Approximating American options and other financial contracts using barrier derivatives

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Despite more than twenty-five years of efforts, valuation of the American put and similar free-boundary problems remains a perplexing enigma. These problems have escaped true analytical solutions and still must be solved by numerical methods. Over the years, a number of approximations have been proposed. This paper describes another approximation method based on barrier options. The exercise policy is approximated by a simple class of functions, and the best policy within that class is selected by standard optimization techniques. The resulting value is generally a very tight lower bound to the true option value. The advantages of this method are its simplicity and speed, even when used in general-purpose computer programs such as spreadsheets.

1. INTRODUCTION

Despite more than twenty-five years of efforts, the American put problem has escaped a true analytical solution, and the problem can still only be solved approximately.

This paper analyzes an approximation for the American put and some other derivatives which is based on barrier options. Barrier options are contingent claims whose value depends upon their behavior at various boundaries. The prototypical barrier option is the down-and-out call, which is a call option that is canceled if the stock price ever falls below the knock-out barrier. Barrier options were introduced into the academic literature by Merton (1973). In this paper we also utilize barrier options that make a payment at a barrier. Such options were first studied by Ingersoll (1977).

Section 2 of this paper reviews previous approximations for pricing American options. Section 3 introduces the barrier option approach to valuing American options. Sections 4 and 5 develop the various barrier options required for our approximation and derive approximations using constant and exponential barriers. Section 6 shows how to compute the deltas and other Greek measures for the barrier approximations. Finally, Section 7 extends the results to other types of contracts.

2. A REVIEW OF AMERICAN OPTION APPROXIMATIONS

The first approach to valuing American options, proposed by Brennan and Schwartz (1977), Parkinson (1977), and others, was a direct numerical evaluation of the Black–Scholes partial differential equation using finite differences. These methods are still commonly employed because they are versatile and easily understood. Unfortunately finite-difference methods
are also expensive in terms of computer time, even after employing such enhancement techniques as control variates or convergence extrapolation.

One of the first approximation schemes developed to reduce this time-consuming task was proposed by Johnson (1983). Johnson derived an econometric approximation to the American put by fitting an empirical function to the numerical results generated by the binomial model. Since his approximating function is analytical, this method is easy to use in any type of computer implementation, and Johnson confirmed its accuracy for relatively short term puts \((r < 0.12)\). Blomeyer (1986) extended Johnson's results to puts with a single dividend during the option's life. However, the nature of this method requires a completely new estimation for each new problem even if the changes are merely quantitative.

Johnson's and Blomeyer's approximations have an econometric basis. Subsequent approximations have usually been based on a theoretical relation between the American put and some similar derivative contract with a known analytical solution. Barone-Adesi and Whaley (1987) (hereafter BW) used Merton's (1973) solution for perpetual American options. Geske and Johnson (1984) (hereafter GJ) approximated the American put as a modified American or Bermuda option and used Geske's (1979) compound option formula to evaluate it.

The BW approximation assumes the early exercise premium has a particular functional dependence on time and solves for its stock price dependence using the perpetual equation. The resulting approximation is

\[
P_{BW}(S, \tau) = P_{BS}(Se^{-q\tau}, \tau) + \frac{S^*}{\lambda} \left[1 - e^{-q\tau} \Phi(d)\right] \left(\frac{S}{S^*}\right)^{-\lambda},
\]

where

\[
d = \frac{-\ln(S^*/X) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},
\]

\[
\lambda = \frac{\gamma - 1}{2} + \frac{1}{2} \sqrt{(\gamma - 1)^2 + \frac{8r}{\sigma^2} \frac{1 - e^{-r\tau}}{1 - e^{-r\tau}}},
\]

\[
\gamma = \frac{2(r - q)}{\sigma^2},
\]

and \(P_{BS}(S, \tau)\) is the Black–Scholes function for a European put with a maturity of \(\tau\) written on a share of stock with price \(S\). The put is exercised (and worth \(X - S\)) if \(S \leq S^*\). Here \(S^*\) is the estimated optimal exercise point; it is the solution to

\[
X - S^* = P_{BW}(S^*, \tau).
\]

A similar approximation holds for American calls on stock with a continuous dividend or options on assets like futures or foreign exchange rates that can be treated as if they paid continuous dividends.\(^1\)

The BW method is simple to implement, requiring only the numerical solution of this

\(^1\) As shown in Cox, Ingersoll, and Ross (1981) a futures price can be treated as the price of an asset which pays a continuous dividend at a yield equal to the interest rate. A foreign exchange rate is like the price of an asset which pays a continuous dividend at a yield equal to the foreign interest rate.
one nonlinear equation (2). Since the early exercise premium on long-term American options has only weak time-dependence, this approximation is accurate for very long term options. Indeed, in the limit, as $\tau \to \infty$, $P_{BS}(S, \tau)$ goes to zero and the term in brackets goes to unity, leaving the exact perpetual solution. It is also accurate for short-term options since, for these options, the premium for American exercise is small and most reasonable exercise policies are close to optimal. It is less useful for options of intermediate term, where the assumed dependence may differ markedly from the actual functional form. However, exactly what maturities should be considered intermediate is an empirical question.

The GJ method assumes the put can only be exercised at certain points in its life and uses a modified American (or Bermuda) option as an approximation to the American option. The GJ formula for two possible exercise dates is

$$P_{GJ2} = X e^{-\tau r/2} \Phi(d_2) - S e^{-\tau r/2} \Phi(-d_2),$$

$$+ X e^{-\tau r} \Phi_2(d_2 - \sigma \sqrt{1/2}, -d_1 + \sigma \sqrt{r; -1/\sqrt{2}} - S e^{-\tau r} \Phi_2(d_2, -d_1; -1/\sqrt{2}),$$

where

$$d_1 = \frac{\ln(S/X) + \tau(r - q + \frac{1}{2}\sigma^2)}{\sigma \sqrt{r}}; \quad d_2 = \frac{\ln(S/S_0) + \frac{1}{2} \tau(r - q + \frac{1}{2}\sigma^2)}{\sigma \sqrt{\frac{1}{2}}}.$$

$\Phi_2(\cdot)$ is the standard bivariate normal distribution function, and $S^*$ is the stock price at or below which the put is exercised halfway through its life. $S^*$ is the solution to the nonlinear equation

$$S^* = X - P_{BS}(S^* e^{-\tau r/2}, \frac{1}{2} \tau).$$

The first line in (3) is the present value of exercising the put halfway through the option's life if the stock price then is less than the critical value $S^*$. The second line gives the present value of the put's payoff at maturity, provided it is in the money and was not exercised at the halfway point.

The GJ approximation can also be implemented with additional early exercise dates. Unfortunately, if more than a single early exercise date is employed, it then requires a series of numerical integrations of the multivariate normal distribution function. In addition one nonlinear equation must be solved for each possible early exercise date. With only one early exercise date, the required computer time is comparable with that of BW. Its accuracy is also comparable for short-term options, but it is less accurate for long-term options. On the other hand, since the modified American option is just an American option with a restriction, we can definitely sign the approximation error; the GJ approach must underestimate the put's true value.

The GJ approximation can be improved by using Richardson extrapolation as explained in the original paper. This is a linear extrapolation in the time between exercise points. For a European option, the time until exercise is $T - \tau$; for the GJ2 approach, the time between exercise points is $\frac{1}{2}(T - \tau)$; for an American option, the time between exercise points is zero, so

$$P_{Am} \approx P_{GJ2} + (P_{GJ2} - P_{BS}) = 2P_{GJ2} - P_{BS}.\quad (5)$$

More recently Ho, Stapleton, and Subrahmanyam (1994) have cited evidence that the $n$-point modified American option values are approximately exponential in $n$ and have
therefore suggested using an exponential extrapolation from just the European \((n = 1)\) and two-date GJ solution:\(^2\)

\[
P_{\text{Am}} \approx P_{\text{HSS}} = P_{\text{GJ}2}^2 / P_{\text{BS}}.
\]  

Both extrapolations use only the European solution and the two-date solution \(P_{\text{GJ}2}\) above, neither of which requires numerical integration. HSS extrapolation improves the accuracy for long-term options but not sufficiently to make GJ competitive with BW. One primary disadvantage of the extrapolations is that we no longer know the sign of the approximation error.

Bunch and Johnson (hereafter BJ) suggest altering the Geske–Johnson approach by optimizing the early exercise dates.\(^3\) That is, the option owner is restricted to exercising the put on two dates; however, these dates may be selected (in advance) and need not be the maturity date and halfway to the maturity date. The approximate value of the put as given by the BJ approach is

\[
P_{\text{BJ}} = \max_{\tau_1 < \tau_2 < \tau} \left[ X e^{-r\tau_2} \Phi((-D_2 + \sigma \sqrt{\tau_2} - S e^{-q\tau_2} \Phi(-D_2))
\right.
\]

\[
+ X e^{-r\tau_1} \Phi_2(D_2 - \sigma \sqrt{\tau_2}, -D_1 + \sigma \sqrt{\tau_2}, -\sqrt{\tau_2} / \tau_1)
\]

\[
- S e^{-q\tau_1} \Phi_2(D_2, -D_1; -\sqrt{\tau_2} / \tau_1) \right].
\]  

where

\[
D_1 \equiv \frac{\ln(S/X) + (r - q + \frac{1}{2} \sigma^2)\tau_1}{\sigma \sqrt{\tau_1}} \quad \text{and} \quad D_2 \equiv \frac{\ln(S/S^*) + (r - q + \frac{1}{2} \sigma^2)\tau_2}{\sigma \sqrt{\tau_2}}.
\]

Here \(S^*\) is the stock price at or below which the put is exercised on its early exercise date. It is the solution to the nonlinear equation

\[
S^* = X - P_{\text{BS}}(S^* e^{-q(\tau_1 - \tau_2)}, \tau_1 - \tau_2).
\]  

The first line in (7) is the present value of exercising the put on its early exercise date if the stock price then is less than the critical value \(S^*\). The second line gives the present value of the put's payoff at its later exercise date, provided it is in the money and was not exercised at the halfway point. Note that the later exercise date may be before the option's maturity date; however, the put will always be exercised then if it is in the money since it cannot be exercised at maturity if \(\tau_1 < \tau\). The Richardson and HSS extrapolations can also be applied to the BJ method.

\(^2\) Ho, Stapleton, and Subrahmanyan reason as follows: \(P_{\text{GJ}1} \approx A^{1/n} P_{\infty}\), where the fraction \(A\) depends on the variables affecting the option's price, but not on the number of early exercise dates in the specific approximation. Then \(P_{\text{GJ}2} \approx A P_{\infty}\) and \(P_{\text{GJ}2} \approx \sqrt{A} P_{\infty}\). Solving for \(P_{\infty}\) (the American solution) and the unknown \(A\) gives \(P_{\infty} \approx P_{\text{GJ}2}^2 / P_{\text{GJ}1}\). The HSS extrapolation is a linear extrapolation in the logs of the Bermuda solutions.

\(^3\) See also Ingersoll (1988).
3. THE BARRIER OPTION APPROACH TO VALUING AMERICAN OPTIONS

The American put problem has escaped a true analytical solution because of the need to simultaneously determine the put's value and the optimal exercise policy. The difference between the values of otherwise identical American and European puts can be substantial because the likelihood of early exercise is substantial.

We adopt the usual Black–Scholes assumptions that the markets are open continuously and that there are no impediments such as transactions costs. The basis asset price follows the continuous diffusion process

\[ \frac{dS}{S} = (\mu - q) dt + \sigma d\omega. \]

The parameter \( q \) is the yield on a continuously paid dividend, the foreign interest rate if \( S \) is a foreign exchange rate or the interest rate if \( S \) is a futures price.

The value of any contingent claim, \( F \), based on this asset is the solution to the standard partial differential equation

\[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + (r - q) S \frac{\partial F}{\partial S} - rF + \frac{\partial F}{\partial t} = 0 \]

subject to the claim-specific boundary conditions. For a put, the condition at maturity is \( F(S, T) = \max(X - S, 0) \). The stock price boundary conditions are \( F(\infty, t) = 0 \) and the early exercise boundary condition is \( F(K^*(t), t) = X - K^*(t) \), where \( K^*(t) \) is the optimal exercise policy. The difficulty arises because the optimal policy \( K^*(t) \) is not known and cannot be determined without simultaneously solving the pricing problem.\(^4\)

The value of a perpetual American put can be determined analytically because we know the optimal exercise policy is constant over time. We can solve the pricing problem for an arbitrary fixed exercise policy and then determine the option's value by choosing that policy parameter \( K^* \) which maximizes the value.

The barrier method of valuation is similar. First a parametric family \( \mathcal{K} \) of exercise policies is chosen. Each policy in the family \( K(t, \theta) \) is characterized by a vector of parameters, \( \theta \). The exercise rule for any given policy is to exercise the first time the stock price falls to or below the barrier, \( S_t \leq K(t, \theta) \). To implement the barrier approximation, the value of the put is determined as a function of the parameters, and the resulting option value is maximized with respect to the parameters.

The policy corresponding to the selected parameters is approximately optimal, and the corresponding value is the put's approximate value. If we could examine all feasible exercise policies, then this approximation would be exact. Similarly, if the family \( \mathcal{K} \) happened to include the optimal policy, the approximation would be exact though we would not necessarily know this to be the case. Otherwise, the policy selected will be suboptimal and the approximation will result in a lower bound for the option's price.

While the optimal policy is not known, we do know that it must be optimal to exercise

\(^4\) For some similar problems like the pricing of a callable convertible discount bond, the optimal policies can be determined ex ante and the contracts can be priced. See, for example, Ingersoll (1977).
any option that is in the money at maturity, so \( K^*(T) = X \). However, it generally will not
be possible to ensure this condition for parametric families of policies. For example, it is
true for a constant policy only if \( K(t) \) is always equal to \( X \), but this policy cannot be best
within the class of constant policies since it leads to a value of zero for the put.
Consequently it is useful to model only the early exercise policy and then explicitly
append the at-maturity policy.\(^5\)

Let \( \tau_K \) denote the first time the stock price falls to or below the exercise barrier. The
value of the put under any exercise policy \( K(t) \), whether optimal or not, can then be
expressed as

\[
PV[X - S_{\tau_K}; \ \tau_K < T] + PV[\max(X - S_T, 0); \ \tau_K \geq T].
\]

The present value is an expected discounted value as usual, but here the expectation is
over the random (stopping) time \( \tau_K \) as well as the stock price.

For exercise policies continuous in time, a diffusion process which is also is continuous
cannot jump across the barrier, so if the put is exercised at time \( \tau_K \) prior to maturity, the
stock price then must be \( K(\tau_K) \). Therefore, under Black–Scholes conditions, (11) can be
simplified to

\[
PV[X - K(\tau_K); \ \tau_K < T] + PV[\max(X - S_T, 0); \ \tau_K \geq T].
\]

where \( \tau_K \) is now the first time at which \( S_t = K(t) \).

If the put is not exercised prior to maturity, then the value from the payoff at maturity,
the second term, is

\[
PV[X - S_T; \ {S_T < X \cap \tau_K \geq T}].
\]

The first inequality, \( S_T < X \), is the ‘in-the-money’ condition. The second inequality,
\( \tau_K \geq T \), ensures that the put was not exercised before maturity.

It will be convenient to compute this present value in two parts as

\[
X D(S, t; T; \ {S_T < X \cap \tau_K \geq T}) - S(S, t; T; \ {S_T < X \cap \tau_K \geq T}),
\]

where \( D(\cdot; T; \mathcal{E}) \) represents the present value of receiving one dollar at time \( T \) if and only
if the event \( \mathcal{E} \) has occurred and \( S(\cdot; T; \mathcal{E}) \) represents the current value of receiving one
share of stock at time \( T \) if and only if the event \( \mathcal{E} \) has occurred. Contracts like \( D \) that pay
either one dollar or nothing have been termed digital or binary options after their ‘off–on’
nature. Similarly, we shall call the \( S \) contract, which converts into one share of stock
under certain conditions, a digital share.\(^6\)

\(^5\) If the stock pays dividends continuously at a yield \( q > r \), then even the optimal exercise policy
\( K^*(t) \) is not continuous at \( X \). Rather \( \lim_{t \to 0} K^*(t) = rX/q \neq K^*(T) = X \). This limit can be verified as
follows. Consider an investor holding the stock and the put just before maturity. Using the put to
sell the stock would realize \( X \). Holding one more instant until maturity would realize \( X e^{-\eta r dt} + Sq dt \).
The indifference point is where these are equal, \( K^*(dt) = rX/q \).

\(^6\) See Ingersoll (1997) for a development of this digital approach to pricing and the derivation of the
digital option \( D(\cdot) \) and share \( S(\cdot) \) values, as well as the first-touch digital \( T(\cdot) \) below, for common
over a specific event.
Using the usual risk-neutral technique, the present value of early exercise is

\[ \int_0^T e^{-r(t-s)}[X - K(t)]\psi(t; S, r, q, \sigma), \]

where \( \psi(t; \cdot) \) is the risk-neutral probability distribution function of the first passage time of the stock price to the barrier \( K(t) \). The parameters of the risk-neutral probability distribution are the risk-neutral drift \( r - q \) and the volatility \( \sigma \).

We will employ the following notation for this type of first-touch derivative contract. The quantity \( T(s, t; T, H, k, (r, q, \sigma)) \) denotes the value at time \( t \) of receiving a (single) payment \( H(t) \) the first time \( t \) that the stock price hits the ‘barrier’ \( K(t) \), provided that this occurs before time \( T \). For the put option, the barrier is the exercise policy \( K(t) \) and the payment is \( H(t) = X - K(t) \).

Combining these two expressions gives the value of the American put simply as

\[ P_{Am} = XD(s, t; T; \{S_T < X \cap K \geq T\}) \]
\[ - S(s, t; T; \{S_T < X \cap K \geq T\}) \]
\[ + T(s, t; T, X - K(t), K(t)). \]

This same expression also gives the (smaller) value for any suboptimal policy \( K(t) \), provided that it is continuous in time (except possibly at \( T \)). In our approximations, we shall confine our attention to such policies.

We can thus express the put value as the functional

\[ P_{Am} \geq P_{barr} = \max_{k \in \mathcal{K}} \left[ XD(s, t; T; \{S_T < X \cap K \geq T\}) \right. \]
\[ \left. - S(s, t; T; \{S_T < X \cap K \geq T\}) \right] \]
\[ + T(s, t; T, X - K(t), K(t)). \]

If the set of policies considered, \( \mathcal{K} \), is all functions (or all nondecreasing continuous functions), then the resulting put value will be exact. If the set \( \mathcal{K} \) is restricted, then the resulting value will be an approximation providing a lower bound to the put price. The values \( T, D, \) and \( S \) of these digital can be determined from the values of barrier options, as discussed in the next section.

4. A CONSTANT BARRIER APPROXIMATION FOR AMERICAN PUTS

The simplest class of exercise policies includes just the constant functions \( K(t) = k \). With a constant policy, the put is exercised the first time the stock price falls to some price \( k \). The payoff at that time is \( X - k \). If the stock price never falls to \( k \), then the option is exercised for \( X - S_T \) if it is in the money at maturity. This policy, in effect, transforms the American put into a capped put, like those traded on the CBOE and AMEX.

To value the put under a constant exercise policy, we require the digitals for the event \( \{S_T < X \cap t_k \geq T\} \). Since the boundary is a constant, this latter condition is equivalent to \( \{S_{\min}(0,T) > k\} \). Merton (1973) used the Black–Scholes methodology to price a down-and-out option. For this option, the payoff event is \( \{S_T > X \cap S_{\min} > k\} \). Extending his
analysis to stocks that pay dividends continuously, we can express the risk-neutral probability of our payoff event as

\[ \Pr\{S_T < X \cap S_{\text{min}} > k\} = \Phi(h_1(S/k)) - \Phi(h_1(S/X)) + (k/S)^{\gamma-1}\left[\Phi(h_1(k^2/XS) - \Phi(h_1(k/S))\right]. \tag{18} \]

where

\[ h_1(z) \equiv \ln z + (r - q - \frac{1}{2}\sigma^2)(T - t) \sigma\sqrt{T - t} \quad \text{and} \quad \gamma \equiv 2\left(\frac{r - q}{\sigma^2}\right). \]

As shown in Ingersoll (1997), the values of the digital option and share for this event and the value of the first-touch digital can readily be determined from this probability. They are

\[ D(S, t; T; S_T < X \cap S_{\text{min}} > k) = \]

\[ e^{-r(T - t)}\left[\Phi(h_1(S/k)) - \Phi(h_1(S/X)) + (k/S)^{\gamma-1}\left[\Phi(h_1(k^2/XS) - \Phi(h_1(k/S))\right]\right]. \tag{19a} \]

\[ S(S, t; T; S_T < X \cap S_{\text{min}} > k) = \]

\[ S e^{-r(T - t)}\left[\Phi(h_2(S/k)) - \Phi(h_2(S/X)) + (k/S)^{\gamma+1}\left[\Phi(h_2(k^2/XS) - \Phi(h_2(k/S))\right]\right]. \tag{19b} \]

\[ T(S, t; T; X - k, k) = \]

\[ (X - k)[(k/S)^{b-\beta}\Phi(\eta_1(k/S, \beta)) - (k/S)^{b+\beta}\Phi(\eta_2(k/S, \beta))]. \tag{19c} \]

where

\[ h_2(z) \equiv h_1(z) + \sigma\sqrt{T - t}, \]

\[ \eta_1(z, \beta) \equiv \ln z - \beta\sigma^2(T - t) \sigma\sqrt{T - t}, \quad \eta_2(z, \beta) \equiv \eta_1(z, \beta) + 2\beta\sigma\sqrt{T - t}, \]

\[ \gamma \equiv 2\left(\frac{r - q}{\sigma^2}\right), \quad b \equiv \frac{r - q - \frac{1}{2}\sigma^2}{\sigma^2}, \quad \beta \equiv \sqrt{b^2 + 2r/\sigma^2}. \]

The constant exercise policy approximation for the American put is therefore

\[ P_{\text{con barr}} = \max_k\{XD(S, t; T; S_T < X \cap S_{\text{min}} > k) - S(S, t; T; S_T < X \cap S_{\text{min}} > k) + T(S, t; T; X - k, k)\}. \tag{20} \]

The maximization can be performed in a number of ways. As a function of \(k\), the value in (20) is unimodal, so a simple gradient approach works well. For example, the maximization routines embedded in spreadsheet applications handle the problem easily.

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7 After adding a dividend to Merton’s expression, we can use

\[ \Pr\{S_{\text{min}} > k\} = \Pr\{S_T > k \cap S_{\text{min}} > k\} \]

\[ \Pr\{S_T < X \cap S_{\text{min}} > k\} = \Pr\{S_{\text{min}} > k\} - \Pr\{S_T > X \cap S_{\text{min}} > k\} \]

to compute the required probability.

8 For \(k = X\), the put is exercised as soon as the option is at the money, so it is worthless. For \(k = 0\), the put is never exercised prior to maturity and is worth the same as a European put. The best constant exercise policy is some value between 0 and X.
Barrier approximation

Figure 1 illustrates this procedure for a one-year put option with a strike price of 100 written on a share of stock with a price of 100 and a volatility of 30%. The interest rate is 7% and the dividend yield is 2%. The figure shows the present value of early exercise, 
\((100 - k)T(\cdot)\), and the present value of exercise at maturity, 100\(D(\cdot) - S(\cdot)\), along with the value of the put.

For a policy of \(k = 100\), both of these present values are zero, the former because the option is immediately exercised for zero and the latter because there is no chance for exercise at maturity. As we lower \(k\), the value of exercise at maturity rises monotonically because the probability of early exercise falls. The value of exercise at maturity rises at first with a lowered \(k\) because the value received, \(X - k\), rises. Around \(k = 83\), the value of early exercise begins to drop because the increase in the payoff received is offset by the increased likelihood that the barrier will not be reached or will be reached later.

The put value is maximized at 9.68 for a policy of \(k^* = 75.17\). As noted previously, 9.68 is an underestimate of the actual value, since following a constant policy, though feasible, is suboptimal. The actual put value, computed with a binomial model of 1000 steps, is 9.75. The error of 7 cents is less than 1%.

The actual optimal exercise point with one year to maturity is approximately $70. This rises over time, reaching $75.17 with approximately 6 months remaining in the option's life. This 'error' in giving the optimal policy is of no particular concern, however. We cannot directly compare the two policies. It is only through the metric of price that the suboptimality of the policy can be measured.

Stated another way, the exercise policy number, 75.17, is not by itself important. What is important is the answer to the question: should exercise occur now? And since the stock price is above \(k^*\) as well as \(K^*(t)\), the answer to this question is no, which is correct. Furthermore, as shown in Figure 2, the best constant policy depends on the current stock price, so even if the stock price were at $75.17, the best constant exercise policy would be to hold the option. For example, if the stock price were $75 instead of $100, the best constant policy would be to wait until the stock price dropped to $71.42 to exercise. And, of course, if the stock price were actually $71.42, the best constant policy would be to wait for a still lower price.
Constant exercise policies for different stock prices

Parameters
\[ X = 100 \quad \tau = 1 \]
\[ \sigma = 30\% \quad q = 2\% \]
\[ r = 7\% \]

This sequence has a limit or fixed point at which the best constant policy is just equal to the prevailing stock price. In this case the limit is $70.55, so under a best constant policy the put would be exercised only if the stock price were at or below $70.55 with one year to maturity. This is just a little above the true optimal policy.\(^9\)

The lesson here is that the best constant policy is not truly optimal and therefore need not have self-fulfillment embedded within, as does an optimal policy determined by dynamic programming. It is rational to anticipate that the best constant policy will change over time as conditions change.

The reason for the difference in these best constant policies can be see in Figure 3. The higher the current stock price, the later will be the likely time at which it crosses the optimal exercise boundary. It is most important that the best constant policy mimic the optimal policy at the most likely times of exercise. Since the optimal price at which to exercise increases over time rising to \( X \) (or \( rX/q \)) at the maturity date, the best constant policy will be increasing in the current stock price.

In fact, the difference between the best constant policies proves the not surprising result that no constant exercise policy is truly optimal, since the actual optimal policy must be independent of the current stock price.

How accurate is the constant barrier approximation for the American put value? The correct value for this put, as computed with a binomial tree of 1000 steps, is 9.75. The BW approximations is 9.79. The GJ approximation is 9.46; both the Richardson and HSS extrapolations of the GJ approximation are 9.76.

Clearly the constant exercise policy is not close enough to the true exercise policy to provide an answer as accurately as those other approximations. The approximation can be improved by searching for the best policy in a wider class of functions. The next section discusses some improvements.

\(^9\) Even with a tree of 1000 binomial steps, the optimal policy cannot be determined any more accurately than to say it is in this range. However, it must be less than the fixed point value of $70.55 just determined. The option value must be larger under the true optimal exercise policy than under the best constant exercise policy, so its value must exceed 100 - 70.55 = 29.45. But in that case, it is worth more alive than exercised, so the optimal exercise point must be at a smaller stock price.
5. THE EXPONENTIAL BARRIER APPROXIMATION FOR AMERICAN PUTS

The barrier approximation method can be improved by using a more-inclusive class of functions from which to select the approximately optimal one. For example, we could use continuously varying functions to model the exercise policy. The obvious choice is an exponential barrier, $K(t) = k_0 e^{rt}$, since options with exponential barriers have known analytical solutions under Black–Scholes conditions. To approximate the put’s exercise boundary, we will want a growing barrier, $\kappa > 0$.

Under the risk-neutral probability measure, the expected stock price at time $t$ is $S_0 e^{rt - \frac{\kappa}{2} t}$, so the stock price tends to recede from a constant lower barrier at the rate $r - q$. The rate of recession from an exponentially growing barrier is $r - q - \kappa$. Therefore, digital options and shares for exponentially growing barriers can be valued using the formulas given in (19) after replacing the dividend yield $q$ by $q + \kappa$ in the definitions of $\gamma$, $h_1$, and $h_2$ (in the ratio $k/S$, the parameter $k$ is measured at its current value $k_t$):\(^{10}\)

\[
D(S, t; S_T < X \cap \tau_K \geq T) = e^{-\kappa(T-t)} \left[ \Phi(h'_1(S/k_t)) - \Phi(h_1(S/X)) \right] \\
\quad \quad + (k_t/S)^{-1} \left[ \Phi(h_1(k_t^2/XS)) - \Phi(h'_1(k_t/S)) \right], \quad (21a)
\]

\[
S(S, t; S_T < X \cap \tau_K \geq T) = S e^{-\kappa(T-t)} \left[ \Phi(h'_2(S/k_t)) - \Phi(h_2(S/X)) \right] \\
\quad \quad + (k_t/S)^{-1} \left[ \Phi(h_2(k_t^2/XS)) - \Phi(h'_2(k_t/S)) \right]. \quad (21b)
\]

\(^{10}\) Formally we introduce the modified stock price $W_t = S e^{-\kappa t}$. If $S_T < X$, then $W_T < X e^{-\kappa T}$. If $S_T$ is above $K(t) = k_0 e^{rt}$, then $W_T > K(t)e^{-\kappa t} = k_0$. So the barrier for $W_T$ is constant. The risk-neutral dynamics of $W$ are $\frac{dW}{W} = (r - q - \kappa) dt + \sigma dW$, which are lognormal, so (19) can be applied immediately with a barrier of $k_0$ after changing the dividend yield to $q + \kappa$ and the strike price to $X e^{-\kappa T}$. Reexpressing the results in terms of $S$ and $X$ gives (21). Note that the terms which depend on $X$ in (21) use $h_t(z)$ rather than $h'_t(z)$ since the changing barrier effects cancel.
where

\[ h'_1(z) \equiv \frac{\ln z + (r - q - \kappa - \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}}, \quad h'_2(z) \equiv h'_1(z) + \sigma \sqrt{T - t}, \]

\[ \gamma' \equiv 2 \left( \frac{r - q - \kappa}{\sigma^2} \right). \]

To value the early exercise, we want a first-touch digital with a barrier of \( k_0 e^{rt} \) and a barrier payment of \( X - S = X - k_0 e^{rt} \). This can be calculated in two parts. The value of a constant payment \( X \) made at the first touch of an exponentially growing barrier can be determined, like the digital option and share, by replacing the dividend yield \( q \) by \( q - \kappa \) in the definitions of \( b \) and \( \beta \) in (19c), giving

\[ T(S, t; T; X, k_0 e^{rt}) = X\left[(k_i/S)^{\nu - \beta} \Phi(\eta_1(k_i/S, \beta)) - (k_i/S)^{\nu + \beta} \Phi(\eta_2(k_i/S, \beta'))\right], \tag{22} \]

where \( \eta_{1,2} \) are as defined below (19).

To handle the growing portion \( k_0 e^{rt} \) of the payment, a similar technique can be used. Discounting a payment increasing exponentially at the rate \( \kappa \) at an interest rate \( r \) is equivalent to discounting a constant payment at the rate \( r - \kappa \). So this time we replace \( r \to r - \kappa \) and \( q \to q + \kappa \), giving

\[ T(S, t; T; k_0 e^{rt}, k_0 e^{rt}) = k_i\left[(k_i/S)^{\nu - \beta} \Phi(\eta_1(k_i/S, \beta)) - (k_i/S)^{\nu + \beta} \Phi(\eta_2(k_i/S, \beta'))\right], \tag{23} \]

where \( \eta_{1,2} \) are as defined below (19).

The exponential barrier approximation for an American put is then

\[ P_{Am} \equiv P_{exp \text{barr}} = \max\left[T(S, t; T; X, k_0 e^{rt}) - T(S, t; T; k_0 e^{rt}, k_0 e^{rt}) + X(D(S, t; T; S_T < X \cap \tau_K \geq T) - S(S, t; T; S_T < X \cap \tau_K \geq T)\right]. \tag{24} \]

The exponential barrier approximation has similar properties to the constant barrier approximation. Again, at different stock prices, distinct values of \( k_i \) and \( \kappa \) will give the best estimate. Any put values estimated using the exponential barrier approximation will, of course, still underestimate the true put value since the best exponential policy is still suboptimal. However, these estimates must be higher than the constant barrier estimates since constant barriers are included in the class of exponential barriers, \( \kappa = 0 \).

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11 The present value of a single payment \( X(t) = x_0 e^{rt} \) received at a random time \( t \) is

\[ T(S, t; T; x_0 e^{rt}, K(\cdot); r, q, \sigma) = \int_0^T e^{-(r - \theta)(t - \tau)} \Phi(t; S, r; K(\cdot); r - \theta - (q - \theta), \sigma) \]

\[ = T(S, t; T; x_0, K(\cdot); r - \theta, q - \theta, \sigma). \]

We must also make the adjustment \( q \to q + \kappa \) for the exponential barrier. In this case, the payment and the barrier are both growing at the same rate \( \kappa \), so \( \kappa \) is added to and subtracted from \( q \), leaving it the same, while \( r \to r - \kappa \).
Table 1 compares the two barrier approximations with other estimates which have been published. The parameter values used are $X = 100$, $r = 7\%$, $q = 0$, and $\sigma = 30\%$. Correct put values are determined using a binomial model with 1000 steps and a control variate correction. Put values with maturities of 0.5, 1, 2, 5, 10, and 20 years are estimated for stock prices of $S = 90, 100, \text{ and } 110$.

All of these estimation methods require comparable amounts of computer time. In each case, the most time-consuming part of the estimation is the optimization search. The GJ, BW, and constant barrier methods are a little faster since they all involve only a single parameter in the optimization compared with two parameters for the exponential barrier and three for the BJ method. Nevertheless, even the optimizers built into spreadsheets handle these tasks in negligible time.

The exponential barrier approximation is always within 2.6 cents of the correct answer; its largest percentage error is 0.22%. The average absolute error is 2 cents. The constant barrier approximation is always within 10.8 cents of the correct value; its largest percentage error is 0.99%. Its average absolute error is 6 cents.

In every case but one, the exponential barrier approximation yields the best estimate. The one exception is the two-year option for a stock price of 90; in that case the Barone-Adesi and Whaley approximation is the closest estimate. Nevertheless the average absolute error in the BW approximation is more than ten times as large as the average absolute error of the exponential barrier approximation and more than three times as large as the average absolute error of the constant barrier approximation. Looking at just short-term puts, those with maturities of two years or less, the exponential barrier approximation is still the best. Its average absolute error is just over 1 cent. The constant barrier and BW approximations are similar. The other estimation methods fare far worse. For long-term options, those with maturities above two years, the average absolute errors for the constant and exponential barrier approximations are 5 cents and 2 cents. The average absolute error for the BW estimation is 30 cents. Again, the other methods have much larger errors. In particular, the extrapolated answers for long-term options must be regarded with suspicion.

6. DELTA AND OTHER GREEK MEASURES

The barrier approximation estimates the value of the option as a portfolio of digital contracts. Since the delta (gamma, vega, or theta) of a portfolio is equal to the sum of the

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12 A European put option was used as a control variate with the correction

$$\hat{P}_{\text{Am}} = P_{\text{binomial}, \text{Am}} + P_{\text{Eu}} - P_{\text{binomial}, \text{Eu}}.$$  

The European value was computed with the Black–Scholes model. The control variate correction was never greater than 0.003. Option values for stock prices between the binomial nodes were computed with linear interpolation. Since the put's value is convex in the stock price, linear interpolation will give values which are slightly above the true value. This will overstate very slightly the errors in the barrier approximation which must give values below the true price. The effect on the other approximation methods is unknown.

13 Puts with long maturities are not commonly found, but these longer-maturity estimates serve to calibrate the approximation scheme which can be applied to other long-maturity derivatives, such as convertible bonds, as well.
Table 1. Comparison of different approximations for American put prices: $S = 90, 100, 110; X = 100; r = 7\%; q = 0; \sigma = 30\%$.

<table>
<thead>
<tr>
<th>$S = 90$</th>
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<th>$S = 110$</th>
<th>Barrier Approx.</th>
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<td>----------</td>
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<tr>
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<tr>
<td>20</td>
<td>15.19</td>
<td>4.02</td>
<td>8.38</td>
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</tbody>
</table>

Absolute Dollar Errors
- maximum: 16.13, 10.39, 4.66, 2.26, 6.63, 33.10, 0.10, 0.03
- average: 4.95, 2.79, 0.75, 0.76, 0.18, 1.88, 1.20, 0.46, 0.06, 0.02

Absolute Percentage Errors
- maximum: 76.72, 49.45, 22.19, 14.90, 26.95, 31.55, 157.47, 3.28, 0.99, 0.22
- average: 30.46, 16.92, 4.36, 4.70, 11.90, 6.71, 27.47, 1.28, 0.55, 0.12

Exact price is computed with a binomial model of 1000 steps and a control variate correction using a European put. GJ is the Geske-Johnson price computed with equation (3). BJ is the optimized Geske-Johnson price computed by the Bunch–Johnson method in equation (7). Rehrdsn is the extrapolated GJ or BJ price computed with equation (5). HSS is the Ho–Stapleton–Subrahmanyan extrapolation computed with equation (6). BW is the Barone-Adesi–Whaley price computed with equation (1). Barrier approximations are computed by (20) and (24) in this paper.
deltas (gammas, vegas, or thetas) of the component parts, the approximate delta of a put as computed from the constant barrier approximation is

\[ \Delta_{\text{con,barr}} \approx X \cdot \Delta_D(S_t; T; \tau > T) - \Delta_S(S_t; T; \tau > T) + \Delta_T(S_t; T; \tau > T). \]  

(25)

Similarly, the delta of a put computed from the exponential barrier approximation is

\[ \Delta_{\text{exp,barr}} \approx \Delta_T(S_t; T; \tau > T) - \Delta_S(S_t; T; \tau > T) + \Delta_T(S_t; T; \tau > T). \]  

(26)

Note that, in each case, the deltas of the digitalis evaluated at the value-maximizing parameters are computed; we do not compute new parameters which maximize the delta.\(^{14}\)

The deltas for the digitalis used in the approximations are

\[ \Delta_D = e^{-r(T-t)} \left\{ \frac{\phi(h_1(S/k^*)) - \phi(h_1(S/X))}{S \sigma \sqrt{T-t}} - \frac{1}{k^*} \left( \frac{k^*}{S} \right)^{1+\gamma} \left[ (1-\gamma) \left( \phi(h_1(k^*/S)) - \phi(h_1(k^{*2}/SX)) \right) - \frac{\phi(h_1(k^*/S)) - \phi(h_1(k^{*2}/SX))}{\sigma \sqrt{T-t}} \right) \right\}, \]  

(27a)

\[ \Delta_S = e^{-r(T-t)} \left\{ \phi(h_2(S/k^*)) - \phi(h_2(S/X)) + \frac{\phi(h_2(S/k^*)) - \phi(h_2(S/X))}{\sigma \sqrt{T-t}} + \left( \frac{k^*}{S} \right)^{1+\gamma} \left( \gamma \left( \phi(h_2(k^*/S)) - \phi(h_2(k^{*2}/SX)) \right) + \frac{\phi(h_2(k^*/S)) - \phi(h_2(k^{*2}/SX))}{\sigma \sqrt{T-t}} \right) \right\}, \]  

(27b)

\[ \Delta_T(S_t; T; \tau = 0) = \frac{k^{b+\beta}}{S^{1+b+\beta}} \left( \beta + \phi \left( \frac{\eta_1(k/S, \beta)}{\sigma \sqrt{T-t}} \right) \right) - \frac{k^{b+\beta}}{S^{1+b+\beta}} \left( \beta + \phi \left( \frac{\eta_2(k/S, \beta)}{\sigma \sqrt{T-t}} \right) \right). \]  

(27c)

These equations hold for all of the digitalis employed here. We use \( \gamma \) or \( \gamma' \) and \( k \) or \( k^* \) for the digital options and shares as appropriate. For the first-touch digital we use \( b, b' \), or \( b'' \) and \( \beta, \beta', \) or \( \beta'' \), as required. Each first-touch digital delta is multiplied by the payment received, \( X - k, X, \) or \( -k \).

Table 2 compares the deltas as computed from the barrier approximations with the true deltas and the GJ and BW approximations. Again the barrier approximations give the closest estimates. Their average absolute errors are less than half that of the HSS extrapolation of the GJ method and less than one-quarter the size of the errors from the BW approximation (which gave the next-best price estimates).

\(^{14}\) The deltas computed in (25) and (26) are the deltas of the approximating portfolios. Since the maximizing parameters depend on the stock price, it might seem that using derivatives like

\[ \Delta = \left[ X \left( \frac{\partial D}{\partial S} \frac{\partial S}{\partial S} + \frac{\partial T}{\partial S} \right) + \left( X \left( \frac{\partial D}{\partial k} \frac{\partial S}{\partial k} + \frac{\partial T}{\partial k} \right) \right) \right] \frac{\partial k^*}{\partial S} \]

would better approximate the delta. Recall, however, that \( k^* \) is the maximizing value of \( k \), so the term in the brackets is equal to zero at \( k = k^* \), and these two estimates are equal.
Table 2. Comparison of different approximations for American put deltas: \( S = 90, 100, 110; \ X = 100; \, r = 7\%; \, q = 0; \, \sigma = 30\%.

<table>
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<th>( S = 90 )</th>
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<th>Exact</th>
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<th>GJ</th>
<th>Rhrdsn</th>
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<th>BJ</th>
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<th>BW</th>
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<th>Rhrdsn</th>
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<th>Rhrdsn</th>
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<td>-0.0371</td>
<td>-0.0942</td>
<td>-0.1514</td>
<td>-0.2316</td>
<td>-0.1914</td>
<td>-0.3438</td>
<td>-0.7738</td>
<td>-0.2194</td>
<td>-0.2248</td>
<td>-0.2247</td>
<td></td>
</tr>
</tbody>
</table>

**Absolute Dollar Errors**
- maximum: 0.3112, 0.2289, 0.1465, 0.0234
- average (\( \tau < 2 \)): 0.0567, 0.0242, 0.0090, 0.0119
- average (\( \tau \geq 5 \)): 0.1978, 0.1248, 0.0520, 0.0162

**Absolute Percentage Errors**
- maximum: 1.48, 1.09, 0.70, 0.22
- average (\( \tau < 2 \)): 0.54, 0.24, 0.09, 0.11
- average (\( \tau \geq 5 \)): 1.16, 0.72, 0.29, 0.10

Barrier approximation deltas are computed with the methods of this paper using (25) or (26) and (27). Deltas for other approximations are computed as \(|P(5 + 0.01) - P(5)|/0.01\) with their prices determined as in Table 1.
7. APPLICATIONS TO OTHER DERIVATIVE CONTRACTS

Barrier approximations can be applied to many other derivative contracts with American-style exercise or other decisions affecting the value. Call options on stock indices or other basis assets whose dividend may be assumed to be paid continuously are obvious examples. In fact the results here can be applied immediately by invoking put-call equivalence for the Black–Scholes model

\[ C_{BS}(S; t; T, X, r, \sigma, q) = P_{BS}(X; t; T, S, q, \sigma, r) \]  

(28)

Convertible bonds and many OTC equity derivatives can also be handled in such a direct fashion. In other cases, the basic method can be adapted to other derivatives.

To illustrate, consider shout contracts which give the right to initiate a call or put option with a strike price based on the prevailing stock price. Shouts come in many varieties, each with their own special provisions. A fixed-tenor shout converts into an at-the-money (or a given percentage in-the-money or out-of-the-money) option with a given time to maturity upon the owner’s ‘shout’. A fixed-maturity shout converts into an option with a given maturity date. Usually fixed-maturity shouts are marketed as so-called modified shout options which are ordinary options plus the right to lock in the current in-the-money and convert to an at-the-money option with the original maturity date. That is, once the owner shouts, the payoff on the contract is

\[ X - S_{\text{shout}} + \max(S_{\text{shout}} - S_T, 0), \]  

(29)

where \( S_{\text{shout}} \) is the stock price at the time of the shout. The payoff is \( \max(X - S_T, 0) \) as usual if there is no shout.

Suppose the owner follows the policy of shouting the first time the stock price falls to \( k < X \).\(^{15}\) The payoff on the contract is then

<table>
<thead>
<tr>
<th>( S_{\text{min}} \leq k )</th>
<th>( S_{\text{min}} &gt; k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_T \geq X )</td>
<td>( X - k )</td>
</tr>
<tr>
<td>( X &gt; S_T ) ( &gt; k )</td>
<td>( X - k )</td>
</tr>
<tr>
<td>( k \geq S_T )</td>
<td>( X - k + k - S_T = X - S_T )</td>
</tr>
</tbody>
</table>

There are a number of ways to represent this payoff with digital shares and options. Perhaps the simplest is

\[
\text{Shout} \geq \max_k \left\{ X[e^{-\sigma(T-t)} - \mathcal{D}(S; t; T; S_{\text{min}} > k \land S_T \geq X)] - k \mathcal{D}(S; t; T; S_{\text{min}} \leq k < S_T) \right. \\
\left. - \mathcal{S}(S; t; T; k < S_{\text{min}} \leq S_T < X) - \mathcal{S}(S; t; T; S_T \leq k) \right\}.
\]  

(30)

Here a payment of \( X \) is always received unless no shout occurred, \( S_{\text{min}} > k \), and the contract expires out of the money, \( S_T > X \). If no shout occurred and the stock price ends

\(^{15}\) It cannot be optimal to shout when the contract is out of the money. The payoff then is \( X - k + \max(k - S_T, 0) = \max(X - S_T, X - k) \), which is dominated by the payoff of not shouting, \( \max(X - S_T, 0) \).
below $X$ or a shout occurred and the stock price ends below $k$, a share of stock must be given up. Conversely, if a shout occurred and the final stock price is above $k$, then $k$ is given up (for a net payment of $X - k$).

The first digital option value can be determined from Merton’s down-and-out call formula

$$
D(S, t; T; S_{\text{min}} > k \cap S_T \geq X) = e^{-n(T-t)}[\Phi(h_1(S/X)) - (k/S)^{y-1}\Phi(h_1(k^2/SX))].
$$

(31)

The second digital option value can be determined from (31):

$$
D(S, t; T; S_{\text{min}} \leq k < S_T) = D(S, t; T; S_T > k) - D(S, t; T; S_{\text{min}} > k \cap S_T \geq X)
= e^{-n(T-t)}(k/S)^{y-1}\Phi(h_1(k/S)).
$$

(32)

The first digital share value in (30) was given in (19b). The second digital share value comes immediately from the Black–Scholes European put formula

$$
S(S, t; T; S_T \leq k) = Se^{-q(T-t)}\Phi(-h_2(S/k)).
$$

(33)

An approximate value for the shout can also be computed using a modification of the GJ or BJ methods. In either case there is a single shout date $T^\circ$. Since this is the only chance to shout, one should be made if the stock price is less than $X$. If there is a shout, then there is a guaranteed payoff at maturity of $X - S_T$ plus an additional possible payoff of $\text{max}(0, S_T - S_T)$. The value of the former on the shout date is $(X - S_T)e^{-\gamma(T-T^\circ)}$. The latter is an at-the-money put, so its value on the shout date is $P_{BS}(S_T e^{-\gamma(T-T^\circ)}, T - T^\circ; S_T)$. Even if there is no shout, there is still a payoff of $\text{max}(0, X - S_T)$. The value of the shout contract can be determined by discounting the third component’s expected value at $T$ and the first two components’ expected values at $T^\circ$. The present value is

$$
\text{Shout}_{\text{GJ}} = XD(S, t; T; S_T > X \cap S_T < X) - S(S, t; T; S_T > X \cap S_T < X)
+ e^{-\gamma(T-T^\circ)}[XD(S, t; T^\circ; S_T \leq k) - S(S, t; T^\circ; S_T \leq k)]
+ S(S, t; T^\circ; S_T \leq X)P_{BS}(e^{-\gamma(T-T^\circ)}, T - T^\circ; 1).
$$

(34)

where

$$
D(S, t; T; S_T > X \cap S_T < X) = e^{-n(T-t)}\Phi(h_1(S/X), -h_1(S/X), -\rho),
$$

$$
S(S, t; T; S_T > X \cap S_T < X) = Se^{-q(T-t)}\Phi(h_2(S/X), -h_2(S/X), -\rho).
$$

$$
D(S, t; T^\circ; S_T \leq X) = e^{-n(T-t)}\Phi(h_1(S/X)),
$$

$$
S(S, t; T^\circ; S_T \leq k) = Se^{-q(T-t)}\Phi(h_2(S/X)),
$$

$$
\rho = \sqrt{(T^\circ - \gamma)/(T - \gamma)}.
$$

The final line relies on the homogeneity of the Black–Scholes function; namely a put option is equivalent in value to $P_{BS}(e^{-\gamma(T-T^\circ)}, T - T^\circ; X/S)$ shares of stock.

The approximation based on the Bunch-Johnson method is similar. Instead of specifying the shout date to be halfway through the option’s life, $T^\circ$ is chosen to maximize the value in (32).
Table 3 compares the barrier approximation for a shout contract with the values computed from formulas based on modifications of the GJ and BJ methods. The barrier method clearly provides a much better approximation than Bermuda-option-based methods of GJ and BJ. The average absolute error is a little more than one-third of the extrapolated GJ absolute errors and about one-fifth of the extrapolated BJ errors.

More exotic contracts may necessitate more complicated barrier digitalis for pricing. For example, an up-and-in put option will be exercised when it is sufficiently in the money, but only after the stock price has risen to the knock-in barrier (or in strike). To value this contract, we require digitalis for the following event: the stock price rises to \( H \), the in strike, and subsequently falls to \( K^*(t) \), the optimal exercise policy.

Let \( \tau_H \) denote the first time the stock price is equal to \( H \) and \( \tau_{HK} \) denote the first time after \( \tau_H \) that the stock price is equal to \( K^*(t) \). The payoff when the latter occurs is

\[
\text{Payoff} = \max(0, S - K^*)
\]

### Table 3. Comparison of different approximations for shout contract prices: \( S = 90, 100, 110; X = 100; r = 7\%; q = 0; \sigma = 30\% \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>Amer. put</th>
<th>Exact shout</th>
<th>GJ</th>
<th>BJ</th>
<th>GJ</th>
<th>BJ</th>
<th>Barrier</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S = 90 )</td>
<td>0.25</td>
<td>11.15</td>
<td>14.42</td>
<td>13.00</td>
<td>15.31</td>
<td>14.17</td>
<td>17.64</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>12.36</td>
<td>15.79</td>
<td>14.11</td>
<td>16.64</td>
<td>15.14</td>
<td>18.70</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>13.26</td>
<td>16.76</td>
<td>14.81</td>
<td>17.42</td>
<td>15.59</td>
<td>18.99</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>14.01</td>
<td>17.48</td>
<td>15.26</td>
<td>17.90</td>
<td>15.78</td>
<td>18.95</td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>14.53</td>
<td>18.01</td>
<td>15.55</td>
<td>18.17</td>
<td>15.70</td>
<td>18.49</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>15.12</td>
<td>18.42</td>
<td>15.72</td>
<td>18.31</td>
<td>15.82</td>
<td>18.50</td>
</tr>
<tr>
<td></td>
<td>1.75</td>
<td>15.49</td>
<td>18.74</td>
<td>15.81</td>
<td>18.34</td>
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<td>18.45</td>
</tr>
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<td></td>
<td>2.00</td>
<td>15.99</td>
<td>18.97</td>
<td>15.84</td>
<td>18.30</td>
<td>15.87</td>
<td>18.35</td>
</tr>
<tr>
<td>( S = 100 )</td>
<td>0.25</td>
<td>5.25</td>
<td>6.83</td>
<td>6.19</td>
<td>7.27</td>
<td>6.19</td>
<td>7.28</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>7.04</td>
<td>9.06</td>
<td>8.10</td>
<td>9.51</td>
<td>8.10</td>
<td>9.51</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>8.25</td>
<td>10.54</td>
<td>9.31</td>
<td>10.90</td>
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<td>10.90</td>
</tr>
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<tr>
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<td>9.90</td>
<td>12.46</td>
<td>10.76</td>
<td>12.53</td>
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<td>13.59</td>
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<tr>
<td>( S = 110 )</td>
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<td>2.08</td>
<td>2.71</td>
<td>2.35</td>
<td>2.66</td>
<td>2.42</td>
<td>2.80</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>3.74</td>
<td>4.84</td>
<td>4.22</td>
<td>4.84</td>
<td>4.28</td>
<td>4.97</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>4.95</td>
<td>6.36</td>
<td>5.52</td>
<td>6.36</td>
<td>5.59</td>
<td>6.49</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>5.93</td>
<td>7.53</td>
<td>6.50</td>
<td>7.49</td>
<td>6.56</td>
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<td>7.25</td>
<td>8.34</td>
<td>7.31</td>
<td>8.47</td>
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<tr>
<td></td>
<td>1.50</td>
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<td>7.84</td>
<td>9.01</td>
<td>7.91</td>
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</tr>
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<td>9.87</td>
<td>8.31</td>
<td>9.52</td>
<td>8.38</td>
<td>9.66</td>
</tr>
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<td>10.40</td>
<td>8.68</td>
<td>9.92</td>
<td>8.76</td>
<td>10.07</td>
</tr>
</tbody>
</table>

**Absolute Errors**

| average (all) | 1.65 | 0.33 | 1.46 | 0.62 | 0.12 |
| average (in the money) | 2.31 | 0.32 | 1.53 | 0.41 | 0.09 |
| average (at the money) | 1.54 | 0.21 | 1.53 | 0.21 | 0.14 |
| average (out of the money) | 1.99 | 0.16 | 1.93 | 0.13 | 0.12 |
| maximum | 3.13 | 0.39 | 3.10 | 0.32 | 0.17 |

American put and exact shout prices are computed with a binomial model of 1000 steps and control variate correction using a European put. Barrier approximation is computed with (30) in this paper.
\( X - K^*(t) \). The value of this put is then

\[
P_{U\&1} = X \cdot D(S, t; T; \tau_H < T \cap \tau_{HK^*} > T \cap S_T < X) \]

\[
- S(S, t; T; \tau_H < T \cap \tau_{HK^*} > T \cap S_T < X) \]

\[
+ T(S, t; T; X - K^*(t) \text{ after } t = \tau_H). \tag{35}
\]

For the digital share and option, the condition \( \tau_H < T \) confirms that the option has been knocked in, while \( \tau_{HK^*} > T \) ensures it has not been exercised prior to maturity. As before, the last two arguments of \( T(\cdot) \) are the payoff at the exercise barrier and a description of the exercise barrier. In this case the exercise barrier only applies after \( \tau_H \), the time at which knock-in occurs.\(^{16}\)

The barrier approximation for the up-and-in put will be the maximum value within a class of restricted policies. For example, for constant exercise policies

\[
P_{U\&1} \geq P_{U\&1, \text{const}} = \max_k \left\{ X \cdot D(S, t; T; \tau_H < T \cap \tau_{HK^*} > T \cap S_T < X) \right. \]

\[
- S(S, t; T; \tau_H < T \cap \tau_{HK^*} > T \cap S_T < X) \]

\[
+ T(S, t; T; X - k \text{ after } t = \tau_H) \right\} \tag{36}
\]

where \( \tau_{HK} \) is the first time the stock price hits the constant policy barrier \( k \) after the first time it has hit the knock-in barrier \( H \).

As shown in the Appendix, the values for these digitals are

\[
D(S, t; T; \tau_H < T \cap \tau_{HK^*} > T \cap S_T < X) = e^{-r(T-t)} \left[ \left( \frac{k}{H} \right)^{y-1} \Phi(h_1(Sk/H^2)) + \left( \frac{H}{S} \right)^{y-1} \Phi(-h_1(H^2/Sk)) \right]. \tag{37a}
\]

\[
S(S, t; T; \tau_H < T \cap \tau_{HK^*} > T \cap S_T < X) = Se^{-q(T-t)} \left[ \left( \frac{k}{H} \right)^{y+1} \Phi(h_2(Sk/H^2)) + \left( \frac{H}{S} \right)^{y-1} \Phi(-h_2(H^2/Sk)) \right]. \tag{37b}
\]

\[
T(S, t; T; X - k \text{ after } t = \tau_H) = \left( \frac{k}{S} \right)^{b-\beta} \left( \frac{k}{H} \right)^{2\beta} \Phi(\eta_2(Sk/H^2, \beta)) + \left( \frac{k}{H} \right)^{b+\beta} \left( \frac{H}{S} \right)^{2\beta} \Phi(\eta_1(Sk/H^2, \beta)). \tag{37c}
\]

Table 4 shows the barrier approximation prices for an up-and-in put contract. The Bermuda-option-based methodology of Geske and Johnson cannot be adapted to up-and-in options, so a comparison is not possible. Nevertheless, the errors are very small, averaging just over 3 cents. The largest error is 8 cents. The largest percentage error is 1.7%. The accuracy of the approximation is about the same regardless of the level of the stock price in the range 90 to 105. Table 4 also shows the best constant exercise policies for the up-and-in and regular American puts. The up-and-in exercise prices are always higher than those for the corresponding ordinary put. Recall that it is most important that the best approximate policy mimic the optimal policy at the most likely times of

\(^{16}\) As with a regular put, the optimal exercise barrier will have \( K^*(T) = X \), so that exercise is assured if a knocked-in option expires in the money. Nevertheless, we separately express the value of exercise at maturity using a digital share and option to cover the case \( q > r \), when the optimal exercise boundary is not continuous at \( T \) and for symmetry with the suboptimal representation in \((3.1)\).
Table 4. Comparison of different approximations for shout contract prices: $S = 90, 100, 105$; $X = 100; H = 110; r = 7\%; \sigma = 30\%$.

<table>
<thead>
<tr>
<th>$S = 90$</th>
<th>Amer. put</th>
<th>Up-and-in puts</th>
<th>Barrier</th>
<th>‘Best’ exercise price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Amer.</td>
<td>Euro.</td>
<td>Approx.</td>
<td>Error</td>
</tr>
<tr>
<td>0.25</td>
<td>11.15</td>
<td>0.13</td>
<td>0.13</td>
<td>0.13</td>
</tr>
<tr>
<td>0.50</td>
<td>12.36</td>
<td>0.71</td>
<td>0.69</td>
<td>0.70</td>
</tr>
<tr>
<td>0.75</td>
<td>13.26</td>
<td>1.41</td>
<td>1.36</td>
<td>1.40</td>
</tr>
<tr>
<td>1.00</td>
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<td>2.09</td>
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<td>3.38</td>
<td>3.12</td>
<td>3.34</td>
</tr>
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<td>3.58</td>
<td>3.89</td>
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<td>15.99</td>
<td>4.44</td>
<td>3.99</td>
<td>4.39</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$S = 100$</th>
<th>Amer. put</th>
<th>Up-and-in puts</th>
<th>Barrier</th>
<th>‘Best’ exercise price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Amer.</td>
<td>Euro.</td>
<td>Approx.</td>
<td>Error</td>
</tr>
<tr>
<td>0.25</td>
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<td>2.85</td>
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<td>5.83</td>
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<td>5.77</td>
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<td>5.64</td>
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</table>

<table>
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<tr>
<th>$S = 105$</th>
<th>Amer. put</th>
<th>Up-and-in puts</th>
<th>Barrier</th>
<th>‘Best’ exercise price</th>
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</thead>
<tbody>
<tr>
<td>$r$</td>
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<td>Euro.</td>
<td>Approx.</td>
<td>Error</td>
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Absolute Errors

maximum 0.08
average 0.03

American put and exact up-and-in put prices are computed with a binomial model of 1000 steps and a control variate correction using a European put. Barrier approximation is computed with (34) in this paper.

exercise. On average, an ordinary put will be exercised earlier than an otherwise identical up-and-in put, because the stock price may fall to the exercise barrier before it hits the knock-in price. Since the optimal exercise boundary rises over time, an ordinary put will therefore be exercised at a lower stock price on average.

Nevertheless, the actual optimal policy for exercising an up-and-in put is identical to the optimal policy for a regular put, since once the knock-in price has been hit, it becomes a regular put (and before the knock-in price has been hit, it cannot be exercised). Of course, even under these constant exercise policies, the actual exercise of the up-and-in put will not differ from a regular put once the in-strike has been reached, because the barrier-pricing method constantly reestimates the barrier.

8. CONCLUSIONS

The valuation of American puts has long baffled financial economists. They must be priced by numerical methods or with approximations. This paper describes an
approximation method based on barrier options. The method is intuitive and can be implemented in many general-purpose computer programs, such as spreadsheets. The numerical computation is quite rapid since it involves only a gradient search over a few variables for the maximum of a convex function. For the options considered here, the resulting values are very tight lower bounds to the true option values.

The present method can be extended to value a wide variety of contracts beyond simple puts and calls. For example, Ingersoll (1998) shows its application to installment options. Other applications are immediate.

APPENDIX

Digitals for up-and-in put option

Let \( W_t \) be a diffusion process evolving according to

\[
dW_t = \mu \, dt + \sigma \, d\omega_t, \tag{A1}
\]

with constant drift \( \mu \) and variance \( \sigma^2 \) per unit time. Let \( \tau_u \) and \( \tau_l \) be stopping times for this process defined as the first time that \( W_t = u > W_0 \) and the first time after \( \tau_u \) that \( W_t = l < u \).

**Lemma 1** For \( x \geq l \),

\[
\Pr \{ \tau_u \leq T \land W_T > x \mid W_0 = w \} = \exp \left( \frac{2\mu}{\sigma^2} \left( \frac{x}{\sigma \sqrt{2}} \right) \right) \Phi \left( \frac{w + 2(l - u) - x + \mu T}{\sigma \sqrt{T}} \right). \tag{A2}
\]

**Proof.** The event \( \tau_u \leq T \) is the event that the diffusion process \( W_t \) rises above \( u \) and then falls below \( l \) before time \( t \). A typical path for this event is shown in Figure 4. Note that \( W_t \) does hit \( l \) before it hits \( u \), but this is ignored. We require a touch at \( l \) after the first touch at \( u \).

The figure also shows the reflection of the early portion of this path prior to its first touch at \( u \). Since the original path starts below the barrier by the amount \( u - w \), the reflected path starts the same distance above the barrier at \( u + (u - w) = 2u - w \). Clearly, there is a one-to-one correspondence between the original paths that start at \( w \), hit \( u \), hit \( l \), and finish above \( x \), and the reflected paths that start at \( 2u - w \), hit \( l \), and finish above \( x \). Now reflect this reflected path before its first touch at \( l \). This doubly reflected path starts at \( l - (2u - w - l) \). Again, there is a one-to-one correspondence between paths which start at \( 2u - w \), hit \( l \), and finish above \( x \), and paths which start at \( w - 2(u - l) \) and finish above \( x \).

While the doubly reflected and original paths can be put into a one-to-one correspondence, the probabilities of the two path types are typically different because some 'up' and 'down' movements have been swapped. As in the single-barrier case, the proportional adjustment in probability required to switch the 'up' and 'down' steps is equal to \( \exp(2\mu/\sigma^2) \cdot (\text{barrier} - \text{start}) \). In this case, the two adjustments are
exp(2μ/σ^2) · (u - w) and exp(2μ/σ^2) · (l + w - 2u). Therefore

\[
\Pr[\tau_{ul} \leq T \cap W_T > x \mid W_0 = w] = \exp\left(\frac{2\mu}{\sigma^2}(u - w) + \frac{2\mu}{\sigma^2}(l + w - 2u)\right) \Pr[W_T > x \mid W_0 = w - 2(u - l)]
\]

\[
= \exp\left(\frac{2\mu - 2u}{\sigma^2}\right)\left[1 - \Phi\left(\frac{x - [w - 2(u - l) + \mu T]}{\sigma \sqrt{T}}\right)\right]. \quad (A3)
\]

Using \(\Phi(z) = 1 - \Phi(-z)\) gives (A2). □

**Lemma 2.** The probability that the diffusion process \(W_t\) rises above \(u\) and then falls below \(l\) before time \(T\), i.e. the probability of \(\{\tau_{ul} \leq T\}\) is

\[
\Pr[\tau_{ul} \leq T \mid W_0 = w]
\]

\[
= \exp\left(\frac{2\mu - 2u}{\sigma^2}\right)\Phi\left(\frac{w + l - 2u + \mu T}{\sigma \sqrt{T}}\right) + \exp\left(\frac{2\mu - 2u}{\sigma^2}\right)\Phi\left(\frac{w + l - 2u - \mu T}{\sigma \sqrt{T}}\right). \quad (A4)
\]

**Proof.** The event \(\{\tau_{ul} \leq T\}\) can be separated into two mutually exclusive subevents, \(\{\tau_{ul} \leq T \cap W_T > l\}\) and \(\{\tau_{ul} \leq T \cap W_T \leq l\}\). The probability of the first subevent is given by Lemma 1 with \(x = l\); this probability is the first term in (A4). In the second subevent, \(W_T \leq l\) guarantees that \(\tau_{ul} \leq T\), so the second subevent is equivalent to \(\{\tau_u < T \cap W_T \leq l\}\).
This subevent is therefore a standard single-barrier probability
\[
\Pr(\tau_{ul} \leq T \wedge W_T \leq l) = \Pr(\tau_u < T \wedge W_T \leq l) \\
= \exp\left(\frac{2\mu(u - w)}{\sigma^2}\right) \Phi\left(\frac{w + l - 2u - \mu T}{\sigma \sqrt{T}}\right). \quad (A5)
\]
Combining terms gives (A4). \(\square\)

**Lemma 3** The event that the diffusion process \(W_t\) rises above \(u\) and then does not fall below \(l\) before time \(T\) is \(\{\tau_u < T \wedge \tau_{ul} > T\}\). Then, for \(x \geq l\),
\[
\Pr(\tau_u < T \wedge \tau_{ul} > T \wedge W_T \leq x \mid W_0 = w) \\
= \exp\left(\frac{2\mu(u - w)}{\sigma^2}\right) \left[ \Phi\left(\frac{w + x - 2u - \mu T}{\sigma \sqrt{T}}\right) - \Phi\left(\frac{w + l - 2u - \mu T}{\sigma \sqrt{T}}\right) \right] \\
+ \exp\left(\frac{2\mu(l - w)}{\sigma^2}\right) \left[ \Phi\left(\frac{w + 2(l - w) - x + \mu T}{\sigma \sqrt{T}}\right) - \Phi\left(\frac{w + l - 2u + \mu T}{\sigma \sqrt{T}}\right) \right]. \quad (A6)
\]

**Proof.** The events \(\{\tau_u < T \wedge \tau_{ul} > T\}\) and \(\{\tau_u < T \wedge \tau_{ul} \leq T\}\) = \(\{\tau_{ul} \leq T\}\) are mutually exclusive and their union is the event \(\{\tau_u < T\}\). So
\[
\Pr(\tau_u < T \wedge \tau_{ul} > T \wedge W_T \leq x) \\
= \Pr(\tau_u < T \wedge W_T \leq x) - \Pr(\tau_{ul} \leq T \wedge W_T \leq x) \\
= \Pr(\tau_u < T \wedge W_T \leq x) - \left[ \Pr(\tau_{ul} \leq T) - \Pr(\tau_{ul} \leq T \wedge W_T > x) \right]. \quad (A7)
\]
The first term is the standard single-barrier probability as given in (A5) for \(x = l\). The second probability is given in (A4) in Lemma 2, and the third probability is given in (A2) in Lemma 1. Combining terms gives (A6).

The digital asset required to approximate the up-and-down put option with a knock-in barrier of \(H\) under the constant exercise policy of \(k\), can be determined from these three lemmas. We assume the standard Black–Scholes conditions. The stock price evolves according to a lognormal diffusion
\[
dS = (\mu - q) dt + \sigma S dw.
\]
The stopping times \(\tau_H\) and \(\tau_{hk}\) for this process are the first time the stock price reaches the knock-in barrier at \(H\) and the first time after \(\tau_H\) that the stock price falls to \(k\) and is exercised according to a constant exercise policy.

**Theorem 1** The exercise-at-maturity event for the up-and-down put option is
\[
\mathcal{E}_1 = \{\tau_H < T \wedge \tau_{hk} > T \wedge S_T < X\}. \quad (A8)
\]
The values of a digital option and a digital share for this event are
\[
D(S, t; T; \mathcal{E}_1) = e^{-\alpha(T-t)} \left\{ (H/S)^{-1} \left[ \Phi(h_1(H^2/Sk)) - \Phi(h_1(H^2/SX)) \right] \\
+ (k/H)^{-1} \left[ \Phi(h_1(Sk^2/H^2 X)) - \Phi(h_1(Sk/H^2)) \right] \right\}, \quad (A9a)
\]
\[
S(S, t; T; \mathcal{E}_1) = S e^{-\alpha(T-t)} \left\{ (H/S)^{-1} \left[ \Phi(h_2(H^2/Sk)) - \Phi(h_2(H^2/SX)) \right] \\
+ (k/H)^{-1} \left[ \Phi(h_2(Sk^2/H^2 X)) - \Phi(h_2(Sk/H^2)) \right] \right\}. \quad (A9b)
\]
Proof. Apply Lemma 3 for \( w = \ln S, u = \ln H, l = \ln k, x = \ln X \), and \( \mu = r - q - \frac{1}{2} \sigma^2 \) to derive

\[
\Pr(\mathcal{E}_1) = (H/S)^{r-1}\left[\Phi(h_1(H^2/Sk)) - \Phi(h_1(H^2/SX))\right] + (k/H)^{r-1}\left[\Phi(h_1(Sk^2/H^2X)) - \Phi(h_1(Sk/H^2))\right]
\]

(A10)

as the risk-neutral probability of exercise at maturity. The value of the digital option for this event is this risk-neutral probability multiplied by the discount factor, which is \((A9a)\). The digital share for this event can be valued by a change to the stock price numeraire as shown by Ingersoll (1997). This is accomplished mechanically by replacing the dividend yield \( q \) by \( q - \sigma^2 \) in \( h_1 \) and \( \gamma \) and replacing the discount factor \( e^{-r(T-t)} \) by \( Se^{-q(T-t)} \), giving \((A9b)\). \( \square \)

**Theorem 2** The early exercise event for the up-and-in put option is

\[
\mathcal{E}_2 = \{\tau_H < T \cap \tau_H \leq T\} = \{\tau_H \leq T\}.
\]

(A11)

The values of a digital option and a digital share for this event are

\[
\mathcal{D}(S, t; T; \mathcal{E}_2) = e^{-r(T-t)}[(k/H)^{r-1}\Phi(h_1(Sk/H^2)) + (H/S)^{r-1}\Phi(-h_1(H^2/Sk))],
\]

(A12a)

\[
\mathcal{S}(S, t; T; \mathcal{E}_2) = Se^{-q(T-t)}[(k/H)^{r+1}\Phi(h_2(Sk/H^2)) + (H/S)^{r+1}\Phi(-h_2(H^2/Sk))].
\]

(A12b)

Proof. Apply Lemma 2 for \( w = \ln S, u = \ln H, l = \ln k, x = \ln X \), and \( \mu = r - q - \frac{1}{2} \sigma^2 \) to derive

\[
\Pr(\mathcal{E}_2) = \Pr(\tau_H \leq T)
\]

\[
= (k/H)^{r-1}\Phi(h_1(Sk/H^2)) + (H/S)^{r-1}\Phi(-h_1(H^2/Sk))
\]

(A13)

as the risk-neutral probability of early exercise. The value of the digital option for this event is this risk-neutral probability multiplied by the discount factor, which is \((A12a)\). The digital share for this event can be valued by a change to the stock price numeraire, just as in Theorem 1, giving \((A12b)\). \( \square \)

Under a constant exercise policy, the up-and-in put will be exercised prior to maturity for \( X - k \) if it is knocked in, \( \tau_H < T \), and the stock price subsequently falls to \( k \) before maturity, \( \tau_H < T \). We need a first-touch digital for time \( \tau_H \). We examine the case when there are no dividends on the underlying stock first.

**Lemma 4** If the stock does not pay dividends, the value of the first-touch digital for \( \{X - k, k \text{ after } t = \tau_H\} \) is

\[
T(S, t; T; X - k, k \text{ after } t = \tau_H)
\]

\[
= \frac{X - k}{k}S[(k/H)^{2r/\sigma^2+1}\Phi(h_2(Sk/H^2)) + (H/S)^{2r/\sigma^2+1}\Phi(-h_2(H^2/Sk))].
\]

(A14)

Proof. When the stock does not pay dividends,

\[
T(S, t; T; X - k, k \text{ after } t = \tau_H) = \frac{X - k}{k}S(S, t; T; \tau_H < T).
\]

(A15)
To confirm (A15), note that the digital shares are worth \([(X - k)/k]S_T\) if \(T_h < T\). But if this is true, the first-touch digital will have paid \(X - k\) at \(T_h\). This money could have been used then to purchase exactly \((X - k)/k\) shares. Since the shares do not pay dividends, this purchase will be worth \([(X - k)/k]S_T\) at maturity, exactly like the digital shares. Now applying Theorem 2 gives (A14). \( \square \)

This construction is not valid when the stock pays dividends, since the proceeds from the first-touch digital invested in stock will earn dividends but the digital shares will not. Theorem 3 handles this general case by creating a derivative asset based on \(S\) which does not pay dividends and has similar constant barriers.

**Theorem 3** The value of a first-touch digital for early exercise is

\[
T(S, t; T; X - k, k \text{ after } t = T_h) = (k/S)^{h - \beta}(k/H)\Phi(\eta_1(Sk/H^2, \beta)) + (k/S)^{h + \beta}\Phi(\eta_1(Sk/H^2, \beta)). \tag{A16}
\]

**Proof.** Create a new derivative asset worth

\[
V = S^{b-a}, \tag{A17}
\]

where

\[
b = \frac{r - q}{\sigma^2} - \frac{1}{2} \quad \text{and} \quad \beta = \sqrt{b^2 + 2r/\sigma^2}.
\]

By Ito’s lemma, the dynamics of \(V\) are

\[
dV = (\beta - b)S^{b-a-1}dS + \frac{1}{2}(\beta - b - 1)(\beta - b)S^{b-a-2}S^2\sigma^2 dt
\]

\[
= (\beta - b)[(\mu - q + \frac{1}{2}(\beta - b - 1)\sigma^2)V dt + (\beta - b)\sigma V d\omega]. \tag{A18}
\]

Substituting \(r\) for \(\mu\) and using \(\beta^2 - b^2 = 2r/\sigma^2\) gives the risk-neutral dynamics of \(V\) as

\[
dV = rV dt + (\beta - b)\sigma V d\omega. \tag{A19}
\]

Since the risk-neutral expected rate of growth in the variable \(V\) is \(r\), we may treat it as if it were the price of an asset with no dividends.\(^{17}\) Therefore, the construction in (A15) is valid for a first-touch digital based on \(V\). The barriers for \(V\) corresponding to \(H\) and \(k\) are \(H^{b-a}\) and \(k^{b-a}\), and, since \(\beta > b\), these are upper and lower barriers, respectively. Furthermore, since \(S = H\) if and only if \(V = H^{b-a}\), and \(S = k\) if and only if \(V = k^{b-a}\), the stopping times \(T_h\) and \(T_k\) apply to this asset as well. More importantly, first-touch digitals written on \(V\) and \(S\) are identical if they make the same payments.

The value of the \(V\) digital share can be determined from Theorem 2 as

\[
S(V, t; T; \mathcal{E}_T) = V[(k/H)^{\beta-a}(y_{+1})\Phi(h_2(Vk^{b-a}/H^{2(b-a)}))
\]

\[
+(H^{b-a}/V)^{y_{-1}}\Phi(-h_2(H^{2(b-a)}/Vk^{b-a}))). \tag{A20}
\]

\(^{17}\) In fact \(V\) is the price of a derivative asset with a payoff of \(S_T^{b-a}\) at time \(T\). Since \(V \equiv S^{b-a}\) at all times, it doesn’t matter when this contract matures: it is always worth \(S^{b-a} = V\).
Substituting for $V = S^{\beta - b}$ and $\gamma = 2r/[(\beta - b)\sigma]^{2}$ and simplifying terms gives

$$S(S, t; T; \xi_{2}) = (Sk/H)^{\beta - b}(k/H)^{\beta + b}\Phi(\eta_{2}(Sk/H^{2}, \beta)) \quad + H^{\beta - b}(H/S)^{\beta + b}\Phi(\eta_{2}(Sk/H^{2}, \beta)). \quad (A21)$$

Now, using Lemma 4, we get

$$T(S, t; T; X - k, k \text{ after } t = \tau_{H})$$

$$= T(V, t; T; X - k, k^{\beta - b} \text{ after } t = \tau_{H})$$

$$= \frac{X - k}{k^{\beta - b}} S(V, t; T; \xi_{2})$$

$$= (k/S)^{\beta - b}(k/H)^{2\beta}\Phi(\eta_{2}(Sk/H^{2}, \beta)) + (k/S)^{\beta + b}(H/k)^{2\beta}\Phi(\eta_{2}(Sk/H^{2}, \beta)), \quad (A22)$$

which is (37c) in the text.

REFERENCES


