemporal substitution; the larger is \( b_1 \), the more easily can past consumption substitute for current consumption. The strength of intertemporal habit formation is measured by \( b_2 \). The larger is \( b_2 \), the more regret does past consumption induce.
The conditions for intertemporal risk aversion or preference are not easy to assess even for the linear felicity case, the cross partial derivative for \( \tau > t \) is

\[
\frac{\partial^2 \mathcal{T}}{\partial C(t) \partial C(\tau)} = e^{-\rho \tau} a \left[ b_1 \theta_1(\tau - t) - b_2 \theta_2(\tau - t) \right] \nu'' \\
+ \sum_{s=\tau+1}^{\tau} e^{-\rho s} \left[ b_1 \theta_1(s - t) - b_2 \theta_2(s - t) \right] \\
\times \left[ b_1 \theta_1(s - \tau) - b_2 \theta_2(s - \tau) \right] \nu'' ,
\]

(11)

where subscripts on \( \nu \) indicate the time at which the felicity function is evaluated, and primes denote derivatives. Clearly intertemporal substitution alone, \( b_2 = 0 \), or its dominance over intertemporal complementarity, \( b_1 \theta_1(s) > b_2 \theta_2(s) \) (\( \forall s \)), leads to global intertemporally risk-averse behavior. Intertemporal complementarity alone can create either intertemporal risk aversion or preference. The reason for this difference is that there are two effects of intertemporal dependence. The immediate (lag 1) effect, on the first line of (11), is negative for substitution and positive for complementarity. The delayed effect, on the second and third lines of (11), is negative for either pure substitution or complementarity, but can be positive when both characteristics are present.

To illustrate these effects of intertemporal complementarity and substitution on intertemporal risk aversion consider the following example. The state variable \( x \) has a single component with a weighting function concentrated at the first two lags, \( \theta(1) = \delta > 0 \) and \( \theta(2) = 1 - \delta \geq 0 \). The rate of time preference is zero. The felicity function is \( \nu(C + bx) \). The consumer is faced with a choice among two gambles. The first gives equal chances at consumption streams \( \langle H, H, C \rangle \) or \( \langle L, L, C \rangle \) where \( H > L \). The second gives equal chances at consumption streams \( \langle H, L, C \rangle \) or \( \langle L, H, C \rangle \). As stated earlier, a consumer with intertemporal risk aversion will prefer the latter.

The expected utility of these gambles are, respectively,

\[
\frac{1}{2} \left[ \nu(H) + \nu(H + b \delta H) + \nu(C + b \delta H + b(1 - \delta) H) \right] + \frac{1}{2} \left[ \nu(L) + \nu(L + b \delta L) + \nu(C + b \delta L + b(1 - \delta) L) \right],
\]

\[
\frac{1}{2} \left[ \nu(H) + \nu(L + b \delta H) + \nu(C + b \delta L + b(1 - \delta) H) \right] + \frac{1}{2} \left[ \nu(L) + \nu(H + b \delta L) + \nu(C + b \delta H + b(1 - \delta) L) \right].
\]

(12)

The first and fourth terms of these two expressions match. The second and fifth terms come from the effect of the most recent consumption and can be expressed as the simple lottery for an individual with a single-period utility.
function $\nu$,

\begin{align*}
\text{Intertemporally More Risky} & \quad \text{Intertemporally Less Risky} \\
\frac{\mu_1 + \sigma(1 + \delta b)}{2} & \quad \frac{\mu_1 + \sigma(1 - \delta b)}{2} \\
\frac{\mu_1 - \sigma(1 + \delta b)}{2} & \quad \frac{\mu_1 - \sigma(1 - \delta b)}{2}
\end{align*}

where

\[ \mu_1 \equiv (1 + \delta b) \frac{H + L}{2}, \quad \sigma \equiv \frac{H - L}{2}. \]

The positivity of $\delta$ implies $|1 + \delta b| \geq |1 - \delta b|$ for $b \geq 0$, so based just on this comparison, intertemporal substitution leads to intertemporal risk aversion while intertemporal complementarity leads to intertemporal risk preference.

The third and sixth terms come from the effect of twice lagged consumption. They can be expressed as

\begin{align*}
\text{Intertemporally More Risky} & \quad \text{Intertemporally Less Risky} \\
\frac{\mu_2 + \sigma b}{2} & \quad \frac{\mu_2 + \sigma b(1 - 2\delta)}{2} \\
\frac{\mu_2 - \sigma b}{2} & \quad \frac{\mu_2 - \sigma b(1 - 2\delta)}{2}
\end{align*}

where

\[ \mu_2 \equiv C + b \frac{H + L}{2}, \quad \sigma \equiv \frac{H - L}{2}. \]

If $\delta = 1$ (that is only the immediate lag is important), then these last two lotteries are the same, and only the previous comparison matters. Otherwise $|1 - 2\delta| < 1$, so all investors will tend to weakly prefer the intertemporally less risky gamble. For investors who display intertemporal substitution these two desires align. For investors who display intertemporal complementarity these two desires are at odds.

4. The consumption portfolio problem

This section analyzes the consumption portfolio problem for an investor with an intertemporally dependent utility function. To compare our answers to those obtained by others we will work in a continuous time model whose standard solutions are known in closed form. The state variables $x_i(t)$ measuring the effect of past consumption on current felicity are integrals of past consumption. For computational ease we confine our attention to exponentially smoothed averages,

\[ x_i(t) = \kappa_i e^{-\kappa_i t} \int_{-\infty}^{t} e^{\kappa_i \tau} C(\tau) \, d\tau = \kappa_i \int_{0}^{\infty} e^{-\kappa_i s} C(t - s) \, ds, \quad (13) \]
dynamic budget constraint \( dW(t) = ([r + w(t)(u - r)]W(t) - C(t)) \, dt + w(t)\sigma W(t) \, d\omega \), where \( w(t) \) is the fraction of wealth invested in the risky asset at time \( t \).

We first solve this portfolio consumption problem for an investor who is infinitely-lived and has a constant rate of time preference \( \rho \).\(^{10}\) We introduce the derived utility function

\[
J(W, x) = \max_{C, w} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} u[C(s), x(s)] \, ds. \tag{15}
\]

The choice variables are the consumption expenditure and the fraction of wealth invested in the risky asset. Note that in determining the derived utility, the subjective discounting is done only from time \( t \). This together with an infinite horizon and time-invariant dynamics ensures that the derived utility function \( J(\cdot) \) is not explicitly time-dependent.

The partial differential equation for \( J \) derived from the intertemporal budget constraint and the evolution of \( x \) is

\[
0 = u(C, x) + \frac{1}{2} w^2 \sigma^2 W^2 J_{WW} + \left( [r + w(\mu - r)]W - C \right) J_W \\
- \rho J + \kappa_1 (C - x_1) J_{x_1} + \kappa_2 (C - x_2) J_{x_2}. \tag{16}
\]

The first-order conditions of the consumer’s maximization problem are

\[
0 = u_C - J_W + \kappa_1 J_{x_1} + \kappa_2 J_{x_2}, \tag{17a}
\]

\[
0 = w \sigma^2 W^2 J_{WW} + (\mu - r) W J_W. \tag{17b}
\]

The former is a modification of the standard envelope condition. The latter gives the usual result that the optimal portfolio weight is proportional to the ratio of expected excess return to variance and inversely proportional to the investor’s relative risk aversion.

\(^{10}\)The following assumptions are made about the parameters:

(i) \( \rho > \max(0, \gamma(r + \frac{1}{2}\lambda^2/(1 - \gamma))) \),

(ii) \( a + \kappa_1 \phi_1 - \kappa_2 \phi_2 > 0 \) where \( \phi_i = b_i/(r + \kappa_i) \),

(iii) \( W(0) > \frac{1}{a + \kappa_1 \phi_1 - \kappa_2 \phi_2} \left[ \frac{\eta}{r} - \phi_1 x_1(0) + \phi_2 x_2(0) \right] \).

The first inequality is the standard transversality condition. It assures that the expectation in (15) is bounded for the optimal plan. Conditions (ii) and (iii) are required to guarantee that some feasible plan exists. If either is violated, there exists no consumption portfolio plan which assures consumption can exceed the subsistence level of \( [\eta - b_1 x_1(t) + b_2 x_2(t)]/a \) at every point in time. The right-hand side of (iii) is the subsistence level of wealth as defined later in (20).
To obtain a closed form solution we now assume that the felicity function is in the HARA or linear risk tolerance class,\textsuperscript{11}
\[
\nu(aC + b_1x_1 - b_2x_2) = \frac{1}{\gamma} (aC + b_1x_1 - b_2x_2 - \eta)^{\gamma}.
\] (18)

The solution to (16) for HARA felicity is\textsuperscript{12}
\[
J(W, x) = \frac{1}{\gamma} B^{\gamma-1} \left[ (a + \kappa_1 \phi_1 - \kappa_2 \phi_2)W + \phi_1 x_1 - \phi_2 x_2 - \frac{\eta}{r} \right]^{\gamma}.
\] (19)

where
\[
B \equiv \frac{1}{1 - \gamma} \left[ \rho - \gamma r - \frac{1}{2} \lambda^2 \frac{\gamma}{1 - \gamma} \right] > 0,
\]
\[
\lambda \equiv \frac{\mu - r}{\sigma} > 0, \quad \phi_i \equiv \frac{b_i}{r + \kappa_i} > 0.
\]

The parameter $\lambda$ is the usual measure of the price of risk. The positivity of $B$ follows from the standard transversality condition.

At any time the investor holds three 'resources' which contribute to utility. The first resource is wealth. The other two 'resources' are past levels of consumption which contribute directly to future utility through the taste state variables $x_i$. In the absence of any future consumption, $x_i$ decays deterministically at the rate $\kappa_i$. Therefore, using the standard dividend growth model we can value the future contribution of $x_i$ to the argument of the derived utility function as $\pm b_i/(r + \kappa_i) = \pm \phi_i$ per unit, where the positive and negative signs apply to $x_1$ and $x_2$, respectively. This is the interpretation of the terms $\pm \phi_i x_i$ in $J(\cdot)$.

The contribution of wealth to utility is proportional to $a + \kappa_1 \phi_1 - \kappa_1 \phi_1$. This quantity can be interpreted as follows. Suppose all remaining wealth were consumed immediately. This would have a direct contribution to the

\textsuperscript{11}The HARA class includes the power ($\eta = 0$) and logarithmic ($\eta = \gamma = 0$) as special cases. [Quadratic and negative exponential utility are other special cases which can be obtained with a more general formulation. See Merton (1971)]. $\eta$ determines the subsistence level.

\textsuperscript{12}For log utility the derived utility of wealth function, $J(\cdot)$, is the limit as $\gamma \to 0$. For exponential utility, $J(\cdot)$ is
\[
J(W, x) = -\frac{1}{r} \exp \left[ -r(a + \kappa_1 \phi_1 - \kappa_2 \phi_2)W - r\phi_1 x_1 + r\phi_2 x_2 \right],
\]
and in the latter case there is no subsistence wealth; $\hat{W} = -\infty$. 
argument of the felicity function of \(aW\). In addition, from (13) \(x_1\) and \(x_2\) would increase by \(\kappa_1 W\) and \(\kappa_2 W\), respectively. Since we know that \(x_i\) contributes \(\pm \phi_i\) per unit, the contribution of wealth within the felicity function is equal to \(a + \kappa_1 \phi_1 - \kappa_2 \phi_2\) per unit of wealth.\(^{13}\)

The derived utility function is defined only for wealth in excess of a subsistence level,\(^{14}\)

\[
W > \hat{W}(x) = \frac{1}{a + \kappa_1 \phi_1 - \kappa_2 \phi_2} \left( \frac{\eta}{r} - \phi_1 x_1 + \phi_2 x_2 \right).
\]  

(20)

In the absence of any intertemporal utility dependencies, the subsistence level of consumption is \(\eta/a\). To ensure this level of consumption for all time requires wealth equal to the present value of a perpetual annuity at this rate, i.e., \(\eta/ar\). With the assumed intertemporal dependencies, felicity is defined only if consumption at time \(t\) is at a rate of at least \([\eta - b_1 x_1(t) + b_2 x_2(t)]/a\). However, this level of consumption at time \(t\) affects \(x\) at later times and, therefore, the later subsistence levels of consumption. \(\hat{W}(x)\) is the present value of all future subsistence consumption given the current levels of the averages \(x(t)\).\(^{15}\) The term \(a + \kappa_1 \phi_1 - \kappa_2 \phi_2\) in the denominator serves to convert this present value into wealth 'units'.

The effect of intertemporal substitution is to reduce the subsistence level of wealth, while complementarity increases it. If substitution is the short-term effect, then high consumption in the recent past (high \(x_1\)) will decrease the subsistence level while high consumption over a long period (high \(x_2\)) will increase it. An investor can abide lower current consumption with high consumption in the recent past since the latter is still contributing to current

\(^{13}\)Given our other assumptions, marginal utility of wealth is positive if and only if \(a + \kappa_1 \phi_1 - \kappa_2 \phi_2 > 0\). This condition is guaranteed by (ii) in footnote number 10.

\(^{14}\)If this subsistence constraint is met at time \(t\), then the optimal portfolio behavior described later will assure that it is met at all future dates as well. This could be proved by examining the stochastic process for wealth under the optimal controls to show that \(\hat{W}(\cdot)\) is inaccessible. However, since the marginal utility of wealth is infinite at the subsistence level, it is clear that the optimal behavior will invest a sufficient quantity in the risk-free asset to maintain wealth strictly above the subsistence level.

\(^{15}\)To see this consider the case with just substitution. Let \(C'_t\) denote the minimum required level of consumption at time \(s\), \(C'_t = (\eta - bx)/a\). Let \(x'_t\) denote the level of the taste state variable at time \(s\) conditional on \(x'_s = x_s\), if the above minimum is consumed each period from \(t\) through \(s\). Then from (13), \(dx'_t/\eta = \kappa(C'_t - x'_t)\) and

\[
x'_s = \frac{\eta}{a + b} \left[ 1 - e^{-\kappa s}\phi e^{b(x'_s - s)} \right] + x_s e^{-\kappa s}\phi e^{b(x'_s - s)}.
\]

The subsistence level of wealth is

\[
\hat{W}_t = \frac{1}{a} \int^t e^{-\kappa (t-s)} [\eta - bx'_s] ds = \frac{1}{a + \kappa \phi} \left[ \frac{\eta}{r} - \phi x_s \right],
\]

which is the quantity given in (20) for \(b_2 = 0\).
wealth.\textsuperscript{17} The greater is the current influence of intertemporal substitution (larger $x_1$), the smaller is the relative risk aversion. The greater is the current influence of intertemporal complementarity (larger $x_2$), the larger is the relative risk aversion. When wealth is large compared to the subsistence level, utility is close to isoelastic with relative risk aversion $1 - \gamma$. For lower levels of wealth, risk aversion is larger approaching $\infty$ in the limit as $W \downarrow \hat{W}$.

A comparative static increase in $\gamma$ will reduce relative risk aversion. From (21) it is obvious that the remaining comparative statics for risk aversion are the same as those for the subsistence level of wealth.

The optimal portfolio and consumption rules are

\begin{align}
  w^* &= \frac{1}{R(W, x)} \frac{\mu - r}{\sigma^2}, \\
  C^* &= \frac{1}{a} \left[ B(a + \kappa_1 \phi_1 - \kappa_2 \phi_2)W + (B\phi_1 - b_1)x_1 \\
  &\quad - (B\phi_2 - b_2)x_2 + \frac{r - B}{r - \eta} \right].
\end{align}

The investor's portfolio holdings can be separated into three parts. The dollar holding of the safe asset is $(1 - w^*)W = \hat{W} + (1 - w^*) (W - \hat{W})$, where $w^*$ is the optimal risky holding for an investor with a time-additive isoelastic utility function, $U(C, t) = e^{-\rho t} C^\gamma / \gamma$. The dollar holding of the risky asset is $w^*W = w^*((W - \hat{W})$. Thus, the subsistence level of wealth is invested in the safe asset and the remainder is split between the safe and risky assets based on a risk aversion of $1 - \gamma$.

The investor's response to comparative static changes in the returns parameters of the risky asset are the usual ones. An increase in the expected rate of return, $\mu$, or a decrease in the standard deviation, $\sigma$, will elicit an increase in the wealth allocated to the risky asset. The effect of a change in the interest rate is uncertain. The natural tendency (i.e., price-substitution effect) is to shift from the risky asset into the safe asset at higher interest rates. However, even in the absence of intertemporal dependencies, risk aversion can increase or decrease with a change in $r$. Thus, we cannot be sure that this 'natural' tendency is followed. The remaining comparative statics effects on the optimal portfolio are the same as those for the subsistence level of wealth.

\textsuperscript{17}In the case of exponential utility of consumption, relative risk aversion is independent of the two consumption state variables; however, it is still affected by the presence of intertemporal effects $R(W, x) = (a + \kappa_1 \phi_1 - \kappa_2 \phi_2)W$. 
Optimal consumption is linear in wealth and the state variables. Intertemporal dependencies introduce three types of effects into consumption behavior. First, past consumption is a direct surrogate for current consumption. An increase in past consumption \((x_1)\) decreases the marginal felicity of current consumption by altering the subsistence level of consumption as discussed previously. This decrease in marginal felicity is equivalent to an increase in the 'price' of consumption, so the investor consumes less currently substituting in its place consumption at a later date. This effect contributes the term \(-b_1 x_1/a\) (and \(b_2 x_2/a\) for intertemporal complementarity) to (22b).

Along with the surrogate effect, intertemporal dependencies have prosperity effects on consumption. The decrease in subsistence wealth from a higher \(x_1\) is like an increase in income or wealth – a part of which the investor allocates to current consumption.

The net prosperity and surrogate effect for intertemporal substitution is \([B/(r + \kappa_1) - 1]b_1 x_1/a\), which can be either negative or positive. The prosperity effect (higher consumption in the past increases current consumption) dominates if \(\rho - r - \frac{1}{2} \gamma x^2/(1 - \gamma) - (1 - \gamma) \kappa_1 > 0\). Thus, a low interest rate tends to favor the prosperity effect as the 'price' of later consumption is then higher. Similarly, a low rate of time preference tends to favor the surrogate effect. A long memory for intertemporal substitution (small \(\kappa_1\)) also favors the prosperity effect. For intertemporal complementarity the opposite is true.

Finally, intertemporal dependencies indirectly affect consumption by altering the marginal propensity to consume. The marginal propensity to consume out of wealth is \(B(a + \kappa_1 \phi_1 - \kappa_2 \phi_2)/a\), which is positive. The more tolerant the investor is to substitution providing for felicity (the larger is \(b_1\) and therefore \(\phi_1\)), the higher is the marginal propensity to consume. With a high tolerance for substitution an investor can risk consuming more currently as this will sustain felicity in the future. Similarly, a longer memory for substitution, smaller \(\kappa_1\), will lead to a lower marginal propensity to consume. The intuition here is with a long memory even a small amount of consumption leads to a high level of lifetime utility, and at the margin it is better to save more wealth to cover future emergencies.

On the other hand, the higher is the aversion to regret (the larger is \(b_2\)), the lower is the marginal propensity to consume since failure to maintain this level will cause later distress. A longer memory for regret, smaller \(\kappa_2\), results in a higher marginal propensity to consume.

An increase in the appreciation of consumption, \(a\) (or equivalently proportionate decreases in both the tolerance for substitution, \(b_1\), and the strength of regret, \(b_2\)), will increase or decrease the marginal propensity to consume as the strength of complementarity, \(\kappa_2 \phi_2\), exceeds or falls short of the.

\(^{18}\text{This surrogate effect is like the substitution effect in standard price theory. The prosperity effect is similar to the income effect. We will not use these terms to avoid confusion with intertemporal substitution.}\)
strength of substitution, $\kappa_1\phi_1$. Recall that $\kappa_1\phi_1 - \kappa_2\phi_2$ is the present value of the contribution of the intertemporal effects of consuming one unit of wealth, while $a$ is the direct contribution. If the former quantity is negative, then in the optimization process the investor is compromising between the benefit from consumption, $a$, and the harm, $\kappa_1\phi_1 - \kappa_2\phi_2$. An increase in $a$ will shift this compromise in the direction of more consumption. For $\kappa_1\phi_1 - \kappa_2\phi_2 > 0$, the intuition is just the opposite.

Thus, the response to a windfall gain in wealth by an investor with a high tolerance to intertemporal substitution or a short 'substitution memory span' will be large, and he or she will consume initially at a rate that he or she will not expect to be able to maintain. The response to a windfall gain by an investor with great intertemporal complementarity or a short 'memory span for regret' will be small, and he or she will consume initially at a rate upon which he will expect to be able to improve. The remaining comparative statics for the marginal propensity to consume are the same as in the standard model. All of the comparative statics for the marginal propensity to consume and the subsistence level of wealth are summarized in table 1.

<table>
<thead>
<tr>
<th>Increase in the parameter and description</th>
<th>Effect on</th>
<th>Marginal propensity to consume</th>
<th>Subsistence wealth$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ Appreciation of consumption</td>
<td>$\uparrow$ for $\kappa_2\phi_2 &gt; \kappa_1\phi_1$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$b_1$ Tolerance for substitution</td>
<td>$\uparrow$</td>
<td>$\downarrow$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$b_2$ Strength of regret</td>
<td>$\downarrow$</td>
<td></td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$\kappa_1$ Forgetfulness for substitution</td>
<td>$\uparrow$</td>
<td></td>
<td>$\uparrow$ for $x_1 &gt; r\hat{W}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\downarrow$ for $x_1 &lt; r\hat{W}$</td>
</tr>
<tr>
<td>$\kappa_2$ Forgetfulness for regret</td>
<td>$\downarrow$</td>
<td></td>
<td>$\uparrow$ for $x_2 &gt; r\hat{W}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\downarrow$ for $x_2 &lt; r\hat{W}$</td>
</tr>
<tr>
<td>$\rho$ Rate of time preference</td>
<td>$\uparrow$</td>
<td></td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$\mu$ Expected rate of return on risky asset</td>
<td>$\downarrow$ for $\gamma &gt; 0^b$</td>
<td></td>
<td>$\downarrow$ for $\gamma &lt; 0^b$</td>
</tr>
<tr>
<td>$\sigma$ Risk of risky asset</td>
<td>$\uparrow$ for $\gamma &gt; 0$</td>
<td></td>
<td>$\downarrow$ for $\gamma &lt; 0$</td>
</tr>
<tr>
<td>$r$ Interest rate</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
</tr>
</tbody>
</table>

$^a$Comparative statics for risk aversion have opposite sign.

$^b$If $\mu < r$, then this comparative static has the opposite sign.
Fig. 2. This figure illustrates the consumption for an investor with exponential utility with intertemporal complementarity. The return on wealth is exactly equal to its expected value each period except for two instants. At time 2, wealth increases by 25. At time 10, wealth decreases by 25. The utility function is \(-\exp\left[-(C - 0.3x_2)\right]\) with a rate of time preference \(\rho = 16\%\) and forgetfulness of \(\kappa_2 = 0.2\).

The consumption pattern of the investor characterized by intertemporal complementarity is plotted in fig. 2. For this investor the opposite story is true. The initial level of consumption is higher due to lower risk aversion. The immediate effects of the unexpected changes in wealth are smaller than in the standard case, but they would accumulate to the same long-run net change. The trend is more gradual than in fig. 1 since we have assumed complementarity is a longer-term effect (\(\kappa_1 > \kappa_2\)). With intertemporal complementarity investors do not consume the same fraction of a windfall since failure to maintain this high level of consumption would cause regret. This extra savings permits them to consume at a higher rate after an unexpected decrease in wealth.

The consumption pattern of the investor characterized by both intertemporal complementarity and substitution is plotted in fig. 3. For this investor substitution characterizes short-run behavior while complementarity characterizes longer-term behavior. Again the long-run total change in consumption caused by a windfall is the same. The investor responds to the windfall in wealth by increasing consumption by an intermediate amount. Consumption then falls in the short run, while substitution dominates and then later rises as complementarity takes effect. For different parameter values the optimal response to a shock can result in an oscillating consumption stream in this case.

As these figures illustrate, intertemporal substitution increases and intertemporal complementarity decreases the volatility of consumption. This
Fig. 3. This figure illustrates the consumption for an investor with exponential utility with both intertemporal substitution and complementarity. The return on wealth is exactly equal to its expected value each period except for two instants. At time 2, wealth increases by 25. At time 10, wealth decreases by 25. The utility function is \(- \exp[-(C + 0.9x_1 - 0.3x_2)]\) with a rate of time preference \(\rho = 16\%\) and forgetfulness of \(\kappa_1 = 1, \kappa_2 = 0.2\).

has implications for the question of 'excess' volatility in the stock market raised by Shiller (1981) and others. The dynamics of wealth and consumption are

\[
dW = \mu_W W \, dt = \frac{\lambda}{R(\cdot)} W \, d\omega, \tag{23a}
\]

\[
dC^* = \mu_c \, dt + \frac{B \lambda}{R(\cdot)} \left[ 1 + \frac{\kappa_1 \phi_1 - \kappa_2 \phi_2}{a} \right] W \, d\omega. \tag{23b}
\]

In the standard model the ratio of the variance of consumption to the variance of wealth is \(B^2\). [Note that the true value of \(B\) as given in (19) is independent of any intertemporal dependencies]. However, if the market is characterized by intertemporal substitution or complementarity, this ratio is

\[
\frac{\text{var}(dC)}{\text{var}(dW)} = B^2 \left[ 1 + \frac{\kappa_1 \phi_1 - \kappa_2 \phi_2}{a} \right]^2. \tag{24}
\]

Therefore, if we ignored intertemporal dependencies and used the ratio of standard deviations to infer an estimate of \(B\) and from it \(\lambda\) or \(\gamma\), we would misestimate their true values. We could easily come to the conclusion that...
the relative variabilities of the stock market and of consumption were inconsistent with estimated risk aversion.

5. Horizon effects for a finitely-lived investor

The qualitative properties of the consumption portfolio problem remain unaltered if the investor has a finite horizon $T$. The only changes are that the subsistence level of wealth, the marginal propensity to consume, and the effects of substitution and complementarity become time-dependent. The equation for the derived utility is the same with the partial derivative $\partial J/\partial t$ replacing the term $-\rho J$ in (16).

We will examine the case of a logarithmic felicity investor characterized by intertemporal complementarity, i.e., $u(C, x, t) = e^{-\rho t} \ln(aC - bx - \eta)$. Intertemporal substitution can be studied by changing the sign of $b$ throughout. In this case, the derived utility function is:

$$J(W, x, t) = e^{-\rho t} \beta(t) \ln[(a - \kappa \phi)(W - \hat{W})] + A(t),$$

(25)

where

$$\beta(t) \equiv \frac{1}{\rho} \left[1 - e^{-\rho(T-t)}\right],$$

$$\hat{W}(x, t) \equiv \frac{1}{a - \kappa \phi} \left[\frac{\eta}{r} \left(1 - e^{-r(T-t)}\right) + \phi \Phi(t) x \right.$$ \nonumber

$$\left. - \frac{\phi \eta}{a - b} e^{-r(T-t)} \left(1 - e^{-\kappa(1-b/a)(T-t)}\right)\right],$$

$$\Phi(t) \equiv 1 - e^{-(\kappa + r - \kappa b/a)(T-t)},$$

and $\phi \equiv b/(r + \kappa)$ as in (19). As $T \uparrow \infty$, $\beta \to \rho^{-1}$ and $\hat{W} \to (\eta/r + \phi x)/(a - \kappa \phi)$, so this reduces to (19) [with $b_1 = \kappa_1 = 0$].

\textsuperscript{19}We examine the logarithmic case for simplicity of notation. Subsistence wealth and the function $\Phi(t)$ are identical for all HARA utility functions so the intertemporal effects are the same. Only the function $\beta(t)$ changes.

\textsuperscript{20}As before we must have $a - \kappa \phi > 0$ to ensure positive marginal utility. The function $A(t)$ is the solution to the ordinary differential equation

$$0 = \frac{1}{2} \lambda^2 + r - \rho A + A' + \frac{1}{\beta(t)} \left[\beta'(t) A - 1 + \ln\left(\frac{a}{\beta(t)(a - \kappa[\phi + \Phi(t)])}\right)\right].$$

However, as it is a separate additive term in $J$, it can be ignored.
important when the horizon is close, and past consumption leads to increased current consumption. For a more distant horizon, the prosperity effect becomes more important. Past consumption has the strongest positive effect on current consumption when the horizon is a little over three years away. With just more than eight years until the horizon, the prosperity effect begins to dominate and past consumption tends to depress current consumption.

Recall that the prosperity effect measures the tendency to reduce current consumption with a decrease in prosperity (an increase in \(x\)) in order to save for the future. Near the horizon this effect has little influence on consumption since there is only a short time remaining for which to save. With a more distant horizon it becomes increasingly important. For an investor with a small rate of time preference or a long memory (small \(\kappa\)), the prosperity effect dominates even at short horizons. In the first case the future is relatively more important and in the second the current harm is remembered longer. With a large interest rate it requires a more distant horizon for the prosperity effect to become dominant because saving for the future is cheaper.

The remaining distinctive features of the horizon are captured in the subsistence level of wealth. The first term in \(\dot{W}\) is proportional to the annuity value of the required stream \(\eta\) from time \(t\) until time \(T\). The same term is found in the absence of intertemporal dependencies.

The second term is the direct effect of \(x\) on subsistence wealth. This is the finite-horizon analog of the effect considered in the previous section. Since \(\Phi(t) < 1\), subsistence wealth is affected less by intertemporal effects with a finite horizon.

The third term is entirely the result of the horizon. It vanishes as \(T \uparrow \infty\). It also has little separate influence if the memory span is short (\(\kappa\) is large implying \(\phi = 0\)). Intuitively, with a short memory even a nearby horizon does not affect current behavior. This term is the result of interactions between the two components of subsistence consumption, \(hx\) and \(\eta\).\(^{21}\)

6. The Hindy–Huang–Kreps specification

In the Hindy–Huang–Kreps (HHK) utility specification, consumption in any period contributes to felicity in many periods but is of no special importance to immediate felicity. As a consequence, an investor does not

\(^{21}\)Note for either \(\eta = 0\) or \(b = 0\) (implying \(\phi = 0\)), this term disappears. As shown in a previous footnote \(\phi x\) is the present value of a perpetual flow \(x\) shrinking at the rate \(\kappa\) given that consumption is at the minimum possible level and \(\eta = 0\). \(\phi \Phi(t)x\) is this present value for a finite horizon \(T\). So the first two terms in the definition of \(W\) represent the wealth required to finance future consumption which meets both the subsistence amount \(\eta\) and the intertemporal complementarity needs. The third term is the subtraction of the 'double counting' since consuming an 'additional' \(\eta\) each period affects future values of \(x\).
care if his or her consumption stream is smooth so long as it is smooth ‘on average’. For our model this is represented by setting $a = 0$ – but for $a = 0$ optimal consumption as given in (22b) is unbounded. The reason for this degeneracy is that consumption rather than being a part of the direct goal becomes only a means of smoothing average consumption. Average consumption can be smoothed relative to wealth most effectively by employing positive and negative consumption impulses. As there is no motivation to smooth consumption per se, it can and does fluctuate wildly in the optimal plan for smoothing average consumption; hence $|C(t)| = \infty$ almost everywhere.

To understand the difference between the standard results and those under the HHK specification, we must restrict negative consumption. With this restriction imposed, average consumption can still be quickly increased, but cannot be quickly decreased. As a consequence the investor is less willing to consume.

To study this problem in more detail consider the case $a = 0$, $b_1 = 1$, $b_2 = 0$. The first constraint puts us in the HHK framework. The second constraint is merely a normalization. The next focuses our concentration on intertemporal substitution by ignoring complementarity. When $a = 0$, intertemporal substitution must be present to ensure positive marginal utility of consumption. Intertemporal complementarity can also be present but is not required.

We will assume that the felicity function is isoelastic, $\nu(z) = z^\gamma/\gamma$, to obtain a closed form solution. We also impose the standard transversality conditions $\rho > 0$ and $\rho > \gamma [r + \frac{1}{2}\lambda^2/(1 - \gamma)]$.22

The partial differential equation for $J$ remains as given in (16) (with just a single state variable $x$) as does the first-order condition for the optimal portfolio (17b). Recognizing the nonnegativity of consumption, the Kuhn–Tucker conditions for optimal consumption are

$$0 \geq \kappa J_x - J_w, \quad 0 = C(\kappa J_x - J_w).$$

That is, the investor will refrain from consuming whenever the marginal utility of wealth is greater than $\kappa J_x$ and will consume sufficient wealth to drive marginal utility up to this level otherwise. The optimal consumption plan will be characterized by two regions. When wealth is small (relative to $x$), current consumption will be zero, and current felicity will derive solely from the intertemporal substitution of past consumption. At a sufficiently high level of wealth, $W^*(x)$, consumption will occur in the form of an impulse decreasing wealth and hence increasing its marginal utility to $\kappa J_x$. $W^*(x)$ is chosen by the investor as a part of the optimal plan.

In the region where optimal consumption is zero, the resulting partial differential equation evaluated at the optimal portfolio policy is

$$
1 - \frac{1}{\gamma} J^2 \frac{J_W}{J_{WW}} + rWJ_W - \rho J - \kappa xJ_x, \quad W < W^*(x).
$$

(29)

Note that consumption flow is now absent from the $J_W$ term as there is no consumption in the region $W < W^*(x)$. The boundaries are characterized by $J_W = \kappa J_x |_{W = W^*(x)}$, which is the second of the two first-order condition above, and

$$
J(0, x) = \frac{1}{\gamma} \int_0^\infty e^{-\mu t} (xe^{-kt})^\gamma dt = \frac{1}{\gamma(r + \kappa\gamma)} x^\gamma.
$$

(30)

The latter equation measures the utility over the remainder of the investor’s life derived from past consumption given no consumption in the future.\textsuperscript{23}

To solve this problem we make the change of variables $z = W/x$ and $J(W, x) \equiv x^\gamma I(z)$. Substitution into (29) gives

$$
0 = \frac{1}{\gamma} - \frac{1}{2}(I')^2 I'' + (r + \kappa) z I' - (\rho + \gamma \kappa) I,
$$

(31)

with the boundary conditions $I(0) = [\gamma(\rho + \kappa\gamma)]^{-1}$ and $I' = \gamma I - z^* I'$. The value of $z^*$ is the ratio of wealth to $x$ at which more consumption is initiated \textit{i.e.}, $z^* = W^*(x)/x$. It is chosen by the investor as a part of the optimization problem.

The solution to (31) is of the form $I(z) = [\gamma(\rho + \kappa\gamma)]^{-1} + Az^\alpha$. Substituting this into (31), solving for $\alpha$, determining $z^*$, and reexpressing the solution in original terms give\textsuperscript{24}

$$
J(W, x) = \frac{1}{\xi} \left[ \frac{1}{\gamma} x^\gamma + \left( \frac{1 - \alpha}{\alpha - \gamma} \right)^{1-\alpha} \kappa^\alpha W^\alpha x^{\gamma-\alpha} \right], \quad W < W^*(x),
$$

(32)

\textsuperscript{23}If $\rho + \kappa\gamma \leq 0$, this integral is not defined, and past consumption is never sufficient by itself to guarantee lifetime utility above $-\infty$. When this is true, the optimal program will never consume all wealth, so $W = 0$ is inaccessible and no explicit boundary condition is required.

\textsuperscript{24}To verify that $\alpha$ is a real note

$$
\theta^2 - 4\xi(r + \kappa) = (\frac{1}{\xi} r + \kappa + \xi)^2 - 4\xi(r + \kappa)
$$

$$
= \left( \frac{1}{\xi} r - \kappa + \xi \right)^2 + 2\xi r + \kappa > 0.
$$

The standard transversality conditions are sufficient to guarantee that $\alpha > \gamma$ as shown in the appendix. A transversality violation ($\alpha \leq \gamma$) leads to an optimal policy that accumulates wealth forever just as in the standard model.
becomes more like that in the standard model. In the limit it mimics the standard model exactly.

An increase in the rate of time preference, \( p \), also increases \( \alpha \). As stated above an investor with a higher \( \alpha \) consumes sooner. This is consistent with a higher time preference.

The comparative statics for \( \alpha \) with respect to the parameters of the returns distributions are

\[
\frac{\partial \alpha}{\partial \mu} \geq 0, \quad \frac{\partial \alpha}{\partial \sigma} \leq 0, \quad \frac{\partial \alpha}{\partial \lambda} \geq 0,
\]

as \( \alpha \leq 0 \). The direction of the inequalities in \( \frac{\partial \alpha}{\partial \mu} \) and \( \frac{\partial \alpha}{\partial \lambda} \) assumes that the price of risk is positive, \( \lambda > 0 \), which is true if the risky asset’s expected return exceeds the risk-free rate.

The ambiguity in the last three comparative statics is due to prosperity and surrogate effects. With a positive price of risk, the investor holds some funds in the risky asset. Therefore, an increase in \( \mu \) makes future consumption cheaper on average. The investor will therefore tend to substitute consumption in the future for earlier consumption. On the other hand, the total potential consumption increases as well; this increase in ‘income’ leads to increased consumption in each period. In this model earlier (later) consumption is favored by an increase (decrease) in \( \alpha \). So the prosperity effect dominates if \( \alpha < 0 \), while the surrogate effect dominates if \( \alpha > 0 \). The comparative static with respect to the asset’s variance has just the opposite interpretation in certainty-equivalent terms.

7. Conclusion

This paper examined the continuous-time consumption portfolio problem for an investor whose utility function is not time-additive. The effects of both intertemporal substitution and complementarity were explored. With intertemporally dependent utility, unexpected changes in wealth can have both long- and short-term affects on consumption. In addition to altering the consumption choice, intertemporal dependencies affect the optimal portfolio of the investor as well.

The pattern of temporal effects induced by intertemporally-dependent utility may account for some of the time series properties in asset returns found by other researchers which appear to be due to irrational behavior of traders. Most of the rational expectations literature assumes that utility functions are quite simple. Additional research needs to be devoted to exploring the role of information and its revelation in the market when investors have tastes which cannot be described simply.
Appendix

A.1. Hindy–Huang–Kreps desirata

In this section we verify that the utility representation used in section 6 satisfies the HHK desirata. Recall that they were: (i) 'Two patterns of consumption that have almost equal accumulated consumption at every point in time should be close [in utility].' (ii) '...sizeable shifts in consumption across small amounts of time are regarded as insignificant.' (iii) '...the shift of an increasingly larger amount of consumption over a decreasingly smaller amount of time' has no utility effect.

The first requirement is obviously met. Lifetime utility $T$ is continuous in felicity: $u(x(t)) (\forall t)$, felicity is continuous in $x(t)$: $u(\cdot) = [x(t)]^\gamma / \gamma$, and $x(t)$ is continuous in $C(\tau)$. Thus lifetime utility is continuous in the consumption flow at each date.\textsuperscript{26}

To check on the second and third conditions consider two consumption streams:

$$C'(t) = C(t) + \Gamma \delta(t - t'), \quad C''(t) = C(t) + \Gamma \delta(t - t''),$$

(A.1)

where $C(t)$ is an arbitrary stream and $\delta$ is a Dirac delta function giving an impulse at time $t'$ or $t''$. The state variable $x'(t)$ measuring average past consumption is

$$x'(t) = \begin{cases} x(t), & t \leq t', \\ x(t) + \Gamma \kappa e^{-\kappa(t-t')}, & t > t', \end{cases}$$

(A.2)

where $x(t)$ is the state variable for the consumption stream $C(t)$ and similarly for $x''(t)$. For $t' < t''$, $T' < T''$ as the $C''$ gets the same felicity at an earlier date. The difference in lifetime utility for the two consumption streams is

$$T'' - T' = \frac{1}{\gamma} \int_{t'}^\infty e^{-\rho t} \left[ (x(t) + \kappa \Gamma e^{-\kappa(t-t')})^\gamma - x^\gamma(t) \right] dt$$

$$+ \frac{1}{\gamma} \int_{t'}^{\infty} e^{-\rho t} \left[ (x(t) + \kappa \Gamma e^{-\kappa(t-t')})^\gamma - (x(t) + \kappa \Gamma e^{-\kappa(t-t')})^\gamma \right] dt.$$  

(A.3)

\textsuperscript{26}A problem that can arise is for consumption paths with $x(t) = 0$ for $t \in [0, s)$. For such plans lifetime utility is $-\infty$. To avoid this initialization problem, we assume that $C(0) > 0$ for all paths considered. This guarantees that $x(t) > 0 (\forall t > 0)$. This constraint is obviously met for any optimal consumption plan.
This difference is bounded by the value for \( x(t) = 0 \),

\[
\mathcal{V}'' - \mathcal{V}' < \frac{(\kappa \Gamma)^\gamma}{\gamma(\rho + \kappa \gamma)} (e^{-\rho \tau'} - e^{-\rho \tau})
\]

\[= \frac{(\kappa \Gamma)^\gamma}{\gamma(\rho + \kappa \gamma)} (\tau' - \tau'') + o((\tau' - \tau'')^2). \tag{A.4}
\]

Thus the difference in utility for the two consumption streams is of the same order as the difference in times at which the impulse of consumption occurs. Furthermore, the difference in utility remains small for increasingly large consumption amounts provided \( \Gamma = o((\tau' - \tau'')^{1/\gamma}) \).

### A.2. Transversality conditions

This section proves that the standard transversality conditions are sufficient transversality conditions for the HHK model. The standard conditions are

\[\rho > 0, \tag{A.5a}\]

\[\rho > \gamma \left( r + \frac{1}{2} \lambda^2 \frac{1}{1 - \gamma} \right). \tag{A.5b}\]

For \( \gamma < 0 \), (A.5a) is the tighter constraint. If \( \alpha \geq 0 \), then \( \alpha > \gamma \) and we're done. If \( \alpha < 0 \), then since \( \partial \alpha / \partial \lambda > 0 \),

\[\alpha > \alpha_{\min} = \alpha|_{\lambda=0} = \frac{\rho + \gamma \kappa}{r + \kappa} = \gamma \left( 1 + \frac{\rho / \gamma - r}{r + \kappa} \right). \tag{A.6}\]

For \( \gamma \geq 0 \), (A.5b) is tighter, and recalling the definition of \( \xi = \rho + \gamma \kappa \) from (32) and rearranging terms gives

\[\xi(1 - \gamma) = (\rho + \gamma \kappa)(1 - \gamma) > \gamma \left[ \frac{1}{2} \lambda^2 + (r + \kappa)(1 - \gamma) \right], \tag{A.7}\]

\[\xi > \gamma \left[ \frac{1}{2} \lambda^2 + \xi + \hat{r}(1 - \gamma) \right] = \gamma(\theta - \gamma \hat{r}),\]

where \( \hat{r} \equiv r + \kappa \).
The comparative statics in the text are then

$$\frac{\partial \alpha}{\partial \gamma} = \kappa \frac{\partial \alpha}{\partial \xi} > 0,$$

$$\frac{\partial \alpha}{\partial \rho} = \frac{\partial \alpha}{\partial \xi} > 0,$$

$$\frac{\partial \alpha}{\partial \mu} = \frac{\lambda}{\sigma} \frac{\partial \alpha}{\partial \theta} \leq 0,$$

$$\frac{\partial \alpha}{\partial \sigma} = -\frac{\lambda^2}{\sigma} \frac{\partial \alpha}{\partial \theta} \geq 0,$$

$$\frac{\partial \alpha}{\partial \lambda} = \lambda \frac{\partial \alpha}{\partial \theta} \leq 0.$$

(A.13)

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