X-CAPM: An extrapolative capital asset pricing model ☆

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ABSTRACT

Survey evidence suggests that many investors form beliefs about future stock market returns by extrapolating past returns. Such beliefs are hard to reconcile with existing models of the aggregate stock market. We study a consumption-based asset pricing model in which some investors form beliefs about future price changes in the stock market by extrapolating past price changes, while other investors hold fully rational beliefs. We find that the model captures many features of actual prices and returns; importantly, however, it is also consistent with the survey evidence on investor expectations.

JEL classification: G02, G12
Keywords: expectations, extrapolation, predictability, volatility

☆ We are grateful to Bill Schwert (the editor), an anonymous referee, Markus Brunnermeier, John Campbell, Juhani Linnainmaa, Alan Moreira, David Solomon, Lawrence Summers, and seminar participants at Caltech, Dartmouth College, Harvard University, the London Business School, the London School of Economics, the Massachusetts Institute of Technology, Oxford University, the University of California at Berkeley, the University of Chicago, the University of Southern California, Yale University, the NBER Summer Institute, and the Stanford Institute for Theoretical Economics for useful feedback; and to Jonathan Ingersoll for many helpful conversations.
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1. Introduction

Recent theoretical work on the behavior of aggregate stock market prices has tried to account for several empirical regularities. These include the excess volatility puzzle of LeRoy and Porter (1981) and Shiller (1981), the equity premium puzzle of Mehra and Prescott (1985), the low correlation of stock returns and consumption growth, and, most importantly, the evidence on predictability of stock market returns using the aggregate dividend-price ratio (Campbell and Shiller, 1988; Fama and French, 1988). Both traditional and behavioral models have tried to account for this evidence.

Yet this research has largely neglected another set of relevant data, namely those on actual investor expectations of stock market returns. As recently summarized by Greenwood and Shleifer (2014) using data from multiple investor surveys, many investors hold extrapolative expectations, believing that stock prices will continue rising after they have previously risen, and falling after they have previously fallen.[1] This evidence is inconsistent with the predictions of many of the models used to account for the other facts about aggregate stock market prices. Indeed, in most traditional models, investors expect low returns, not high returns, if stock prices have been rising: in these models, rising stock prices are a sign of lower investor risk aversion or lower perceived risk. Cochrane (2011) finds the survey evidence uncomfortable, and recommends discarding it.

In this paper, we present a new model of aggregate stock market prices which attempts to both incorporate extrapolative expectations held by a significant subset of investors, and address the evidence that other models have sought to explain. The model includes both rational investors and price extrapolators, and examines security prices when both types are active in the market. Moreover, it is a consumption-based asset pricing model with infinitely lived consumers optimizing their decisions in light of their beliefs and market prices. As such, it can be directly compared to some of the existing

[1] Greenwood and Shleifer (2014) analyze data from six different surveys. Some of the surveys are of individual investors, while others cover institutions. Most of the surveys ask about expectations for the next year's stock market performance, but some also include questions about the longer term. The average investor expectations computed from each of the six surveys are highly correlated with one another and are all extrapolative. They are also *negatively* related to subsequent realized returns, which makes it hard to interpret them as rational forecasts. Earlier studies of stock market investor expectations include Vissing-Jorgensen (2004), Bacchetta, Mertens, and van Wincoop (2009), and Amromin and Sharpe (2013).

research. We suggest that our model can reconcile the evidence on expectations with the evidence on volatility and predictability that has animated recent work in this area.

Why is a new model needed? As Table 1 indicates, traditional models of financial markets have been able to address pieces of the existing evidence, but not the data on expectations. The same holds true for preference-based behavioral finance models, as well as for the first generation belief-based behavioral models that focused on random noise traders. Several papers listed in Table 1 have studied extrapolation of fundamentals. However, these models also struggle to match the survey evidence: after good stock market returns driven by strong cash flows, the investors they describe expect higher *cash flows*, but, because these expectations are reflected in the current price, they do not expect higher *returns*.[2] Finally, a small literature, starting with Cutler, Poterba, and Summers (1990) and DeLong et al. (1990b), focuses on models in which some investors extrapolate prices. Our goal is to write down a more "modern" model of price extrapolation that includes infinite horizon investors, some of whom are fully rational, who make optimal consumption decisions given their beliefs, so that the predictions can be directly compared to those of the more traditional models.

Our infinite horizon continuous-time economy contains two assets: a risk-free asset with a fixed return; and a risky asset, the stock market, which is a claim to a stream of dividends and whose price is determined in equilibrium. There are two types of traders. Both types maximize expected lifetime consumption utility. They differ only in their expectations about the future. Traders of the first type, "extrapolators," believe that the expected price change of the stock market is a weighted average of past price changes, where more recent price changes are weighted more heavily. Traders of the second type, "rational traders," are fully rational: they know how the extrapolators form their beliefs and trade accordingly. The model is simple enough to allow for a closed-form solution.

[2] For example, in the cash-flow extrapolation model of Barberis, Shleifer, and Vishny (1998), investors' expectations of returns remain constant over time, even though their expectations of cash flows vary significantly. More elaborate models of cash-flow extrapolation—for example, models with both extrapolators and rational traders—may, as a byproduct, come closer to matching the survey evidence; here, we present an alternative approach that may be simpler and more direct. Models in which investors try to learn an unknown cash-flow growth rate face similar challenges to models of cash-flow extrapolation. We discuss learning-based models in more detail in Section 2.1.

We first use the model to understand how extrapolators and rational traders interact. Suppose that, at time *t*, there is a positive shock to dividends. The stock market goes up in response to this good cash-flow news. However, the extrapolators cause the price jump to be amplified: since their expectations are based on past price changes, the stock price increase generated by the good cash-flow news leads them to forecast a higher *future* price change on the stock market; this, in turn, causes them to push the time *t* stock price even higher.

More interesting is rational traders' response to this development. We find that the rational traders do *not* aggressively counteract the overvaluation caused by the extrapolators. In part, this is because they are risk averse. However, it is also because they reason as follows. The rise in the stock market caused by the good cash-flow news—and by extrapolators' reaction to it—means that, in the near future, extrapolators will continue to have bullish expectations for the stock market: after all, their expectations are based on past price changes, which, in our example, are high. As a consequence, extrapolators will continue to exhibit strong demand for the stock market in the near term. This means that, even though the stock market is overvalued at time *t*, its returns in the near future will not be particularly low—they will be bolstered by the ongoing demand from extrapolators. Recognizing this, the rational traders do not sharply decrease their demand at time *t*; they only mildly reduce their demand. Put differently, they only partially counteract the overpricing caused by the extrapolators.

Using a combination of formal propositions and numerical analysis, we then examine our model's predictions about prices and returns. We find that these predictions are consistent with several key facts about the aggregate market and, in particular, with the basic fact that when its price is high (low) relative to dividends, the stock market subsequently performs poorly (well). When good cash-flow news is released, the stock price in our model jumps up more than it would in an economy made up of rational investors alone: as described above, the price jump caused by the good cash-flow news feeds into extrapolators' expectations, which, in turn, generates an additional price increase. At this point, the stock market is overvalued and its price is high relative to dividends. Since, subsequent to the overvaluation, the stock market performs poorly on average, its price level relative to dividends predicts subsequent price changes in our model, just as it does in actual data. The same mechanism also generates excess volatility—stock market prices are more volatile than can be explained by rational forecasts of future cash flows—as well as negative autocorrelations in price changes at all horizons, capturing the negative autocorrelations we see at longer horizons in actual data.

The model also matches some empirical facts that, thus far, have been taken as evidence for other models. For example, in actual data, surplus consumption, a measure of how current consumption compares to past consumption, is correlated with the value of the stock market and predicts the market's subsequent return. These facts have been taken as support for habit-based models. However, they also emerge naturally in our framework.

Our numerical analysis allows us to quantify the effects described above. Specifically, we use the survey data studied by Greenwood and Shleifer (2014) and others to parameterize the functional form of extrapolation in our model. For one reasonable parameterization, we find that if 50% of investors are extrapolators while 50% are rational traders, the standard deviation of annual price changes is 30% higher than in an economy consisting of rational traders alone.

There are aspects of the data that our model does not address. For example, even though some of the investors in the economy are price extrapolators, the model does not predict the positive autocorrelation in price changes observed in the data at very short horizons. Also, there is no mechanism in our model, other than high risk aversion, that can generate a large equity premium. And while the presence of extrapolators reduces the correlation of consumption changes and price changes, this correlation is still much higher in our model than in actual data.

In summary, our analysis suggests that, simply by introducing some investors with extrapolative expectations into an otherwise traditional consumption-based model of asset prices, we can make sense not only of some important facts about prices and returns, but also, by construction, of the available evidence on the expectations of real-world investors. This suggests that the survey evidence need not be seen as a nuisance, or as an impediment to understanding the facts about prices and returns. On the contrary, the extrapolation observed in the survey data is consistent with the facts about prices and returns, and may be the key to understanding them.

In Section 2, we present our model and its solution, and discuss some of the basic insights that emerge from it. In Section 3, we assign values to the model parameters. In

Section 4, we show analytically that the model reproduces several key features of stock prices. Our focus here is on quantities defined in terms of differences—price changes, for example; given the structure of the model, these are the natural objects of study. In Section 5, we use simulations to document the model's predictions for ratio-based quantities, such as the price-dividend ratio, which are more commonly studied by empiricists. Section 6 concludes. All proofs and some discussion of technical issues are in the Appendix.

2. The model

In this section, we propose a heterogeneous-agent, consumption-based model in which some investors extrapolate past price changes when making forecasts about future price changes. Constructing such a model presents significant challenges, both because of the heterogeneity across agents, but also because it is the change in *price*, an endogenous quantity, that is being extrapolated. By contrast, constructing a model based on extrapolation of exogenous fundamentals is somewhat simpler. To prevent our model from becoming too complex, we make some simplifying assumptions: about the dividend process (a random walk in levels), about investor preferences (exponential utility), and about the risk-free rate (an exogenous constant). We expect the intuitions of the model to carry over to more complex formulations.[3]

We consider an economy with two assets: a risk-free asset in perfectly elastic supply with a constant interest rate r; and a risky asset, which we think of as the aggregate stock market, and which has a fixed per-capita supply of Q. The risky asset is a claim to a continuous dividend stream whose level per unit time evolves as an arithmetic Brownian motion

$$dD_t = g_D dt + \sigma_D d\omega_t \ , \tag{1}$$

where g_D and σ_D are the expected value and standard deviation of dividend changes, respectively, and where ω_t is a standard one-dimensional Wiener process. Both g_D and σ_D are constant. The value of the stock market at time t is denoted by P_t and is determined endogenously in equilibrium.

[3] Several other models of the aggregate stock market make similar assumptions; see, for example, Campbell and Kyle (1993) and Wang (1993). We discuss the constant interest rate assumption in Section 2.1.

There are two types of infinitely lived traders in the economy: "extrapolators" and "rational traders." Both types maximize expected lifetime consumption utility. The only difference between them is that one type has correct beliefs about the expected price change of the risky asset, while the other type does not.

The modeling of extrapolators is motivated by the survey evidence analyzed by Vissing-Jorgensen (2004), Bacchetta, Mertens, and van Wincoop (2009), Amromin and Sharpe (2013), and Greenwood and Shleifer (2014). These investors form beliefs about the future price change of the stock market by extrapolating the market's past price changes. To formalize this, we introduce a measure of "sentiment," defined as

$$S_t = \beta \int_{-\infty}^{t} e^{-\beta(t-s)} dP_{s-dt}, \qquad \beta > 0, \tag{2}$$

where s is the running variable for the integral and where $dP_{s-dt} \equiv P_s - P_{s-dt}$. S_t is simply a weighted average of past price changes on the stock market where the weights decrease exponentially the further back we go into the past. The definition of S_t includes even the most recent price change, $dP_{t-dt} = P_t - P_{t-dt}$. The parameter β plays an important role in our model. When it is high, sentiment is determined primarily by the most recent price changes; when it is low, even price changes in the distant past have a significant effect on current sentiment. In Section 3, we use survey data to estimate β.

We assume that extrapolators' expectation of the change, per unit time, in the value of the stock market, is

$$g^e_{P,t} \equiv \mathbb{E}^e_t[dP_t]/dt = \lambda_0 + \lambda_1 S_t, \tag{3}$$

where the superscript "*e*" is an abbreviation for "extrapolator," and where, for now, the only requirement we impose on the constant parameters λ_0 and λ_1 is that $\lambda_1 > 0$. Taken together, Eqs. (2) and (3) capture the essence of the survey results in Greenwood and Shleifer (2014): if the stock market has been rising, extrapolators expect it to keep rising; and if it has been falling, they expect it to keep falling. While we leave λ_0 and λ_1 unspecified for now, the numerical analysis we conduct later sets $\lambda_0 = 0$ and $\lambda_1 = 1$; we explain in Section 3 why these are natural values to use.

We do not take a strong stand on the underlying source of the extrapolative expectations in (3). One possible source is a judgment heuristic such as representativeness, or the closely related "belief in the law of small numbers" (Barberis, Shleifer, and Vishny, 1998; Rabin, 2002). People who believe in the law of small numbers think that even short samples will resemble the parent population from which they are drawn. As a consequence, when they see good recent returns in the stock market, they infer that the market must currently have a high average return and will therefore continue to perform well.[4]

To compute their optimal consumption-portfolio decision at each moment of time, extrapolators need to form beliefs not only about the expected instantaneous price change, but also about the evolution of future prices. We assume that, in extrapolators' minds, prices evolve according to

$$dP_t = (\lambda_0 + \lambda_1 S_t)dt + \sigma_P d\omega_t^e, \tag{4}$$

where, again from the extrapolators' perspective, ω_t^e is a Wiener process. The drift term simply reflects the expectations in (3), while the instantaneous volatility σ_P is the actual instantaneous volatility that is endogenously determined in equilibrium and that we assume, and later verify, is a constant. Since volatility can easily be estimated from past data, we assume that extrapolators know its true value.

The second type of investor, the rational trader, has correct beliefs, both about the dividend process in (1) and about the evolution of future stock prices.

There is a continuum of both rational traders and extrapolators in the economy. Each investor, whether an extrapolator or a rational trader, takes the risky asset price as given when making his consumption-portfolio decision and has constant absolute risk aversion (CARA) preferences with risk aversion γ and time discount factor δ.[5] At time 0, each extrapolator maximizes

$$\mathbb{E}_0^e\left[-\int_0^\infty \frac{e^{-\delta t-\gamma C_t^e}}{\gamma}dt\right] \tag{5}$$

subject to his budget constraint

[4] Another possible source of extrapolative expectations is the experience effect analyzed by Malmendier and Nagel (2011). One caveat is that, as we show later, the investor expectations documented in surveys depend primarily on recent past returns, while in Malmendier and Nagel's (2011) results, distant past returns also play a significant role.

[5] The model remains analytically tractable even if the two types of investor have different values of γ or δ.

$$dW_t^e \equiv W_{t+dt}^e - W_t^e = (W_t^e - C_t^e dt - N_t^e P_t)(1+rdt) + N_t^e D_t dt + N_t^e P_{t+dt} - W_t^e$$
$$= rW_t^e dt - C_t^e dt - rN_t^e P_t dt + N_t^e dP_t + N_t^e D_t dt \,, \tag{6}$$

where W_t^e, C_t^e, and N_t^e are his time t per-capita wealth, consumption, and number of shares he invests in the risky asset, respectively. Similarly, at time 0, each rational trader maximizes

$$\mathbb{E}_0^r\left[-\int_0^\infty \frac{e^{-\delta t - \gamma C_t^r}}{\gamma} dt\right] \tag{7}$$

subject to his budget constraint

$$dW_t^r \equiv W_{t+dt}^r - W_t^r = (W_t^r - C_t^r dt - N_t^r P_t)(1+rdt) + N_t^r D_t dt + N_t^r P_{t+dt} - W_t^r$$
$$= rW_t^r dt - C_t^r dt - rN_t^r P_t dt + N_t^r dP_t + N_t^r D_t dt \,, \tag{8}$$

where W_t^r, C_t^r, and N_t^r are his time t per-capita wealth, consumption, and number of shares he invests in the risky asset, respectively, and where the superscript "r" is an abbreviation for "rational trader."

Rational traders make up a fraction μ, and extrapolators $1 - \mu$, of the total investor population. The market clearing condition that must hold at each time is

$$\mu N_t^r + (1-\mu) N_t^e = Q \,, \tag{9}$$

where, as noted above, Q is the per-capita supply of the risky asset. Both extrapolators and rational traders observe D_t and P_t on a continuous basis.[6]

Using the stochastic dynamic programming approach developed in Merton (1971), we obtain the following proposition.

Proposition 1. (*Model solution.*) *In the heterogeneous-agent model described above, the equilibrium price of the risky asset is*

$$P_t = A + BS_t + \frac{D_t}{r} \,. \tag{10}$$

The price P_t and sentiment S_t evolve according to

$$dP_t = \left(-\frac{\beta B}{1-\beta B} S_t + \frac{g_D}{(1-\beta B)r}\right) dt + \sigma_P d\omega_t \,, \tag{11}$$

[6] As in any framework with less than fully rational traders, the extrapolators could, in principle, come to learn that their beliefs about the future are inaccurate. We do not study this learning process; rather, we study the behavior of asset prices when extrapolators are unaware of the bias in their beliefs.

$$dS_t = -\frac{\beta}{1-\beta B}\left(S_t - \frac{g_D}{r}\right)dt + \beta\sigma_P d\omega_t, \tag{12}$$

where $\sigma_P = \dfrac{\sigma_D}{(1-\beta B)r}$. *At time t, the value functions for the extrapolators and the rational traders are*

$$J^e(W_t^e, S_t, t) \equiv \max_{\{C_s^e, N_s^e\}_{s\geq t}} \mathbb{E}_t^e\left[-\int_t^\infty \frac{e^{-\delta s-\gamma C_s^e}}{\gamma}ds\right] = -\exp(-\delta t - r\gamma W_t^e + a^e S_t^2 + b^e S_t + c^e),$$
$$J^r(W_t^r, S_t, t) \equiv \max_{\{C_s^r, N_s^r\}_{s\geq t}} \mathbb{E}_t^r\left[-\int_t^\infty \frac{e^{-\delta s-\gamma C_s^r}}{\gamma}ds\right] = -\exp(-\delta t - r\gamma W_t^r + a^r S_t^2 + b^r S_t + c^r). \tag{13}$$

The optimal share demands for the risky asset from the extrapolators and from the rational traders are

$$N_t^e = \eta_0^e + \eta_1^e S_t, \qquad N_t^r = \eta_0^r + \eta_1^r S_t, \tag{14}$$

and the optimal consumption flows of the two types are

$$\begin{aligned} C_t^e &= rW_t^e - \frac{a^e S_t^2 + b^e S_t + c^e}{\gamma} - \frac{\log(r\gamma)}{\gamma}, \\ C_t^r &= rW_t^r - \frac{a^r S_t^2 + b^r S_t + c^r}{\gamma} - \frac{\log(r\gamma)}{\gamma}, \end{aligned} \tag{15}$$

where the optimal wealth levels, W_t^e *and* W_t^r*, evolve as in* (6) *and* (8)*, respectively. The coefficients A, B,* a^e, b^e, c^e, a^r, b^r, c^r, η_0^e, η_1^e, η_0^r, *and* η_1^r *are determined through a system of simultaneous equations.* ■

Comparing extrapolators' beliefs about future prices in (4) with the actual price process in (11), we see that extrapolators' beliefs are incorrect. While extrapolators think that the expected instantaneous price change depends positively on the sentiment level S_t, Eq. (11) shows that it actually depends *negatively* on S_t: in Corollary 2 below, we show that, in the equilibrium we study, $B \in (0, \beta^{-1})$, so that the coefficient on S_t in (11) is negative. Substituting Eq. (4) into the differential form of (2), namely,

$$dS_t = -\beta S_t dt + \beta dP_t, \tag{16}$$

we obtain extrapolators' beliefs about the evolution of sentiment,

$$dS_t = \beta\left(\lambda_0 + (\lambda_1 - 1)S_t\right)dt + \beta\sigma_P d\omega_t^e. \tag{17}$$

Comparing (12) to (17), we see that these beliefs are also incorrect. For example, when $\lambda_0 = 0$ and $\lambda_1 = 1$, extrapolators think that sentiment follows a random walk; in reality, however, it is mean-reverting.

While the extrapolators have incorrect beliefs about the evolution of prices and sentiment, they are fully time-consistent. At time t, an extrapolator's consumption-portfolio decision is a function of two state variables: his wealth W_t^e and sentiment S_t. When computing his time t decision, the extrapolator makes a plan as to what he will do in all future states $\{(W_{t+\tau}^e, S_{t+\tau})\}$. If, at time $t+\tau$, he arrives in state $(W_{t+\tau}^e, S_{t+\tau})$, he is time-consistent: he takes the action that he had previously planned to take in that state. His only error is that, since his beliefs about the evolution of prices and sentiment are incorrect, he misestimates the probability of moving from state (W_t^e, S_t) at time t to state $(W_{t+\tau}^e, S_{t+\tau})$ at time $t+\tau$.

To understand the role that extrapolators play in our model, we compare the model's predictions to those of a benchmark "rational" economy, in other words, an economy where all traders are of the fully rational type, so that $\mu = 1$.[7]

Corollary 1. (*Rational benchmark.*) *If all traders in the economy are rational* ($\mu = 1$), *the equilibrium price of the risky asset is*

$$P_t = -\frac{\gamma\sigma_D^2 Q}{r^2} + \frac{g_D}{r^2} + \frac{D_t}{r}, \tag{18}$$

and therefore evolves according to

$$dP_t = \frac{g_D}{r}dt + \frac{\sigma_D}{r}d\omega_t. \tag{19}$$

The value function for the rational traders is

$$J^r(W_t^r, t) = -\frac{1}{r\gamma}\exp\left(-\delta t - r\gamma W_t^r + \frac{r-\delta}{r} - \frac{\gamma^2\sigma_D^2 Q^2}{2r}\right). \tag{20}$$

The optimal consumption flow is

[7] Another way of reducing our model to a fully rational economy is to set λ_0 and λ_1, the parameters in (3), to g_D/r and zero, respectively. In this case, the rational traders and the extrapolators have the same, correct beliefs about the expected price change of the risky asset.

$$C_t^r = rW_t^r - \frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}, \tag{21}$$

where the optimal wealth level, W_t^r*, evolves as*

$$dW_t^r = \left(\frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}\right)dt + \frac{\sigma_D Q}{r}d\omega_t\ . \tag{22}$$

■

2.1. Discussion

In Sections 4 and 5, we discuss the model's implications in detail. However, the closed-form solution in Proposition 1 already makes apparent its basic properties.

Comparing Eqs. (10) and (18), we see that, up to a constant, the effect of extrapolators on the risky asset price is given by the term BS_t, where, as noted above and stated formally in Corollary 2 below, $B \in (0, \beta^{-1})$. Intuitively, if the sentiment level S_t is high, indicating that past price changes have been high, extrapolators expect the stock market to continue to perform well and therefore push its current price higher.

Eq. (12) shows that, in equilibrium, sentiment S_t follows a mean-reverting process, one that reverts more rapidly to its mean as β increases. Put differently, the mispricing BS_t generated by extrapolators is eventually corrected, and more quickly so for higher values of β. To see why, recall that an overpricing occurs when good cash-flow news generates a price increase that then feeds into extrapolators' beliefs, leading them to push the price still higher. The form of extrapolation in (2), however, means that, as time passes, the price increase caused by the good cash-flow news plays a smaller and smaller role in determining extrapolators' beliefs. As a result, these investors become less bullish over time, and the mispricing corrects. This happens more rapidly when β is high because, in this case, extrapolators quickly "forget" all but the most recent price changes.

Comparing Eqs. (11) and (19), we see that, as noted in the Introduction, the presence of extrapolators amplifies the volatility of price changes—specifically, by a factor of $1/(1 - \beta B) > 1$. And while in an economy made up of rational investors alone, price changes are not predictable—see Eq. (19)—Eq. (11) shows that they *are* predictable in the presence of extrapolators. If the stock market has recently experienced good returns, so that sentiment S_t has a high value, the subsequent stock market return is low on average: the coefficient on S_t in (11) is negative. In short, high valuations in the stock market are followed by low returns, and low valuations are followed by high returns. This anticipates some of our results on stock market predictability in Sections 4 and 5.

Eq. (14) shows that extrapolators' share demand is a positive linear function of sentiment S_t: in Corollary 2 below, we show that the derived parameter η_1^e is strictly positive. In other words, after a period of good stock market performance, one that generates a high sentiment level S_t, extrapolators form more bullish expectations of future price changes and increase the number of shares of the stock market that they hold. With a fixed supply of these shares, this automatically means that the share demand of rational traders varies negatively with sentiment S_t: rational traders absorb the shocks in extrapolators' demand. While extrapolators' beliefs are, by assumption, extrapolative, rational traders' beliefs are contrarian: their beliefs are based on the true price process in Eq. (11) whose drift depends negatively on S_t.

Eq. (21) shows that, in the fully rational economy, optimal consumption is a constant plus the product of wealth and the interest rate. Eq. (15) shows that, when extrapolators are present in the economy, the consumption policy of each type of trader also depends on linear and quadratic terms in S_t. We can show that the derived parameters a^e and a^r in Eq. (15) satisfy $a^e \leq 0$ and $a^r < 0$; we also find that b^e and b^r typically satisfy $b^e < b^r$. The fact that $b^e < b^r$ indicates that extrapolators increase their consumption more than rational traders do after strong stock market returns. After strong returns, extrapolators expect the stock market to continue to rise; an income effect therefore leads them to consume more. Rational traders, on the other hand, correctly perceive low future returns and therefore do not raise their consumption as much. The fact that a^e and a^r are both negative indicates that, when sentiment deviates substantially from its long-run mean, both types increase their consumption. When S_t takes either a very high or a very low value, both types perceive the stock market to be severely misvalued and therefore expect their respective investment strategies to perform well in the future. This, in turn, leads them to raise their consumption.

Since extrapolators have incorrect beliefs about future price changes, it is likely that, in the long run, their wealth will decline relative to that of rational traders. However, the equilibrium price in (10) is unaffected by the relative wealth of the two trader types: under exponential utility, the share demand of each type, and hence also the stock price, are independent of wealth. The exponential utility assumption allows us to abstract from the effect of "survival" on prices, and to focus on what happens when both types of trader play a role in setting prices.

The idea, inherent in our model, that uninformed investors will affect prices even in the long run, is not completely unrealistic. In reality, people earn labor income which can sustain a losing investment strategy for many years; and while some uninformed investors may be forced to exit the financial markets because of poor performance, they are likely to be replaced, at least in part, by a new cohort of naïve traders with little prior experience. In addition, even if the wealth of an uninformed investor declines over time, this process can take a long time (Yan, 2008). We have used numerical simulations to confirm this in the context of our model. We find that, if, at time 0, the extrapolators and rational traders each hold 50% of aggregate wealth, then, after 50 years, for the benchmark parameter values that we lay out in Section 3, the extrapolators on average still hold 40% of aggregate wealth.[8]

At the heart of our model is an amplification mechanism: if good cash-flow news pushes the stock market up, this price increase feeds into extrapolators' expectations about future price changes, which then leads them to push the current price up even higher. However, this then further increases extrapolators' expectations about future price changes, leading them to push the current price still higher, and so on. Given this infinite feedback loop, it is important to ask whether the heterogeneous-agent equilibrium we described above exists. The following corollary provides sufficient conditions for existence of equilibrium.

Corollary 2. (*Existence of equilibrium.*) *When* $1 > \mu > 0$, *the equilibrium described in Proposition* 1 *exists if*

$$r > \beta(\lambda_1 - 2), \qquad 2\beta > r. \tag{23}$$

When $\mu = 0$, *the equilibrium described in Proposition* 1 *exists if* (23) *holds and if*

$$2 > \lambda_1, \qquad r > \lambda_1\beta. \tag{24}$$

[8] Nonetheless, it would be useful to construct a model of price extrapolation in which the relative wealth of informed and uninformed traders affects prices—for example, a model with power utility or Epstein-Zin preferences. However, such a model would be far less tractable than the one we present here; and we conjecture that, if it allows for labor income or for cohorts of new uninformed investors to enter the markets on a regular basis, its predictions will be similar to those of the current model.

In particular, under each set of conditions, there exists an equilibrium in which $B \in (0, \beta^{-1})$, η_1^e *is strictly positive, and* η_1^r *is strictly negative.* ■

Corollary 2 shows that, when *all* investors in the economy are extrapolators, there may be no equilibrium even for reasonable parameter values; loosely put, the feedback loop described above may fail to converge. For example, if $\lambda_1 = 1$ and $\beta = 0.5$, there may be no equilibrium in the case of $\mu = 0$ if the interest rate is less than 50%. However, the corollary also shows that, if $\mu > 0$—in words, if there are any rational traders at all in the economy—the equilibrium is very likely to exist: for most empirically plausible values of r, β, and λ_1, the sufficient conditions in (23) are satisfied.

One of the assumptions of our model is that the risk-free rate is constant. To evaluate this assumption, we compute the aggregate demand for the risk-free asset across the two types of trader. We find that this aggregate demand is very stable over time and, in particular, is only weakly correlated with the sentiment level S_t. This is because the demand for the risk-free asset from one type of trader is largely offset by the demand from the other type: when sentiment S_t is high, rational traders increase their demand for the risk-free asset (and move out of the stock market), while extrapolators reduce their demand for the risk-free asset (and move into the stock market). When sentiment is low, the reverse occurs. This suggests that, even if the risk-free rate were endogenously determined, it would not fluctuate wildly, nor would its fluctuations significantly attenuate the effects we describe here.

Our model is similar in some ways to that of Campbell and Kyle (1993)—a model in which, as in our framework, the risk-free rate is constant, the level of the dividend on the risky asset follows an arithmetic Brownian motion, and infinitely lived rational investors with exponential utility interact with less rational investors. The difference between the two models—and it is an important difference—is that, in Campbell and Kyle (1993), the share demand of the less rational investors is exogenously assumed to follow a mean-reverting process, while, in our model, extrapolators' share demand is derived from their beliefs.

It is also useful to compare our framework to models of rational learning. Wang (1993) considers an economy with informed investors, uninformed investors, and noise traders with exogenous mean-reverting demand for a risky asset. The dividend stream on the risky asset has a time-varying drift that is known to the informed investors but not to the uninformed investors, who instead try to estimate the drift from past dividends and prices.

The model of Wang (1993) captures a number of facts about asset prices, but is less consistent with the survey evidence on expectations. Greenwood and Shleifer (2014) show that, in a regression of the average investor expectation of future returns on the past return and the past change in fundamentals, the coefficient on the past return is strongly positive while the coefficient on the past change in fundamentals is insignificant. In the economy described by Wang (1993), however, the expectations of both the informed and uninformed investors about future price changes typically depend *negatively* on past price changes, after controlling for past fundamentals: if prices go up without a contemporaneous increase in dividends, both investor types infer that noise trader demand has gone up; since this demand is mean-reverting, both investor types forecast low, not high, price changes in the future.

The idea that investors extrapolate past price changes features prominently in classic qualitative accounts of asset bubbles (Kindleberger, 1978; Minsky, 1986; Shiller, 2000). Our model shows formally how, even in the presence of fully rational traders, price extrapolators can generate the most fundamental feature of a bubble, namely a substantial and long-lived overvaluation of an asset class. Our focus on extrapolation differentiates our framework from existing models of bubbles, such as the rational bubble model of Blanchard and Watson (1982) and the heterogeneous-beliefs models of Harrison and Kreps (1978) and Scheinkman and Xiong (2003); in these other models, investors do not hold extrapolative expectations.

While our model shows how extrapolation can lead to overvaluation, the goal of our analysis is not to understand bubbles, but rather, to understand the behavior of the aggregate stock market, and, in particular, the joint behavior of consumption, dividends, and prices. Since we built our model with this goal in mind, it is not surprising that there are several features of bubbles that it does not capture—for example, the persistent momentum in prices while a bubble is forming, the high trading volume at the bubble's peak, and the riding of the bubble by rational investors (Brunnemeier and Nagel, 2004). An extrapolation-based model of bubbles that captures this rich set of facts has yet to be developed.

3. Parameter values

In this section, we assign benchmark values to the basic model parameters. We use these values in the numerical simulations of Section 5. However, we also use them in Section 4. While the core of that section consists of analytical propositions, we can get more out of the propositions by evaluating the expressions they contain for specific parameter values.

For easy reference, we list the model parameters in Table 2. The asset-level parameters are the risk-free rate r; the initial level of the dividend D_0; the mean g_D and standard deviation σ_D of dividend changes; and the risky asset supply Q. The investor-level parameters are the initial wealth levels for the two types of investor, W_0^e and W_0^r; absolute risk aversion γ and the time discount rate δ; λ_0 and λ_1, which link the sentiment variable to extrapolators' beliefs; β, which governs the relative weighting of recent and distant past price changes in the definition of sentiment; and finally, μ, the fraction of rational traders in the investor population.[9]

We set r = 2.5%, consistent with the low historical risk-free rate. We set the initial dividend level D_0 to 10, and given this, we choose $\sigma_D = 0.25;$ in other words, we choose a volatility of dividend changes small enough to ensure that we only rarely encounter negative dividends and prices in the simulations we conduct in Section 5. We set $g_D = 0.05$ to match, approximately, the value of g_D/σ_D in actual data. Finally, we set the risky asset supply Q to 5.

We now turn to the investor-level parameters. We set the initial wealth levels to $W_0^e = W_0^r = 5000$; these values imply that, at time 0, the value of the stock market constitutes approximately half of aggregate wealth. We set risk aversion γ equal to 0.1 so that relative risk aversion, computed from the value function as $RRA = -WJ_{WW}/J_W$ $= r\gamma W$, is 12.5 at the initial wealth levels. And we choose a low time discount rate of δ = 1.5%, consistent with most other asset pricing frameworks.

This leaves four parameters: λ_0, λ_1, β, and μ. As shown in Eq. (3), λ_0 and λ_1 determine the link between sentiment S_t and extrapolators' beliefs. We use $\lambda_0 = 0$ and $\lambda_1 = 1$ as our benchmark values. The integral sum of the weights on past price changes in the definition of sentiment in (2) is equal to one; informally, S_t represents "one unit" of price

[9] For much of the analysis, we do not need to assign specific values to D_0, W_0^e, and W_0^r; the values of these variables are required only for the simulations in Section 5.

change. It is therefore natural for extrapolators to scale S_t by $\lambda_1 = 1$ when forecasting a unit price change in the future. Given this value of λ_1, we set $\lambda_0 = 0$ because this ensures that extrapolators' beliefs are correct "on average": while extrapolators overestimate the subsequent price change of the stock market after good past price changes and underestimate it after poor past price changes, the errors in their forecasts of future price changes over any finite horizon will, in the long run, average out to zero.[10]

The parameter β determines the relative weight extrapolators put on recent as opposed to distant past price changes when forming expectations about the future; a higher value of β means a higher relative weight on recent price changes. To estimate β, we use the time series of investor expectations from the Gallup surveys studied by Greenwood and Shleifer (2014). We describe the estimation procedure in detail in Appendix C. In brief, we run a regression of the average investor expectation of the price change in the stock market over the next year, as recorded in the surveys, on what our model says extrapolators' expectation of this quantity should be at that time as a function of the sentiment level and the model parameters. If the average investor expectation of the future price change that we observe in the surveys depends primarily on recent past price changes, the estimated β will be high. Conversely, if it depends to a significant extent on price changes in the distant past, the estimated β will be low. The estimation makes use of Proposition 2 below, and specifically, Eq. (26), which describes the price change expected by extrapolators over any future horizon.

Proposition 2. (*Price change expectations of rational traders and extrapolators*.) *Conditional on an initial sentiment level* $S_0 = s$, *rational traders' expectation of the price change in the stock market over a finite time horizon* $(0, t_1)$ *is*

$$\mathbb{E}^r[P_{t_1} - P_0 \mid S_0 = s] = B(1 - \mathrm{e}^{-kt_1})\left(\frac{g_D}{r} - s\right) + \frac{g_D t_1}{r}, \tag{25}$$

while extrapolators' expectation of the same quantity is

[10] Another motivation for $\lambda_0 = 0$ and $\lambda_1 = 1$ is that, for these values, the beliefs in Eq. (3) are the beliefs that an investor would hold if he had in mind a particular incorrect but natural model of the world, namely that stock prices follow a random walk with an unobserved drift that itself follows a random walk; see Adam, Beutel, and Marcet (2014) for a model in which investors' extrapolative expectations are grounded in an argument of this type. We have also used the survey evidence to estimate λ_0 and λ_1 and find the estimated values to be close to zero and one; see Appendix C for details. The results we present in Sections 4 and 5 are similar whether we use the estimated values or the values zero and one.

$$\mathbb{E}^e[P_{t_1} - P_0 \mid S_0 = s] = (\lambda_0 + \lambda_1 s)t_1 + \lambda_1(\beta\lambda_0 - ms)\frac{mt_1 + e^{-mt_1} - 1}{m^2}, \tag{26}$$

where $k = \frac{\beta}{1-\beta B}$ *and* $m = \beta(1 - \lambda_1)$. *When* $\lambda_0 = 0$ *and* $\lambda_1 = 1$, (26) *reduces to*

$$\mathbb{E}^e[P_{t_1} - P_0 \mid S_0 = s] = st_1 . \tag{27}$$

■

Eqs. (25) and (26) confirm that the expectations of extrapolators load positively on the sentiment level, while the expectations of rational traders load on it negatively.

When we use the procedure described in Appendix C to estimate β from the survey data, we obtain a value of approximately 0.5. For this value of β, extrapolators' expectations depend primarily on recent past price changes; specifically, when forming their expectations, extrapolators weight the realized annual price change in the stock market starting four years ago only 22% as much as the most recent annual price change. While we pay most attention to the case of $\beta = 0.5$, we also present results for $\beta = 0.05$ and $\beta = 0.75$. When $\beta = 0.05$, the annual price change four years ago is weighted 86% as much as the most recent annual price change, and when $\beta = 0.75$, only 11% as much.[11]

The final parameter is μ, the fraction of rational traders in the investor population. We do not take a strong stand on its value. While the average investor expectation in the survey data is robustly extrapolative, it is hard to know how representative the surveyed investors are of the full investor population. In our analysis, we therefore consider a range of values of μ: 1 (an economy where all investors are fully rational), 0.75, 0.5, and 0.25. We do not consider the case of $\mu = 0$ because Corollary 2 indicates that, when all investors are extrapolators, the equilibrium may not exist for reasonable values of β and λ_1. While we consider four different values of μ, we focus on the lower two values, namely 0.5 and 0.25. The fact that the average investor in the surveys studied by Greenwood and Shleifer (2014)—surveys that include both individual and institutional

[11] When we estimate β, we assume that the surveyed investors correspond to the extrapolators in our model: after all, the presence of extrapolators in our economy is motivated precisely by the survey evidence. If we instead assumed that the surveyed investors correspond to *all* investors in our model, we would likely obtain a similar value of β. Since rational traders' beliefs are the "mirror image" of extrapolators' beliefs, rational traders and extrapolators weight past price changes in a similar way when forming their expectations.

respondents—exhibits extrapolative expectations suggests that many investors in actual financial markets are extrapolators.

For a given set of values of the basic parameters in Table 2, we solve a system of simultaneous equations, as outlined in Appendix A, to compute the "derived" parameters: η_0^e, η_1^e, η_0^r, and η_1^r, which determine the optimal share demands (see Eq. (14)); a^e, b^e, c^e, a^r, b^r, and c^r, which determine the optimal consumption policies (see Eq. (15)); A and B, which specify how the price level P_t depends on the level of the sentiment S_t and the level of the dividend D_t (see Eq. (10)); and finally, σ_P, the volatility of price changes (see Eq. (11)). For example, if $\mu = 0.25$, $\beta = 0.5$, and the other basic parameters have the values shown in Table 2, the values of the derived parameters are:

$$\begin{gathered}
\eta_0^e = 1.54,\ \eta_0^r = 15.39,\ \eta_1^e = 0.51,\ \eta_1^r = -1.54, \\
a^e = -1.22\times 10^{-3},\ a^r = -1.28\times 10^{-3},\ b^e = -7.31\times 10^{-3},\ b^r = 0.042, \\
c^e = 1.63,\ c^r = -3.47,\ A = -117.04,\ B = 0.99,\ \sigma_P = 19.75.
\end{gathered} \tag{28}$$

Before we turn to the empirical implications of the model, we make one more observation about investor expectations. When we say that our model can "match" the survey evidence, or that it is "consistent" with it, we mean that, in our model, a significant fraction of the investor population is comprised of traders—specifically, extrapolators—whose expectation of future price changes depends positively on past price changes. However, we could also define "matching" the survey evidence in a stricter sense, namely to mean that the average expectation of future price changes across *all* investors in the model is extrapolative.

Interestingly, we find that our model can match the survey data even in this stricter sense: the expectation of the future change in price averaged across *all* investors in the model—specifically, the population-weighted average of the expressions in (25) and (26)—depends positively on the sentiment level for *any* $\mu < 1$. In other words, if there are any extrapolators at all in the economy, the average investor expectation is extrapolative. The reason is that, while extrapolators hold extrapolative beliefs and rational traders hold contrarian beliefs, rational traders are less contrarian than extrapolators are extrapolative. After all, the rational traders are contrarian only because the extrapolators are extrapolative; they cannot, therefore, be more contrarian than the extrapolators are extrapolative.

While our model can match the survey evidence even in this stricter sense, we do not focus on this interpretation. Since we do not know how representative the surveyed investors are of the full investor population, it is unclear whether the average expectation of future price changes across all real-world investors is extrapolative.

4. Empirical implications

In this section, we present a detailed analysis of the empirical predictions of the model. A consequence of our assumptions that the dividend level follows an arithmetic Brownian motion and that investors have exponential utility is that it is more natural to work with quantities defined in terms of *differences* rather than ratios—for example, to work with price changes $P_t - P_0$ rather than returns, and with the "price-dividend difference" $P - D/r$ rather than the price-dividend ratio. For example, Corollary 1 shows that, in the benchmark rational economy, it is $P - D/r$ that is constant over time, not P/D. In this section, then, we study the predictions of price extrapolation for these difference-based quantities. In Section 5, we also consider the ratio-based quantities.

We study the implications of the model for the difference-based quantities with the help of formal propositions. For example, if we are interested in the autocorrelation of price changes, we first compute this autocorrelation analytically, and then report its value for the parameter values in Table 2. For two important parameters, μ and β, we consider a range of possible values. Recall that μ is the fraction of rational traders in the investor population, while β controls the relative weighting of near-past and distant-past price changes in extrapolators' forecast of future price changes.

We are interested in how the presence of extrapolators in the economy affects the behavior of the stock market. To understand this more clearly, in the results that we present below, we always include, as a benchmark, the case of $\mu = 1$, in other words, the case where the investor population consists entirely of rational traders.

4.1. Predictive power of D/r J *P for future price changes*

A basic fact about the stock market is that the dividend-price ratio of the stock market predicts subsequent returns with a positive sign; moreover, the ratio's predictive power is greater at longer horizons. In our model, the natural analogs of the dividend-price ratio and of returns are the dividend-price difference $D/r - P$ and price changes, respectively. We therefore examine whether, in our economy, the dividend-price difference predicts subsequent price changes with a positive sign, and whether this predictive power is greater at longer horizons.

It is helpful to express the long-horizon evidence in the more structured way suggested by Cochrane (2011), among others. If we run three univariate regressions—a regression of future returns on the current dividend-price ratio; a regression of future dividend growth on the current dividend-price ratio; and a regression of the future dividend-price ratio on the current dividend-price ratio—then, as a matter of accounting, the three regression coefficients, appropriately signed, must sum to approximately one at long horizons. Empirically, at long horizons, the three coefficients are roughly 1, 0, and 0, respectively. In other words, at long horizons, the dividend-price ratio forecasts future returns—not future dividend growth, and not its own future value.

We can restate this point in a way that fits more naturally with our model, using quantities defined as differences, rather than ratios. Given the accounting identity

$$\frac{D_0}{r} - P_0 = (P_t - P_0) - \left(\frac{D_t}{r} - \frac{D_0}{r}\right) + \left(\frac{D_t}{r} - P_t\right), \tag{29}$$

it is immediate that if we run three regressions—of the future price change, the (negative and scaled) future dividend change, and the future dividend-price difference, on the current dividend-price difference—the three coefficients will sum to one in our economy, at any horizon. To match the empirical facts, our model needs to predict a coefficient in the first regression that, at long horizons, is approximately equal to one.[12] The next proposition shows that this is indeed the case.

Proposition 3. (*Predictive power of $D/r - P$.*) *Consider a regression of the price change in the stock market over some time horizon* $(0, t_1)$ *on the level of $D/r - P$ at the start of the horizon. The coefficient on the independent variable is*[13]

[12] If the coefficient in the first regression is approximately one, this immediately implies that the coefficients in the second and third regressions are approximately zero, consistent with the evidence. The coefficient in the second regression is exactly zero because dividend changes are unpredictable in our economy. The coefficient in the third regression is then one minus the coefficient in the first regression; if the latter is approximately one, the former is approximately zero.

[13] The expectations that we compute in the propositions in Section 4 are taken over the steady-state distribution of sentiment S_t. Ergodicity of S_t guarantees that sample statistics will converge to our analytical results for sufficiently large samples.

$$\beta_{DP}(t_1) \equiv \frac{\mathrm{cov}(D_0/r - P_0, P_{t_1} - P_0)}{\mathrm{var}(D_0/r - P_0)} = 1 - e^{-kt_1}, \tag{30}$$

where $k = \dfrac{\beta}{1-\beta B}$. ■

Proposition 3 shows that, in our model, and consistent with the empirical facts, the coefficient in a regression of the price change in the stock market on the dividend-price difference is positive and increases at longer horizons, rising in value asymptotically toward one. These patterns are clearly visible in Table 3, which reports the value of the regression coefficient in Proposition 3 for several values of μ and β, and for five different time horizons: a quarter, a year, two years, three years, and four years. In the benchmark rational economy (μ = 1), the quantity $D/r - P$ is constant; the regression coefficient in Proposition 3 is therefore undefined.

The intuition for why $D/r - P$ predicts subsequent price changes is straightforward. A sequence of good cash-flow news pushes up stock prices, which then raises extrapolators' expectations about the future price change of the stock market and causes them to push the current stock price even higher, lowering the value of $D/r - P$. Since the stock market is now overvalued, the subsequent price change is low, on average. The quantity $D/r - P$ therefore forecasts price changes with a positive sign.

The table shows that, for a fixed horizon, the predictive power of $D/r - P$ is stronger for low μ: since the predictability of price changes stems from the presence of extrapolators, it is natural that this predictability is stronger when there are more extrapolators in the economy. The predictive power of $D/r - P$ is weaker for low β: when β is low, extrapolators' beliefs are more persistent; as a result, it takes longer for an overvaluation to correct, reducing the predictive power of $D/r - P$ for price changes over any fixed horizon.

4.2. Autocorrelation of P J *D/r*

In the data, price-dividend ratios are highly autocorrelated at short lags. We would like to know if our model can capture this. The natural analog of the price-dividend ratio in our model is the difference-based quantity $P - D/r$. We therefore examine the autocorrelation structure of this quantity.

In our discussion of the accounting identity in (29), we noted that, if we run regressions of the future change in the stock price, the (negative, scaled) future change in dividends, and the future dividend-price difference on the current dividend-price difference, the three regression coefficients we obtain must sum to one. Since the dividend level follows a random walk in our model, we know that the coefficient in the second regression is zero. We also know, from Proposition 3, that the coefficient in the first regression is $1-e^{-kt_1}$. The coefficient in the third regression, which is also the autocorrelation of the price-dividend difference $P - D/r$, must therefore equal e^{-kt_1}. The next proposition confirms this.

Proposition 4. (*Autocorrelation of $P - D/r$.*) *The autocorrelation of $P - D/r$ at a time lag of t_1 is*

$$\rho_{PD}(t_1) \equiv \mathrm{corr}\left(P_0 - \frac{D_0}{r}, P_{t_1} - \frac{D_{t_1}}{r} \right) = e^{-kt_1} \ , \tag{31}$$

where $k = \dfrac{\beta}{1-\beta B}$. ■

We use Proposition 4 to compute the autocorrelation of the price-dividend difference for several pairs of values of μ and β, and for lags of one quarter, one year, two years, three years, and four years. Table 4 reports the results. It shows that, in our model, and consistent with the empirical facts, the price-dividend difference is highly persistent at short horizons, while at long horizons, the autocorrelation drops to zero. The table also shows that the autocorrelation is higher for low values of β: when β is low, extrapolators' beliefs are very persistent, which, in turn, imparts persistence to the price-dividend difference.

4.3. Volatility of price changes and of $P - D/r$

Empirically observed stock market returns and price-dividend ratios are thought to exhibit "excess volatility," in other words, to be more volatile than can be explained purely by fluctuations in rational expectations about future cash flows. We now show that, in our model, price changes and the price-dividend difference—the analogs of returns and of the price-dividend ratio in our framework—also exhibit such excess volatility. In particular, they are more volatile than in the benchmark rational economy described in Corollary 1, an economy where price changes are due only to changes in rational forecasts of future cash flows.

Proposition 5. (*Excess volatility*.) *The standard deviation of price changes over a finite time horizon* (0, t_1) *is*

$$\sigma_{\Delta P}(t_1) \equiv \sqrt{\text{var}(P_{t_1} - P_0)} = \sqrt{\frac{B\sigma_S}{k}\left(B\sigma_S + \frac{2\sigma_D}{r}\right)(1 - e^{-kt_1}) + \frac{\sigma_D^2}{r^2}t_1}\ , \tag{32}$$

while the standard deviation of P – D/r is

$$\sigma_{PD} \equiv \sqrt{\text{var}\left(P - \frac{D}{r}\right)} = \frac{B\sigma_S}{\sqrt{2k}}\ , \tag{33}$$

where $k = \dfrac{\beta}{1 - \beta B}$ *and* $\sigma_S = \dfrac{\beta\sigma_D}{(1 - \beta B)r}$. ■

Table 5 reports the standard deviation of annual price changes and of the price-dividend difference $P - D/r$ for several (μ, β) pairs. Panel A shows that, in the fully rational economy ($\mu = 1$), the standard deviation of annual price changes is 10, in other words, σ_D/r. When extrapolators are present, however, the standard deviation can be much higher: for example, 30% higher when there are an equal number of extrapolators and rational traders in the economy, a figure that, as can be seen in the table and as we explain below, depends little on the parameter β. Similarly, Panel B shows that while the price-dividend difference is constant in the fully rational economy, it exhibits significant volatility in the presence of extrapolators.

The results in Proposition 5 and in Table 5 confirm the intuition in the Introduction for why extrapolators amplify the volatility of stock prices. A good cash-flow shock pushes stock prices up. This leads extrapolators to expect higher *future* price changes and hence to bid current stock prices up even further. Rational investors counteract this overvaluation, but only mildly so: since they understand how extrapolators form beliefs, they know that extrapolators will continue to have optimistic beliefs about the stock market in the near future and therefore that subsequent price changes, while lower than average, will not be *very* low; as a consequence, rational investors do not push back strongly against the overvaluation caused by the extrapolators.

Table 5 shows that, the larger the fraction of extrapolators in the economy, the more excess volatility there is in price changes. More interesting, the amount of excess volatility is largely insensitive to the value of β. This may seem surprising at first: since extrapolators' beliefs are more variable when β is high, one might have thought that a higher β would correspond to higher volatility in price changes. However, another force pushes in the opposite direction: rational traders know that, precisely because extrapolators change their beliefs more quickly when β is high, any mispricing caused by the extrapolators will correct more quickly in this case. As a consequence, when β is high, rational traders trade more aggressively against the extrapolators, dampening volatility. Overall, the value of β has little effect on the volatility of price changes.

Does the higher price volatility generated by extrapolators leave the rational traders worse off? We find that, for many parameter values, it does not. Specifically, if we start with an economy consisting of only rational traders and then gradually add more extrapolators while keeping the per-capita supply of the risky asset constant, the value function of the rational traders *increases* in value. In other words, while, taken alone, the higher price volatility would serve to lower rational traders' utility, this is more than compensated for by the higher profits the rational traders expect to earn by exploiting the extrapolators.[14]

4.4. Autocorrelation of price changes

Empirically, stock market returns are positively autocorrelated at short lags; at longer lags, they are negatively autocorrelated (Cutler, Poterba, and Summers, 1991). We now examine what our model predicts about the autocorrelation structure of the analogous quantity to returns in our framework, namely price changes.

Proposition 6. (*Autocorrelation of price changes.*) *The autocorrelation of price changes over the intervals* $(0, t_1)$ *and* (t_2, t_3), *where* $t_2 \geq t_1$, *is*

[14] DeLong et al. (1990a) obtain a similar result: they find that the introduction of noise traders into the economy raises the utility of rational traders. The reason is that the presence of noise traders expands the investment opportunity set. In our model, the extrapolators do not expand the opportunity set; they change it, by altering the behavior of the risky asset. It is therefore less obvious, in our context, that the presence of extrapolators will raise rational traders' utility, but our calculations indicate that it does. DeLong et al. (1989) point out that, when the supply of capital is endogenous, noise traders may lower rational traders' utility by depressing the capital stock. Since the supply of shares is fixed in our model, we cannot evaluate the importance of this channel.

$$\rho_{\Delta P}(t_1,t_2,t_3) \equiv \mathrm{corr}(P_{t_1}-P_0, P_{t_3}-P_{t_2}) = \frac{\mathrm{cov}(P_{t_1}-P_0, P_{t_3}-P_{t_2})}{\sqrt{\mathrm{var}(P_{t_1}-P_0)\,\mathrm{var}(P_{t_3}-P_{t_2})}}, \tag{34}$$

where

$$\begin{aligned}
\mathrm{cov}(P_{t_1}-P_0, P_{t_3}-P_{t_2}) &= -\frac{B\sigma_S}{2k}\left(B\sigma_S+\frac{2\sigma_D}{r}\right)(e^{-kt_2}-e^{-kt_3})(e^{kt_1}-1),\\
\mathrm{var}(P_{t_1}-P_0) &= \frac{B\sigma_S}{k}\left(B\sigma_S+\frac{2\sigma_D}{r}\right)(1-e^{-kt_1})+\frac{\sigma_D^2}{r^2}t_1,\\
\mathrm{var}(P_{t_3}-P_{t_2}) &= \frac{B\sigma_S}{k}\left(B\sigma_S+\frac{2\sigma_D}{r}\right)(1-e^{-k(t_3-t_2)})+\frac{\sigma_D^2}{r^2}(t_3-t_2),
\end{aligned} \tag{35}$$

with $k=\dfrac{\beta}{1-\beta B}$ *and* $\sigma_S=\dfrac{\beta\sigma_D}{(1-\beta B)r}$. ■

Proposition 6 shows that, in our economy, price changes are negatively autocorrelated at *all* lags, with the autocorrelation tending to zero at long lags. These patterns can also be seen in Table 6, which reports the autocorrelation of quarterly price changes in our model for several pairs of values of μ and β, and at lags of one, two, three, four, eight, and 12 quarters.

It is easy to see why, in our model, price changes are negatively autocorrelated at *longer* lags. Suppose that there is good cash-flow news at time *t*. The stock market goes up in response to this news, which causes extrapolators to expect higher future price changes and hence to push the time *t* stock price even further up. Now that the stock market is overvalued, the long-term future price change is lower, on average. It is intuitive, then, that past price changes would have negative predictive power for price changes that are some way into the future.

Negative autocorrelations are indeed observed in the data, at longer lags; to some extent, then, our model matches the data. However, there is also a way in which our model does not match the data: actual returns are *positively* autocorrelated at the first quarterly lag, while the price changes generated by our model are not.

It may initially be surprising that our model generates negative autocorrelations in price changes even at the shortest lags. The reason for this prediction is that, as laid out in Eqs. (2) and (3), the weights extrapolators put on past price changes when they form expectations decline monotonically the further back we go into the past. Consider again a good cash-flow shock at time t that, as described above, feeds into extrapolators' expectations and amplifies the contemporaneous price change. The weighting scheme in (2) means that, even an instant later, the positive time t price change that caused extrapolators to become more bullish plays a smaller role in determining their expectations; extrapolators therefore become a little less bullish, and there is a price reversal.

The above discussion clarifies why some earlier models of return extrapolation—for example, Cutler, Poterba, and Summers (1990), DeLong et al. (1990b), Hong and Stein (1999), and Barberis and Shleifer (2003)—do generate positive short-term autocorrelation in returns. In these models, the weights extrapolators put on past price changes when deciding on their share demand typically do *not* decline monotonically, the further back we go into the past. In particular, in these models, extrapolators' share demand at time t depends on the *lagged* price change from time $t - 2$ to time $t - 1$; the lagged price change therefore matters more than the most recent price change from $t - 1$ to t in determining share demand. This assumption leads to positive short-term autocorrelation: a price increase at time $t - 1$ feeds into extrapolators' share demand only at time t, generating another price increase at that time. This suggests that an extension of our model in which extrapolators react to past price changes with some delay when forming their expectations may generate both negative long-term *and* positive short-term autocorrelations in price changes. We have analyzed this extension and have confirmed that it generates the conjectured autocorrelation structure. However, we do not pursue this approach here; doing so would complicate the analysis while improving the model's explanatory power in only a minor way.

4.5. Correlation of consumption changes and price changes

Another quantity of interest is the correlation of consumption growth and stock returns. In the data, this correlation is low. We now look at what our model predicts about the analogous quantity in our context: the correlation of consumption changes and price changes.

Proposition 7. (*Correlation between consumption changes and price changes.*) *The correlation between the change in aggregate consumption* $C \equiv \mu C^r + (1-\mu)C^e$ *and the change in price over a finite time horizon* $(0, t_1)$ *is*

$$\operatorname{corr}(C_{t_1} - C_0, P_{t_1} - P_0) = \frac{\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - P_0)}{\sqrt{\operatorname{var}(C_{t_1} - C_0)\operatorname{var}(P_{t_1} - P_0)}}, \tag{36}$$

where

$$\begin{aligned}
&\operatorname{cov}(C_{t_1} - C_0, P_{t_1} - P_0) \\
&= \left(rB\sigma_W - \frac{(2ag_D + rb)(rB\sigma_S + \sigma_D)}{r^2\gamma} \right) \frac{\sigma_S}{k}(1 - e^{-kt_1}) \\
&\quad + \frac{(2a_W g_D + rb_W)\sigma_D\sigma_S}{rk}\left(t_1 - \frac{1 - e^{-kt_1}}{k} \right) + \sigma_D\sigma_W t_1,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\operatorname{var}(C_{t_1} - C_0) = \\
&= \frac{r^2 a_W^2 \sigma_S^2}{k^2}\left(\frac{4g_D^2}{r^2}\Big(t_1 - \frac{1 - e^{-kt_1}}{k}\Big) + \frac{\sigma_S^2}{2k}\Big(t_1 - \frac{1 - e^{-2kt_1}}{2k}\Big) \right) + \frac{r^2 b_W^2 \sigma_S^2}{k^2}\left(t_1 - \frac{1 - e^{-kt_1}}{k} \right) \\
&\quad + \frac{4ra_W b_W \sigma_S^2 g_D}{k^2}\left(t_1 - \frac{1 - e^{-kt_1}}{k} \right) + r^2\sigma_W^2 t_1 + \frac{a^2\sigma_S^2}{\gamma^2 k}\left(\frac{4g_D^2(1 - e^{-kt_1})}{r^2} + \frac{\sigma_S^2(1 - e^{-2kt_1})}{k} \right) \\
&\quad + \frac{b^2\sigma_S^2(1 - e^{-kt_1})}{\gamma^2 k} + \frac{4ab\sigma_S^2 g_D(1 - e^{-kt_1})}{\gamma^2 kr} + \frac{2r\sigma_W\sigma_S(2a_W g_D + b_W r)}{k}\left(t_1 - \frac{1 - e^{-kt_1}}{k} \right) \\
&\quad - \frac{2\sigma_W\sigma_S(2ag_D + br)(1 - e^{-kt_1})}{k\gamma},
\end{aligned} \tag{38}$$

and where the expression for $\operatorname{var}(P_{t_1} - P_0)$ *is given in Proposition* 5. *Note that* $a = \mu a^r + (1-\mu)a^e$, $b = \mu b^r + (1-\mu)b^e$, $a_W = \dfrac{a}{\gamma}$, $b_W = \dfrac{b}{\gamma} - \dfrac{\beta BQ}{1-\beta B} - rBQ$, $\sigma_W = \dfrac{\sigma_D Q}{(1-\beta B)r}$, $k = \dfrac{\beta}{1-\beta B}$, *and* $\sigma_S = \dfrac{\beta\sigma_D}{(1-\beta B)r}$. ■

Panels A and B of Table 7 report the correlation of consumption changes and price changes at a quarterly and annual frequency, respectively, and for several (μ, β) pairs. The numbers are computed using the expressions in Proposition 7. The panels show that, while the presence of extrapolators slightly reduces the correlation of consumption changes and price changes relative to its value in the fully rational economy, the correlation is nonetheless high. As is the case for virtually all consumption-based asset pricing models, then, our model fails to match the low empirical correlation of consumption growth and stock returns.

4.6. Predictive power of surplus consumption

Prior empirical research has shown that a variable called the "surplus consumption ratio"—a measure of how current consumption compares to past consumption—is contemporaneously correlated with the price-dividend ratio of the stock market, and, furthermore, predicts subsequent stock market returns with a negative sign (Campbell and Cochrane, 1999; Cochrane, 2011). These findings have been taken as support for habit-based models of the aggregate stock market. We show, however, that these patterns also emerge from our model.

As we have done throughout this section, we study difference-based quantities—in this case, the surplus consumption *difference* rather than the surplus consumption ratio. Moreover, we focus on the simplest possible surplus consumption difference, namely the current level of aggregate consumption minus the level of aggregate consumption at some point in the past. Proposition 8 computes the correlation between this variable and the current price-dividend difference $P - D/r$.

Proposition 8. (*Correlation between the change in consumption and $P - D/r$.*) *The correlation between the change in aggregate consumption over a finite time horizon* $(0, t_1)$ *and $P - D/r$ measured at time t_1 is*

$$\mathrm{corr}(C_{t_1} - C_0, P_{t_1} - D_{t_1}/r) = \frac{\mathrm{cov}(C_{t_1} - C_0, P_{t_1} - D_{t_1}/r)}{\sqrt{\mathrm{var}(C_{t_1} - C_0)\,\mathrm{var}(P_{t_1} - D_{t_1}/r)}}, \tag{39}$$

where

$$\mathrm{cov}(C_{t_1} - C_0, P_{t_1} - D_{t_1}/r) = \left(\frac{(2a_W g_D + rb_W)r\gamma - (2ag_D + rb)k}{2kr\gamma}\sigma_S + r\sigma_W \right) \frac{B\sigma_S}{k}(1 - e^{-kt_1}) \tag{40}$$

and $\mathrm{var}(P_{t_1} - D_{t_1}/r) = B^2\sigma_S^2/(2k)$. *The expression for* $\mathrm{var}(C_{t_1} - C_0)$ *and the definitions of a, b,* a_W, b_W, σ_W, *k, and* σ_S *are given in Proposition* 7. ■

Proposition 9 examines whether the surplus consumption difference can predict future price changes.

Proposition 9. (*Predictive power of the change in consumption.*) *Consider a regression of the price change in the stock market from t_1 to t_2 on the change in aggregate consumption over the finite time horizon* $(0, t_1)$. *The coefficient on the independent variable is*

$$\beta_{\Delta C}(t_1, t_2) \equiv \frac{\operatorname{cov}\left(C_{t_1} - C_0, P_{t_2} - P_{t_1}\right)}{\operatorname{var}\left(C_{t_1} - C_0\right)}, \tag{41}$$

where

$$\operatorname{cov}(C_{t_1} - C_0, P_{t_2} - P_{t_1})$$
$$= -\left(\frac{(2a_W g_D + rb_W)r\gamma - (2ag_D + rb)k}{2kr\gamma}\sigma_S + r\sigma_W\right)\frac{B\sigma_S}{k}(1 - e^{-k(t_2 - t_1)})(1 - e^{-kt_1}). \tag{42}$$

The expression for $\operatorname{var}(C_{t_1} - C_0)$ *and the definitions of a, b,* a_W, b_W, σ_W, *k, and* σ_S *are given in Proposition* 7. ■

We use Proposition 8 to compute, for several (μ, β) pairs, the correlation between the surplus consumption difference—the change in aggregate consumption over the course of a quarter—and the price-dividend difference at the end of the quarter; the results are in Panel C of Table 7. The panel shows that the two quantities are significantly correlated. Table 8 reports the coefficient on the independent variable in a regression of the price change in the stock market over some interval—one quarter, one year, two years, three years, or four years—on the surplus consumption difference measured at the beginning of the interval; the numbers in the table are based on the expressions in Proposition 9. The table shows that the surplus consumption difference predicts subsequent price changes with a negative sign, and that this predictive power is particularly strong for low values of μ and high values of β. Taken together, then, Panel C of Table 7 and Table 8 show that the surplus consumption difference can be correlated with the valuation level of the stock market and with the subsequent stock price change even in a framework that does not involve habit-type preferences.

What is the intuition for these results? After a sequence of good cash-flow news, extrapolators cause the stock market to become overvalued and hence the price-dividend difference to be high. At the same time, extrapolators' optimistic beliefs about the future lead them to raise their consumption; while the rational traders do not raise their consumption as much, aggregate consumption nonetheless increases, pushing the surplus consumption difference up. This generates a positive correlation between the price-dividend difference and the surplus consumption difference. Since the stock market is overvalued at this point, the subsequent price change in the stock market is low, on average. As a consequence, the surplus consumption difference predicts future price changes with a negative sign.

4.7. The equity premium and Sharpe ratio

Proposition 10 below computes the equity premium and Sharpe ratio of the stock market in our economy.

Proposition 10. (*Equity premium and Sharpe ratio.*) *The equity premium, defined as the per unit time expectation of the sum of the excess price change and dividend, is given by*

$$\frac{1}{dt}\mathbb{E}\left[dP_t + D_t dt - rP_t dt\right] = \frac{(1-rB)g_D - r^2 A}{r}. \tag{43}$$

The Sharpe ratio is

$$\frac{\mathbb{E}\left[\frac{1}{dt}(dP_t + D_t dt - rP_t dt)\right]}{\sqrt{\frac{1}{dt}\mathrm{var}(dP_t + D_t dt - rP_t dt)}} = (1-\beta B)\frac{(1-rB)g_D - r^2 A}{\sigma_D}. \tag{44}$$

■

We use Proposition 10 to compute the equity premium for several (μ, β) pairs; the results are in Panel A of Table 9. The panel shows that the equity premium rises as the fraction of extrapolators in the economy goes up: the more extrapolators there are, the more volatile the stock market is; the equity premium therefore needs to be higher to compensate for the higher risk. Panel B of the table shows that the Sharpe ratio also goes up as μ falls.

5. Ratio-based quantities

In Section 4, we focused on quantities defined in terms of differences: on price changes, and on the price-dividend difference $P - D/r$. Given the additive structure of our model, these are the natural quantities to study. However, most empirical research works with ratio-based quantities such as returns and price-dividend ratios. While these are not the most natural quantities to look at in the context of our model, we can nonetheless examine what the model predicts about them. This is what we do in this section.

Since analytical results are not available for ratio-based quantities, we use numerical simulations to study their properties. In Section 5.1, we explain the methodology behind the simulations. In Section 5.2, we present the results. In brief, the results for the ratio-based quantities are broadly consistent with those for the difference-based quantities in Section 4. However, we interpret these results cautiously: because the ratio-based quantities are not the natural objects of study in our model, they are not as well-behaved as the difference-based quantities examined in Section 4.

5.1. Simulation methodology

To conduct the simulations, we first discretize the model. In this discretized version, we use a time-step of $\Delta t = 1/4$, in other words, of one quarter. As indicated in Section 3, the initial level of the dividend is $D_0 = 10$ and the initial wealth levels are $W_0^e = W_0^r = 5000.$ We further set the initial sentiment level S_0 to its steady-state mean of g_D / r.

We know from Proposition 1 that, at time 0,

$$P_0 = A + BS_0 + \frac{D_0}{r},$$
$$N_0^i = \eta_0^i + \eta_1^i S_0, \quad C_0^i = rW_0^i - \frac{a^i S_0^2 + b^i S_0 + c^i}{\gamma} - \frac{\log(r\gamma)}{\gamma}, \quad i \in \{e, r\}. \tag{45}$$

The proposition also tells us that, for any non-negative integer n,

$$
\begin{aligned}
D_{(n+1)\Delta t} &= D_{n\Delta t} + g_D \Delta t + \sigma_D \sqrt{\Delta t}\, \varepsilon_{(n+1)\Delta t}, \\
S_{(n+1)\Delta t} &= S_{n\Delta t} - \frac{\beta}{1-\beta B}\left(S_{n\Delta t} - \frac{g_D}{r} \right)\Delta t + \frac{\beta \sigma_D \sqrt{\Delta t}}{(1-\beta B) r}\, \varepsilon_{(n+1)\Delta t}, \\
P_{(n+1)\Delta t} &= A + B S_{(n+1)\Delta t} + \frac{D_{(n+1)\Delta t}}{r}, \\
N^i_{(n+1)\Delta t} &= \eta^i_0 + \eta^i_1 S_{(n+1)\Delta t}, \\
W^i_{(n+1)\Delta t} &= W^i_{n\Delta t} + r W^i_{n\Delta t}\Delta t - C^i_{n\Delta t}\Delta t - r N^i_{n\Delta t} P_{n\Delta t}\Delta t \\
&\quad + N^i_{n\Delta t}(P_{(n+1)\Delta t} - P_{n\Delta t}) + N^i_{n\Delta t} D_{(n+1)\Delta t}\Delta t, \\
C^i_{(n+1)\Delta t} &= r W^i_{(n+1)\Delta t} - \frac{a^i S^2_{(n+1)\Delta t} + b^i S_{(n+1)\Delta t} + c^i}{\gamma} - \frac{\log(r\gamma)}{\gamma},
\end{aligned}
\tag{46}
$$

with $i \in \{e, r\}$, and where $\{\varepsilon_{(n+1)\Delta t}, n \geq 0\}$ are independent and identically distributed draws from a standard normal distribution. We make the conventional assumptions that the level of the consumption stream for the interval ($n\Delta t$, $(n+1)\Delta t$) is determined at the beginning of the interval; and that the level of the dividend paid over this interval is determined at the end of the interval.

For a given set of values of the basic model parameters in Table 2, we use the procedure described in Appendix A to compute the values of the derived parameters that determine the optimal consumption and portfolio holdings—parameters such as η^e_1, for example. We then use Eqs. (45) and (46) to simulate 10,000 sample paths for our economy, where each sample path is 200 periods long, in other words, 50 years long. For any quantity of interest—the autocorrelation of stock market returns, say—we compute the value of the quantity for each of the 10,000 paths. In the next section, we report the *average* value of the quantity—for example, the average return autocorrelation—across the 10,000 simulated paths.[15]

5.2. Results

Table 10 presents the model's predictions for ratio-based quantities for μ = 0.25 and for three different values of β. As explained above, for each (μ, β) pair, we

[15] If any of the dividend, the price, aggregate consumption, or aggregate wealth turns negative somewhere on a simulated path, we discard that path. Since the standard deviation of dividend changes σ_D = 0.25 is low relative to the initial dividend level D_0, this is a rare occurrence: we discard only about 1% of paths.

simulate 10,000 paths, each of which is 200 periods long. For each of the 10,000 paths, we compute various quantities of interest—specifically, the quantities listed in the left column of Table 10. The table reports the average value of each quantity across the 10,000 paths. The right-most column reports the empirical value of each quantity computed using U.S. stock market data over the post-war period from 1947 to 2011.[16]

We now discuss each of these quantities in turn. Most of them are simply the ratio-based analogs of the quantities we studied in Section 4: for example, instead of computing the standard deviation of price changes, we compute the standard deviation of returns. However, the simulation methodology also allows us to address some questions that we did not discuss in any form in Section 4, such as whether the consumption-wealth ratio or more complex formulations of the surplus consumption ratio have predictive power for future returns.

Row 1. We report the coefficient on the independent variable in a regression of log excess stock returns measured over a one-year horizon on the log dividend-price ratio at the start of the year. To be clear, as described above, we run this regression in each of the 10,000 paths we simulate; the table reports the average coefficient across all paths, as well as the average R-squared, in parentheses. Consistent with the findings of Section 4.1, the table shows that the dividend-price ratio predicts subsequent returns with a positive sign.

Row 2. We report the autocorrelation of the price-dividend ratio at a one-year lag. Consistent with the results of Section 4.2, the ratio is highly persistent.

Row 3. We compute the excess volatility of returns—specifically, the standard deviation of annual stock returns in the heterogeneous-agent economy divided by the standard deviation of annual stock returns in the benchmark rational economy. Consistent with the findings of Section 4.3, stock returns exhibit excess volatility.

Row 4. We compute the excess volatility of the price-dividend ratio: the standard deviation of the price-dividend ratio in the heterogeneous-agent economy

[16] Returns are based on the Center for Research in Security Prices (CRSP) value-weighted index. For the consumption-wealth ratio, "wealth" is computed using aggregate household wealth from the Flow of Funds accounts, following Lettau and Ludvigson (2001). For the nondurable consumption data, the sample period starts in 1952.

divided by its standard deviation in the benchmark rational economy. Consistent with Section 4.3, the standard deviation of the price-dividend ratio goes up in the presence of extrapolators.

Row 5. We compute the autocorrelation of quarterly log excess stock returns at lags of one quarter and two years. As in Section 4.4, returns are negatively autocorrelated.

Row 6. We report the correlation of annual log excess stock returns and annual changes in log aggregate consumption. As in Section 4.5, this correlation is higher in our model than in actual data.

Row 7. We compute the correlation between the surplus consumption ratio and the price-dividend ratio, where both quantities are measured at a quarterly frequency. Given the greater flexibility afforded by the numerical approach of this section, we use a more sophisticated definition of surplus consumption than in Section 4.6. While this definition is still simpler than that used by Campbell and Cochrane (1999), it captures the spirit of their calculation. Specifically, we define the surplus consumption ratio as:

$$SC_t = \frac{C_t - X_t}{C_t}, \tag{47}$$

where C_t is aggregate consumption, and where the habit level X_t adjusts slowly to changes in consumption:

$$X_t = \sum_{l=1}^{n} w(l;\xi,n) C_{t-l\Delta t}, \quad \text{where } w(l;\xi,n) \equiv \frac{e^{-\xi l \Delta t}}{\sum_{j=1}^{n} e^{-\xi j \Delta t}}. \tag{48}$$

In words, X_t is a weighted sum of past consumption levels, where recent consumption levels are weighted more heavily. For a given ξ, we choose n to be the smallest positive integer for which $\frac{\sum_{j=1}^{n} e^{-\xi j \Delta t}}{\sum_{j=1}^{\infty} e^{-\xi j \Delta t}} = 1 - e^{-\xi n \Delta t} > 90\%.$ In our calculations, we set $\xi = 0.95$ and $n = 10$.[17]

Row 7 of Table 10 shows that, as in Section 4.6, the surplus consumption ratio and price-dividend ratio are positively correlated, consistent with the data.

[17] When $\xi = 0.95$, quarterly consumption one year ago is weighted about 40% as much as current consumption.

Row 8. We report the coefficient on the independent variable in a regression of log excess stock returns over a one-year horizon on the surplus consumption ratio at the start of the year; the surplus consumption ratio is defined in our discussion of Row 7. Consistent with the results in Section 4.6 and with actual data, the surplus consumption ratio predicts subsequent returns with a negative sign.

Row 9. Empirically, the consumption-wealth ratio predicts subsequent stock market returns with a positive sign. We examine whether our model can generate this pattern. We compute the coefficient on the independent variable in a regression of log excess returns over a one-year horizon on the log consumption-wealth ratio at the start of the year. The table shows that, in our model, the ratio indeed predicts subsequent returns, and does so with the correct sign.

What is the intuition for this predictive power? After a sequence of good cash-flow news, extrapolators cause the stock market to become overvalued. This, in turn, increases aggregate wealth in the economy. It also increases aggregate consumption, but not to the same extent: rational traders, in particular, do not increase their consumption very much because they realize that future returns on the stock market are likely to be low. Overall, the consumption-wealth ratio falls. Since the stock market is overvalued, its subsequent return is low, on average. The consumption-wealth ratio therefore predicts subsequent returns with a positive sign.

Row 10. We compute the annual equity premium and Sharpe ratio in our economy.

In summary, while it is more natural, in our framework, to study difference-based quantities rather than ratio-based quantities, Table 10 shows that the ratio-based quantities exhibit patterns that are broadly similar to those that we obtained in Section 4 for the difference-based quantities.

6. Conclusion

Survey evidence suggests that many investors form beliefs about future stock market returns by extrapolating past returns: they expect the stock market to perform well (poorly) in the near future if it has recently performed well (poorly). Such beliefs are hard to reconcile with existing models of the aggregate stock market. We study a heterogeneous-agent model in which some investors form beliefs about future stock market price changes by extrapolating past price changes, while other investors have fully rational beliefs. We find that the model captures many features of actual returns and prices. Importantly, however, it is also consistent with the survey evidence on investor expectations. This suggests that the survey evidence does not need to be seen as a nuisance; on the contrary, it is consistent with the facts about prices and returns and may be the key to understanding them.

Appendix A. Proofs of Proposition 1 and Corollaries 1 and 2

Proof of Proposition 1. To solve the stochastic dynamic programming problem, we need the differential forms of the evolution of the state variables. As noted in (16), the differential form of the definition of sentiment, $S_t = \beta\int_{-\infty}^{t} e^{-\beta(t-s)} dP_{s-dt}$, is

$$dS_t = -\beta S_t dt + \beta dP_t . \qquad \text{(A1)}$$

The term $-\beta S_t dt$ captures the fact that, when we move from time t to time $t + dt$, all the earlier price changes that contribute to S_t become associated with smaller weights because they are further away from time $t + dt$ than from time t; the term βdP_t captures the fact that the latest price change contributes positively to S_t; and the parameter β governs the stickiness of the belief updating. Also, the wealth of each type of trader evolves as

$$\begin{aligned} W_{t+dt}^i &= (W_t^i - C_t^i dt - N_t^i P_t)(1+rdt) + N_t^i D_t dt + N_t^i P_{t+dt} \\ \Rightarrow dW_t^i &= rW_t^i dt - C_t^i dt - rN_t^i P_t dt + N_t^i dP_t + N_t^i D_t dt, \qquad i \in \{e,r\}, \end{aligned} \qquad \text{(A2)}$$

as in Eqs. (6) and (8).

The derived value functions for the extrapolators and the rational traders are

$$J^i(W_t^i, S_t, t) \equiv \max_{\{C_s^i, N_s^i\}_{s\geq t}} \mathbb{E}_t^i\left[-\int_t^{\infty} \frac{e^{-\delta s - \gamma C_s^i}}{\gamma} ds\right], \qquad i \in \{e,r\}. \qquad \text{(A3)}$$

The assumptions that the investors have CARA preferences, that D_t follows an arithmetic Brownian motion, that S_t evolves in a Markovian fashion as in (A1), and that extrapolators' biased beliefs in (3) are linearly related to S_t jointly guarantee that the derived value functions are functions of time, wealth, and sentiment, but nothing else (such as D_t or P_t). We verify this claim and discuss it further after solving the model.

If we define

$$\phi^i(C^i, N^i; W^i, S, t) \equiv -\frac{e^{-\delta t - \gamma C^i}}{\gamma} + \frac{1}{dt}\mathbb{E}_t^i[dJ^i], \qquad i \in \{e,r\}, \qquad \text{(A4)}$$

then, from the theory of stochastic control,[18]

$$0 = \max_{\{C^i, N^i\}} \phi^i(C^i, N^i; W^i, S, t), \qquad i \in \{e,r\}. \qquad \text{(A5)}$$

Using Ito's lemma, (A5) leads to the Bellman equations which state that, along the optimal path of consumption and asset allocation,

$$\begin{aligned} 0 = &-\frac{e^{-\delta t - \gamma C^i}}{\gamma} + J_t^i + J_W^i(rW^i - C^i - rN^i P + N^i g_P^i + N^i D) + \frac{1}{2}J_{WW}^i \sigma_P^2 (N^i)^2 \\ &+ J_S^i(-\beta S + \beta g_P^i) + \frac{1}{2}\beta^2 J_{SS}^i \sigma_P^2 + \beta J_{WS}^i N^i \sigma_P^2, \qquad i \in \{e,r\}, \end{aligned} \qquad \text{(A6)}$$

where g_P^e and g_P^r are the per unit time change in the price of the stock market expected by extrapolators and rational traders, respectively, and where σ_P is the per unit time volatility of the stock price. As stated in (3), $g_P^e = \lambda_0 + \lambda_1 S$, while g_P^r derives from the rational traders' conjecture about the stock price process, which is yet to be determined. In continuous time, the volatility σ_P is essentially observable by computing the quadratic

[18] See Kushner (1967) for a detailed discussion of this topic.

variation; as a result, the two types of trader agree on its value. We assume, and later verify, that σ_P is an endogenously determined constant that does not depend on S or t.

Since the investors have infinite horizons, and since, as we verify later, the evolutions of W^e and W^r do not depend explicitly on the level of the dividend or the stock price, the passage of time only affects the value functions through time discounting. We can therefore write, for $i \in \{e, r\}$,

$$J^i(W_t^i, S_t, t) = e^{-\delta t} I^i(W_t^i, S_t), \quad \text{where } I^i(W_t^i, S_t) \equiv \max_{\{C_s^i, N_s^i\}_{s\geq t}} \mathbb{E}_t^i \left[-\int_t^\infty \frac{e^{-\delta(s-t)-\gamma C_s^i}}{\gamma} ds \right]. \tag{A7}$$

Substituting (A7) into (A6) gives the reduced Bellman equations

$$\begin{aligned} 0 = &-\frac{e^{-\gamma C^i}}{\gamma} - \delta I^i + I_W^i(rW^i - C^i - rN^i P + N^i g_P^i + N^i D) + \frac{1}{2} I_{WW}^i \sigma_P^2 (N^i)^2 \\ &+ I_S^i(-\beta S + \beta g_P^i) + \frac{1}{2}\beta^2 I_{SS}^i \sigma_P^2 + \beta I_{WS}^i N^i \sigma_P^2, \qquad i \in \{e, r\}. \end{aligned} \tag{A8}$$

The first-order conditions of (A8) with respect to C^i and N^i are

$$I_W^i = e^{-\gamma C^i} \tag{A9}$$

and

$$N^i = -\frac{I_W^i}{I_{WW}^i} \frac{g_P^i - rP + D}{\sigma_P^2} - \frac{\beta I_{WS}^i}{I_{WW}^i}, \qquad i \in \{e, r\}. \tag{A10}$$

The first term on the right-hand side of (A10) is the share demand due to mean-variance considerations; the second term is the hedging demand due to sentiment-related risk.

We now conjecture, and later verify, that the true equilibrium stock price satisfies

$$P_t = A + BS_t + \frac{D_t}{r}. \tag{A11}$$

The coefficients A and B are yet to be determined. Assuming that the rational traders know this price equation and the true process for D_t, they can combine (1), (A1), and (A11) to obtain the true evolution of the stock price, namely

$$dP_t = \left(-\frac{\beta B}{1-\beta B} S_t + \frac{g_D}{(1-\beta B) r} \right) dt + \frac{\sigma_D}{(1-\beta B) r} d\omega_t. \tag{A12}$$

Substituting (A12) into (A1) yields

$$dS_t = -\frac{\beta}{1-\beta B} \left(S_t - \frac{g_D}{r} \right) dt + \frac{\beta \sigma_D}{(1-\beta B) r} d\omega_t. \tag{A13}$$

From (A12) and (A13) it is clear that, when $B < \beta^{-1}$, sentiment S_t follows an Ornstein-Uhlenbeck process with a steady-state distribution that is Normal with mean $\frac{g_D}{r}$ and variance $\frac{\beta \sigma_D^2}{2(1-\beta B) r^2}$. In addition,

$$g_P^r = -\frac{\beta B}{1-\beta B} S_t + \frac{g_D}{(1-\beta B) r}, \qquad \sigma_P = \frac{\sigma_D}{(1-\beta B) r}. \tag{A14}$$

That is, if the conjecture in (A11) is valid, rational traders' expected future price change is negatively and linearly related to the sentiment level, and σ_P is a constant.

We noted in Eq. (4) that, in extrapolators' minds, the stock price evolves according to

$$dP_t = \left(\lambda_0 + \lambda_1 S_t\right) dt + \frac{\sigma_D}{(1-\beta B)r} d\omega_t^e \,, \tag{A15}$$

where, again from the perspective of extrapolators, ω_t^e is a Wiener process. At the same time, in order to compute the values of the derived parameters that govern their consumption and portfolio decisions, extrapolators need to be aware of the price equation (A11). Eq. (A11) is consistent with the beliefs in (A15) if extrapolators have incorrect beliefs about the dividend process—specifically, if they believe that dividends evolve according to

$$dD_t = g_D^e dt + \sigma_D d\omega_t^e \,, \tag{A16}$$

where g_D^e is the expected dividend change per unit time perceived by extrapolators. To determine g_D^e, differentiate (A11) and substitute in (A1) and (A16). This gives

$$dP_t = \left(-\frac{\beta B}{1-\beta B} S_t + \frac{g_D^e}{(1-\beta B)r} \right) dt + \frac{\sigma_D}{(1-\beta B)r} d\omega_t^e . \tag{A17}$$

Eq. (A17) is identical to (A15) so long as

$$g_D^e(S_t) = (1-\beta B)\lambda_0 r + \left((1-\beta B)\lambda_1 r + \beta B r \right) S_t \,, \tag{A18}$$

in other words, so long as extrapolators' perceived expected dividend change depends explicitly on S_t.[19]

We guess, and later verify, that $I^e(W^e, S)$ and $I^r(W^r, S)$ take the form

$$I^i(W^i, S) = -\exp(-r\gamma W^i + a^i S^2 + b^i S + c^i), \qquad i \in \{e, r\} . \tag{A19}$$

Substituting (A19) into the optimal consumption rule in (A9) and the optimal share demand in (A10) yields

$$C^i = rW^i - \frac{a^i S^2 + b^i S + c^i}{\gamma} - \frac{\log(r\gamma)}{\gamma} \,, \tag{A20}$$

and

$$N^i = \frac{g_P^i - rP + D}{r\gamma\sigma_P^2} + \frac{\beta(2a^i S + b^i)}{r\gamma} \,, \qquad i \in \{e, r\} . \tag{A21}$$

We denote the share demand of the extrapolators and rational traders as

$$N^i = \eta_0^i + \eta_1^i S, \qquad i \in \{e, r\}. \tag{A22}$$

Substituting g_P^e from (3), (A11), the form of I^e in (A19), the optimal consumption C^e in (A20), and the optimal share demand N^e in (A22) into the reduced Bellman equation (A8) for the extrapolators, we obtain the following quadratic equation in S,

[19] If, instead, extrapolators had correct beliefs about the evolution of D_t, they would be able to use Eq. (A11) to infer the true price process in Eq. (A12).

$$0 = r - \delta - r\left(\gamma(\eta_0^e + \eta_1^e S)(\lambda_0 + \lambda_1 S - rA - rBS) + a^e S^2 + b^e S + c^e + \log(r\gamma)\right)$$
$$+\tfrac{1}{2} r^2\gamma^2\sigma_P^2(\eta_0^e + \eta_1^e S)^2 + (2a^e S + b^e)\left(-\beta S + \beta(\lambda_0 + \lambda_1 S)\right) + \tfrac{1}{2}\beta^2\sigma_P^2(2a^e S + b^e)^2 + \beta^2\sigma_P^2 a^e \quad \text{(A23)}$$
$$-\beta r\gamma\sigma_P^2(2a^e S + b^e)(\eta_0^e + \eta_1^e S),$$
which implies
$$0 = -r\gamma\eta_1^e(\lambda_1 - rB) - ra^e + \tfrac{1}{2} r^2\gamma^2\sigma_P^2(\eta_1^e)^2 + 2\beta a^e(\lambda_1 - 1) + 2\beta^2\sigma_P^2(a^e)^2 - 2\beta r\gamma\sigma_P^2 a^e\eta_1^e, \quad \text{(A24)}$$
$$0 = -r\gamma\eta_0^e(\lambda_1 - rB) - r\gamma\eta_1^e(\lambda_0 - rA) - rb^e + r^2\gamma^2\sigma_P^2\eta_0^e\eta_1^e + 2\beta a^e\lambda_0$$
$$+\beta b^e(\lambda_1 - 1) + 2\beta^2\sigma_P^2 a^e b^e - \beta r\gamma\sigma_P^2(2a^e\eta_0^e + b^e\eta_1^e), \quad \text{(A25)}$$
$$0 = r - \delta - r\gamma\eta_0^e(\lambda_0 - rA) - rc^e - r\log(r\gamma) + \tfrac{1}{2} r^2\gamma^2\sigma_P^2(\eta_0^e)^2 + \beta b^e\lambda_0$$
$$+\tfrac{1}{2}\beta^2\sigma_P^2(b^e)^2 + \beta^2\sigma_P^2 a^e - \beta r\gamma\sigma_P^2 b^e\eta_0^e. \quad \text{(A26)}$$

If extrapolators know the values of A and B, they can use Eqs. (A24), (A25), and (A26) to solve for their optimal share demand N^e, optimal consumption C^e, and value function J^e. Since they are aware of the price equation (A11), the easiest way for them to compute the values of A and B is by using past observations on dividends and prices. Alternatively, if they know the belief structure of the rational traders and the values of μ and Q, they can infer the values of A and B by going through the intertemporal maximization problem for the rational investors (specified below).

We now turn to the rational traders. Substituting (A11), g_P^r from (A14), the form of I^r in (A19), the optimal consumption C^r in (A20), and the optimal share demand N^r in (A22) into the reduced Bellman equation (A8) for the rational traders, we obtain another quadratic equation in S,
$$0 = r - \delta - r\left(\gamma(\eta_0^r + \eta_1^r S)\left(-\frac{\beta B}{1-\beta B} S + \frac{g_D}{(1-\beta B)r} - rA - rBS\right) + a^r S^2 + b^r S + c^r + \log(r\gamma)\right)$$
$$+\tfrac{1}{2} r^2\gamma^2\sigma_P^2(\eta_0^r + \eta_1^r S)^2 - \frac{\beta(2a^r S + b^r)}{1-\beta B}\left(S - \frac{g_D}{r}\right) + \tfrac{1}{2}\beta^2\sigma_P^2(2a^r S + b^r)^2 + \beta^2\sigma_P^2 a^r \quad \text{(A27)}$$
$$-\beta r\gamma\sigma_P^2(2a^r S + b^r)(\eta_0^r + \eta_1^r S),$$
which implies
$$0 = r\gamma\eta_1^r B\left(\frac{\beta}{1-\beta B} + r\right) - ra^r + \tfrac{1}{2} r^2\gamma^2\sigma_P^2(\eta_1^r)^2 - \frac{2\beta a^r}{1-\beta B} + 2\beta^2\sigma_P^2(a^r)^2 - 2\beta r\gamma\sigma_P^2 a^r\eta_1^r, \quad \text{(A28)}$$

$$0 = r\gamma B\eta_0^r\left(\frac{\beta}{1-\beta B} + r\right) - r\gamma\eta_1^r\left(\frac{g_D}{(1-\beta B)r} - rA\right) - rb^r + r^2\gamma^2\sigma_P^2\eta_0^r\eta_1^r - \frac{\beta}{1-\beta B}\left(b^r - \frac{2g_D a^r}{r}\right)$$
$$+2\beta^2\sigma_P^2 a^r b^r - \beta r\gamma\sigma_P^2(2a^r\eta_0^r + b^r\eta_1^r), \quad \text{(A29)}$$
$$0 = r - \delta - r\gamma\eta_0^r\left(\frac{g_D}{(1-\beta B)r} - rA\right) - rc^r - r\log(r\gamma) + \tfrac{1}{2} r^2\gamma^2\sigma_P^2(\eta_0^r)^2 + \frac{\beta g_D b^r}{(1-\beta B)r}$$
$$+\tfrac{1}{2}\beta^2\sigma_P^2(b^r)^2 + \beta^2\sigma_P^2 a^r - \beta r\gamma\sigma_P^2 b^r\eta_0^r. \quad \text{(A30)}$$

Comparing (A22) with (A21) leads to

$$\eta_0^e = \frac{\lambda_0 - rA + \beta\sigma_P^2 b^e}{r\gamma\sigma_P^2}, \qquad \eta_0^r = \frac{1}{r\gamma\sigma_P^2}\left(\frac{g_D}{(1-\beta B)r} - rA\right) + \frac{\beta b^r}{r\gamma}, \tag{A31}$$

$$\eta_1^e = \frac{\lambda_1 - rB + 2\beta\sigma_P^2 a^e}{r\gamma\sigma_P^2}, \qquad \eta_1^r = -\frac{B}{r\gamma\sigma_P^2}\left(\frac{\beta}{1-\beta B} + r\right) + \frac{2\beta a^r}{r\gamma}. \tag{A32}$$

The optimal share demands in (A22) and the market clearing condition in (9) imply

$$\mu\eta_0^r + (1-\mu)\eta_0^e = Q, \tag{A33}$$

$$\mu\eta_1^r + (1-\mu)\eta_1^e = 0. \tag{A34}$$

Eqs. (A24)–(A26) and (A28)–(A34) are the mathematical characterization of the endogenous interaction between the rational traders and the extrapolators. The procedure for solving these simultaneous equations is described below.

The fact that the conjectured forms of P_t, I^e, and I^r in (A11) and (A19) satisfy the Bellman equations in (A8) for all W_t and S_t verifies these conjectures, conditional on the validity of the assumption that I^e and I^r depend explicitly only on W_t and S_t. To verify this last assumption, note that the price equation in (A11), the optimal consumption rules in (A20), and the optimal share demands N_t^e and N_t^r in (A22) jointly guarantee that the evolutions of W_t^e and W_t^r in (A2) depend explicitly only on S_t. Lastly, the derived evolution of the stock price in (A12) verifies the assumption that σ_P is a constant. This completes the verification procedure.

Eqs. (A11), (A12), (A13), (A22), and (A20) confirm Eqs. (10), (11), (12), (14), and (15) in the main text, respectively, and Eqs. (A7) and (A19) together confirm (13). This completes the proof of Proposition 1. ■

Solving the simultaneous equations. We solve Eqs. (A24)–(A26) and (A28)–(A34) in three steps. First, we use (A24), (A28), (A32), and (A34) to determine a^e, a^r, η_1^e, η_1^r, and B. Second, we use (A25), (A29), (A31), and (A33) to determine b^e, b^r, η_0^e, η_0^r, and A. Lastly, we solve each of (A26) and (A30) to obtain c^e and c^r, respectively.

Proof of Corollary 1. When all traders in the economy are fully rational, (A19) reduces to

$$I^r(W_t^r) = -e^{-r\gamma W_t^r} K, \tag{A35}$$

where K is a constant to be determined. Substituting (A35) into (A10) and using $N^r = Q$, we see that the equilibrium stock price is

$$P_t = -\frac{\gamma\sigma_D^2 Q}{r^2} + \frac{g_D}{r^2} + \frac{D_t}{r}. \tag{A36}$$

The third term on the right-hand side of this equation shows that P_t is pegged to the current level of the dividend; the other two terms capture dividend growth and compensation for risk. Substituting (A35) and (A36) into (A8) gives

$$K = \frac{1}{r\gamma}\exp\left(\frac{r-\delta}{r} - \frac{\gamma^2\sigma_D^2 Q^2}{2r}\right). \tag{A37}$$

From (A9), optimal consumption is

$$C_t^r = rW_t^r - \frac{1}{\gamma}\log(r\gamma K) = rW_t^r - \frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}. \tag{A38}$$

From (A2), (A36), and (A38), optimal wealth evolves according to

$$dW_t^r = \left(\frac{r-\delta}{r\gamma} + \frac{\gamma\sigma_D^2 Q^2}{2r}\right)dt + \frac{\sigma_D Q}{r}d\omega_t. \tag{A39}$$

This completes the proof of Corollary 1. ■

Proof of Corollary 2. Our objective is to show that there exists a solution to simultaneous equations (A24)–(A26) and (A28)–(A34). The proof has three steps, which correspond to the three steps in the solution procedure outlined above.

First, we show that Eqs. (A24), (A28), (A32), and (A34) guarantee a solution for a^e, a^r, η_1^e, η_1^r, and B, with $B \in (0, \beta^{-1})$.

Substituting (A32) into (A24) and (A28) gives

$$a^e = -\frac{(1-\beta B)^2(\lambda_1 - rB)^2 r^2}{2\sigma_D^2(r+2\beta-2\beta Br)}, \qquad a^r = -\frac{B^2(1-\beta B)^2 r^2}{2\sigma_D^2(r+2\beta-2\beta Br)}\left(\frac{\beta}{1-\beta B}+r\right)^2. \tag{A40}$$

Eqs. (A40) and (A32) allow us to express a^e, a^r, η_1^e, and η_1^r as functions only of B. Combining (A40) with (A32) and (A34) leads to a nonlinear equation for B,

$$0 = (1-\mu)(\lambda_1 - rB) - \mu B\left(\frac{\beta}{1-\beta B}+r\right) - \frac{(1-\mu)\beta(\lambda_1 - rB)^2}{(r+2\beta-2\beta Br)} - \frac{\mu\beta B^2}{(r+2\beta-2\beta Br)}\left(\frac{\beta}{1-\beta B}+r\right)^2. \tag{A41}$$

It is therefore sufficient to show that there exists $B \in (0, \beta^{-1})$ that satisfies (A41). To do so, denote the right-hand side of (A41) as $f(B)$. Note that $f(0) = \{[r + \beta(2 - \lambda_1)](1 - \mu)\lambda_1\}/(r + 2\beta)$; by (23), this is strictly positive for $\mu < 1$. When $\mu > 0$, (23) also implies that $f(B)$ goes to $-\infty$ as B goes to β^{-1} from below; when $\mu = 0$, (23) and (24) imply that $f(\beta^{-1}) = (\lambda_1\beta - r)(2 - \lambda_1)/(2\beta - r)$ is strictly negative. Continuity of $f(B)$ then guarantees that there exists $B \in (0, \beta^{-1})$ that solves (A41).

Eq. (A40) shows that, when $\mu < 1$, a^r is strictly negative, while a^e is weakly negative. Given this, (A32) and (A34) imply that η_1^r is strictly negative and that η_1^e is strictly positive.

Next, we show that Eqs. (A25), (A29), (A31), and (A33) guarantee a solution for b^e, b^r, η_0^e, η_0^r, and A. We treat a^e, a^r, η_1^e, η_1^r, and B as known coefficients.

Substituting (A31) into (A25) and (A29) gives

$$b^e = \frac{-\sigma_P^{-2}(\lambda_0 - rA)(\lambda_1 - rB) + 2\beta a^e rA}{r+\beta-\beta Br},$$
$$b^r = \frac{1}{r+\beta-\beta Br}\left[\frac{B}{\sigma_P^2}\left(\frac{g_D}{(1-\beta B)r} - rA\right)\left(\frac{\beta}{1-\beta B}+r\right) + 2\beta a^r rA\right]. \tag{A42}$$

Eqs. (A42) and (A31) allow us to express b^e, b^r, η_0^e, and η_0^r as functions only of *A*. Combining (A42) with (A31) and (A33) leads to a linear equation for *A* in which the coefficient on *A* is

$$-(1-\mu)r + \frac{r\beta(1-\mu)(\lambda_1 - rB) + 2(1-\mu)\beta^2 a^e r\sigma_P^2 + 2\mu\beta^2 a^r r\sigma_P^2}{r+\beta-r\beta B} - \frac{\mu r}{1-\beta B}. \tag{A43}$$

To guarantee a solution for *A*, it is sufficient to show that the expression in (A43) is non-zero. Substituting a^e from (A40) into (A43) and rearranging terms gives

$$\begin{aligned} &-\frac{(1-\mu)r}{(r+\beta-\beta Br)(r+2\beta-2\beta Br)}\left[(r+\beta-\beta\lambda_1)(r+2\beta-2\beta Br) + \beta^2(\lambda_1 - rB)^2\right] \\ &-\frac{\mu r}{1-\beta B} + \frac{2\mu\beta^2 a^r r\sigma_P^2}{r+\beta-\beta Br}. \end{aligned} \tag{A44}$$

For $B \in (0, \beta^{-1})$

$$\begin{aligned} &(r+\beta-\beta\lambda_1)(r+2\beta-2r\beta B) + \beta^2(\lambda_1 - rB)^2 \\ &= r[r+2\beta - r\beta B(2-\beta B)] + \beta[r(1-\lambda_1) + \beta(2-2rB+\lambda_1^2 - 2\lambda_1)] \\ &> 2r\beta + \beta[r(1-\lambda_1) + \beta(2-2rB+\lambda_1^2-2\lambda_1)] \\ &> \beta[(r+2\beta) + \beta\lambda_1^2 - (r+2\beta)\lambda_1] > \tfrac{1}{4}(r+2\beta)(2\beta - r) > 0, \end{aligned} \tag{A45}$$

where the last inequality holds because of (23). Given (23), (A45), and the fact that a^r is strictly negative, we know that the expression in (A44) is strictly negative and hence non-zero. This guarantees a solution for *A*, which in turn guarantees a solution for b^e, b^r, η_0^e, and η_0^r.

Finally, it is clear that (A26) and (A30) guarantee a solution for c^e and c^r. This completes the proof of Corollary 2. We note that our proof does not rule out any nonlinear equilibria. ■

Appendix B. Proofs of Propositions 2 to 10

In this Appendix, we present abbreviated proofs of Propositions 2 to 10. In these proofs, we make repeated use of the properties of the process $Z_t \equiv e^{kt}S_t$, where $k = \beta/(1 - \beta B)$; this process evolves according to

$$dZ_t = \frac{ke^{kt}g_D}{r}dt + e^{kt}\sigma_S d\omega_t\ . \tag{A46}$$

Unlike the sentiment process S_t, the Z_t process has a non-stochastic drift term and is therefore easier to analyze.

Proof of Proposition 2. It is straightforward to calculate rational traders' expectations about future price changes. Combining extrapolators' belief about the instantaneous price change, (A15), and the differential definition of the sentiment variable, (A1), we find that extrapolators' belief about the evolution of S_t is

$$dS_t = \beta\left(\lambda_0 + (\lambda_1 - 1)S_t\right)dt + \sigma_S d\omega_t^e \, . \tag{A47}$$

Extrapolators believe that ω_t^e is a standard Wiener process. This means that, from their perspective, the process $Z_t^e \equiv e^{mt} S_t$, where $m = \beta(1 - \lambda_1)$, evolves according to

$$dZ_t^e = e^{mt}\beta\lambda_0 dt + e^{mt}\sigma_S d\omega_t^e \, . \tag{A48}$$

Using the statistical properties of this process, we obtain (26). When $m = 0$, applying L'Hôpital's rule to (26) gives (27). ■

Proof of Proposition 3. From (A11),

$$\mathrm{cov}(D_0 / r - P_0, P_{t_1} - P_0) = -B^2 \,\mathrm{cov}(S_0, S_{t_1} - S_0) - Br^{-1}\,\mathrm{cov}(S_0, D_{t_1} - D_0) \, . \tag{A49}$$

Clearly, $\mathrm{cov}(S_0, D_{t_1} - D_0) = 0.$ Using the properties of the Z_t process, we can show

$$\mathrm{cov}(S_0, S_{t_1} - S_0) = -\frac{\sigma_S^2 (1 - e^{-kt_1})}{2k} \tag{A50}$$

and

$$\mathrm{var}(D_0 / r - P_0) = B^2 \,\mathrm{var}(S_0) = \frac{B^2 \sigma_S^2}{2k} \, . \tag{A51}$$

Eqs. (A49), (A50), and (A51) give (30). ■

Proof of Proposition 4. From (A11),

$$\rho_{PD}(t_1) = \mathrm{corr}(S_0, S_{t_1}) \, . \tag{A52}$$

We can show that

$$\mathrm{cov}(S_0, S_{t_1}) = \mathrm{cov}\left(\mathbb{E}[S_0 \mid S_0 = s], \mathbb{E}[S_{t_1} \mid S_0 = s]\right) = \frac{\sigma_S^2 e^{-kt_1}}{2k} \, . \tag{A53}$$

It is straightforward to also show that $\mathrm{var}(S_0) = \mathrm{var}(S_{t_1}) = \sigma_S^2 / (2k)$. Putting these results together, we obtain (31). ■

Proof of Proposition 5. From (A11),

$$\mathrm{var}(P_{t_1} - P_0) = B^2 \,\mathrm{var}(S_{t_1} - S_0) + 2Br^{-1}\,\mathrm{cov}(S_{t_1} - S_0, D_{t_1} - D_0) + r^{-2}\,\mathrm{var}(D_{t_1} - D_0) \, . \tag{A54}$$

The quantity $\mathrm{var}(S_{t_1} - S_0)$ can be expressed as

$$\mathrm{var}(S_{t_1} - S_0) = \mathbb{E}_s[\mathrm{var}(S_{t_1} - S_0 \mid S_0 = s)] + \mathrm{var}(\mathbb{E}[S_{t_1} - S_0 \mid S_0 = s]), \tag{A55}$$

where the subscript s means that we are taking the expectation over the steady-state distribution of s. We can show

$$\mathrm{var}(S_{t_1} - S_0 \mid S_0 = s) = e^{-2kt_1}\,\mathrm{var}(Z_{t_1} - Z_0 \mid S_0 = s) = \sigma_S^2 e^{-2kt_1} \int_0^{t_1} e^{2kt} dt = \frac{\sigma_S^2 (1 - e^{-2kt_1})}{2k} \, . \tag{A56}$$

Using the properties of the Z_t process, we also find that

$$\mathbb{E}[S_{t_1} - S_0 \mid S_0 = s] = \left(\frac{g_D}{r} - s\right)(1 - e^{-kt_1}) \tag{A57}$$

and

$$\operatorname{cov}(S_{t_1}-S_0, D_{t_1}-D_0)=\frac{\sigma_D\sigma_S(1-e^{-kt_1})}{k}. \tag{A58}$$

Substituting (A56) and (A57) into (A55) gives

$$\operatorname{var}(S_{t_1}-S_0)=\frac{\sigma_S^2(1-e^{-kt_1})}{k}. \tag{A59}$$

Substituting (A58), (A59), and $\operatorname{var}(D_{t_1}-D_0)=\sigma_D^2 t_1$ into (A54) gives (32). Combining (A11) with $\operatorname{var}(S_t)=\sigma_S^2/(2k)$ leads to (33). ■

Proof of Proposition 6. From (A11),

$$\begin{aligned}\operatorname{cov}(P_{t_1}-P_0, P_{t_3}-P_{t_2})&=B^2\operatorname{cov}(S_{t_1}-S_0, S_{t_3}-S_{t_2})+r^{-2}\operatorname{cov}(D_{t_1}-D_0, D_{t_3}-D_{t_2})\\&\quad+Br^{-1}\operatorname{cov}(S_{t_1}-S_0, D_{t_3}-D_{t_2})+Br^{-1}\operatorname{cov}(S_{t_3}-S_{t_2}, D_{t_1}-D_0).\end{aligned} \tag{A60}$$

Using the properties of the Z_t process, we obtain

$$\operatorname{cov}(S_{t_1}-S_0, S_{t_3}-S_{t_2})=\frac{\sigma_S^2}{2k}(e^{-kt_3}-e^{-kt_2})(e^{kt_1}-1) \tag{A61}$$

and

$$\operatorname{cov}(D_{t_1}-D_0, S_{t_3}-S_{t_2})=\frac{\sigma_D\sigma_S}{k}(e^{-kt_3}-e^{-kt_2})(e^{kt_1}-1). \tag{A62}$$

In addition, since the increments in future dividends are independent of any random variable that is measurable with respect to the information set at the current time,

$$\operatorname{cov}(D_{t_1}-D_0, D_{t_3}-D_{t_2})=\operatorname{cov}(S_{t_1}-S_0, D_{t_3}-D_{t_2})=0. \tag{A63}$$

Substituting (A61), (A62), and (A63) into (A60) yields the first equation in (35). The second equation in (35) is derived in Proposition 5, and the third equation can be derived in a similar way. ■

Proof of Propositions 7 to 9. From (A2), (A11), and (A20), we know that aggregate wealth $W\equiv\mu W^r+(1-\mu)W^e$ evolves as

$$dW_t=(a_W S_t^2+b_W S_t+c_W)dt+\sigma_W d\omega_t. \tag{A64}$$

Substituting this into (A20) yields

$$\begin{aligned}C_{t_1}-C_0&=r(W_{t_1}-W_0)-\gamma^{-1}\left(a(S_{t_1}^2-S_0^2)+b(S_{t_1}-S_0)\right)\\&=r\int_0^{t_1}(a_W S_t^2+b_W S_t+c_W)dt+r\sigma_W\int_0^{t_1}d\omega-\gamma^{-1}\left(a(S_{t_1}^2-S_0^2)+b(S_{t_1}-S_0)\right).\end{aligned} \tag{A65}$$

To compute $\operatorname{var}(C_{t_1}-C_0)$, we need to compute the covariance of every pair of terms in the last line of (A65). For example, one of these covariances is[20]

[20] Eq. (A66) makes use of Fubini's theorem. We have checked that the conditions that allow the use of this theorem hold in our context. For more on these conditions, see Theorem 1.9 in Liptser and Shiryaev (2001).

$$\operatorname{cov}\left(\int_0^{t_1} S_t dt, S_{t_1}\right) = \int_0^{t_1} \mathbb{E}_s\left[\mathbb{E}[S_t S_{t_1} \mid S_0 = s]\right] dt - \mathbb{E}_s\left[\mathbb{E}[S_{t_1} \mid S_0 = s]\right] \int_0^{t_1} \mathbb{E}_s\left[\mathbb{E}[S_t \mid S_0 = s]\right] dt$$
$$= \frac{\sigma_S^2 (1 - e^{-kt_1})}{2k^2}. \tag{A66}$$

The other covariance terms can be computed in a similar way. Rearranging and simplifying terms, we obtain (37), (38), (40), and (42). ■

Proof of Proposition 10. Substituting (A11) and (A12) into the definitions of the equity premium and Sharpe ratio gives (43) and (44). ■

Appendix C. Estimating

C.1. Estimation equations

Our objective is to estimate the model parameters β, λ_0, and λ_1 using survey data on investor expectations.

Suppose that we have a time-series of aggregate stock market prices with sample frequency Δ*t*. We approximate the sentiment variable in (2) as

$$S_t(\beta, n) = \sum_{l=0}^{n-1} w(l; \beta, n)(P_{t-l\Delta t} - P_{t-(l+1)\Delta t}), \tag{A67}$$

where $w(l;\beta,n) = \frac{e^{-\beta l \Delta t}}{\sum_{j=0}^{n-1} e^{-\beta j \Delta t}}$. The weighting function is parameterized by β and by *n*, which measures how far back investors look when forming their beliefs. The weights on past price changes sum to one. To compute (A67), we use quarterly price observations on the Standard and Poor's (S&P) 500 index, so that $\Delta t = 1/4$; we also set $n = 60$.

The central assumption of our model is that extrapolators' expected price change (not expected return) is

$$\mathbb{E}_t^e[dP_t]/dt = \lambda_0 + \lambda_1 S_t(\beta). \tag{A68}$$

The expectation in (A68) is computed over the next instant of time, from *t* to *t* + *dt*, not over a finite time horizon. In actual surveys, however, investors are typically asked to state their beliefs about the performance of the stock market over the next year. It is therefore not fully correct to estimate β, λ_0, and λ_1 using (A68). We must instead compute what our model implies for the price change extrapolators expect over a finite horizon. We do this in Proposition 2 and find

$$\mathbb{E}_t^e[P_{t_1} - P_t] = (\lambda_0 + \lambda_1 S_t(\beta))(t_1 - t) + \lambda_1(\beta\lambda_0 - mS_t(\beta)) \frac{m(t_1 - t) + e^{-m(t_1 - t)} - 1}{m^2}, \tag{A69}$$

where $m = \beta(1 - \lambda_1)$. The first term on the right-hand side of (A69) is extrapolators' expected instantaneous price change at time *t*, $\lambda_0 + \lambda_1 S_t(\beta)$, multiplied by the time horizon, $t_1 - t$. (For example, for a six-month horizon, $t_1 - t = 0.5$.) The second term takes into account extrapolators' subjective beliefs about how sentiment will evolve over the interval from *t* to t_1. The parameters β, λ_0, and λ_1 enter (A69) in a nonlinear way.

To determine β, λ_0, and λ_1, we therefore estimate both

$$\mathbb{E}_t^e[P_{t_1} - P_t] = (\hat{\lambda}_0 + \hat{\lambda}_1 S_t(\hat{\beta}, n))(t_1 - t) + \varepsilon_{t_1} \tag{A70}$$

and

$$\mathbb{E}_t^e[P_{t_1} - P_t] = (\hat{\lambda}_0 + \hat{\lambda}_1 S_t(\hat{\beta}, n))(t_1 - t) + \hat{\lambda}_1(\hat{\beta}\hat{\lambda}_0 - mS_t(\hat{\beta}, n))\frac{m(t_1 - t) + e^{-m(t_1 - t)} - 1}{m^2} + \varepsilon_{t_1}, \tag{A71}$$

with $m(\hat{\beta}, \hat{\lambda}_1) = \hat{\beta}(1 - \hat{\lambda}_1)$ and $S_t(\hat{\beta}, n)$ constructed as described above. We also estimate (A71) for the special case where λ_1 is fixed at one. In this case, (A71) becomes

$$\mathbb{E}_t^e[P_{t_1} - P_t] = (\hat{\lambda}_0 + S_t(\hat{\beta}, n))(t_1 - t) + \frac{\hat{\beta}\hat{\lambda}_0(t_1 - t)^2}{2} + \varepsilon_{t_1}. \tag{A72}$$

C.2. Survey data

We estimate Eqs. (A70), (A71), and (A72) using the Gallup survey data studied by Greenwood and Shleifer (2014) and others. Specifically, we identify the surveyed investors with the extrapolators in our model, so that the expectations on the left-hand side of Eqs. (A70), (A71), and (A72) are the average expectation of the surveyed investors about future price changes. To compute these expectations, we start with the "rescaled" investor expectations described in Greenwood and Shleifer (2014). After the rescaling, the reported expectations are in units of percentage expected return on the aggregate stock market over the following 12 months. We convert this series into expected *price* changes by multiplying by the level of the S&P 500 price index at the end of the month in which participants are surveyed. That is,

$$\mathbb{E}_t^e[P_{t_1} - P_t] = \underbrace{\mathbb{E}_t^e[\frac{P_{t_1} - P_t}{P_t}]}_{Survey} \cdot P_t. \tag{A73}$$

The resulting time series of investor expectations comprises 135 data points between October 1996 and November 2011. The data are monthly but there are also some gaps.

We estimate (A70), (A71), and (A72) using nonlinear least squares regression. We report coefficients and *t*-statistics based on Newey-West standard errors with a lag length of six months.

Coefficient	Eq. (A70)	Eq. (A71)	Eq. (A72)
β	0.49	0.44	0.68
[*t*-stat]	[6.50]	[5.77]	[10.73]
λ_0	0.09	0.07	0.07
[*t*-stat]	[30.24]	[35.41]	[36.18]
λ_1	1.35	1.32	
[*t*-stat]	[8.70]	[9.48]	
R-squared	0.77	0.74	0.75

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Table 1
Selected models of the aggregate stock market

		Consumption-based model	*D*/*P* predicts returns	Accounts for volatility	Accounts for equity premium	Accounts for survey evidence
TRADITIONAL						
Habit	Campbell and Cochrane (1999)	Yes	Yes	Yes	Yes	No
Long-run risk	Bansal and Yaron (2004)	Yes	No	Yes	Yes	No
	Bansal, Kiku, and Yaron (2012)	Yes	Yes	Yes	Yes	No
Rare disasters	Rietz (1988), Barro (2006)	Yes	No	No	Yes	No
	Gabaix (2012), Wachter (2013)	Yes	Yes	Yes	Yes	No
LEARNING	Timmermann (1993)	No	Yes	Yes	No	No
	Wang (1993)	Yes	Yes	Yes	No	No
BEHAVIORAL						
Preference-based						
Prospect theory	Barberis, Huang, and Santos (2001)	Yes	Yes	Yes	Yes	No
Ambiguity aversion	Ju and Miao (2012)	Yes	Yes	Yes	Yes	No
Belief-based						
Noise trader risk	DeLong et al. (1990a)	No	Yes	Yes	No	No
	Campbell and Kyle (1993)	Yes	Yes	Yes	No	No
Extrapolation of fundamentals	Barberis, Shleifer, and Vishny (1998)	No	Yes	Yes	No	No
	Fuster, Hebert, and Laibson (2011)	Yes	Yes	Yes	Yes	No
	Choi and Mertens (2013)	Yes	Yes	Yes	Yes	No
	Hirshleifer and Yu (2013)	Yes	Yes	Yes	Yes	No
	Alti and Tetlock (2014)	No	Yes	Yes	No	No
Extrapolation of returns	Cutler, Poterba, and Summers (1990)	No	Yes	Yes	No	Yes
	DeLong et al. (1990b)	No	Yes	Yes	No	Yes
	Hong and Stein (1999)	No	Yes	Yes	No	Yes
	Barberis and Shleifer (2003)	No	Yes	Yes	No	Yes
	This paper	Yes	Yes	Yes	No	Yes

Table 2
Parameter values

The table lists the values we assign to the risk-free rate r; the initial level of the dividend D_0; the per unit time mean g_D and standard deviation σ_D of dividend changes; the risky asset supply Q; the initial wealth levels W_0^e and W_0^r of extrapolators and rational traders, respectively; absolute risk aversion γ; the discount rate δ; the parameters λ_0, λ_1, and β which govern the beliefs of extrapolators; and the fraction μ of rational traders in the investor population.

Parameter	Value
r	2.50%
D_0	10
g_D	0.05
σ_D	0.25
Q	5
W_0^e	5000
W_0^r	5000
γ	0.1
δ	1.50%
λ_0	0
λ_1	1
β	{0.05, 0.5, 0.75}
μ	{0.25, 0.5, 0.75, 1}

Table 3

Predictive power of $D/r - P$ for future price changes

The table reports the model-implied value of the coefficient b in a regression of the stock price change from time t to time $t + k$ (in quarters) on the time t level of $D/r - P$,

$$P_{t+k} - P_t = a + b(D_t / r - P_t) + \varepsilon_{t+k} ,$$

for k =1, 4, 8, 12, and 16, and for several pairs of values of the parameters μ and β. The calculations make use of Proposition 3 in the main text.

		μ			
β	k	1	0.75	0.5	0.25
	1	-	0.014	0.016	0.022
	4	-	0.055	0.064	0.085
0.05	8	-	0.106	0.124	0.162
	12	-	0.155	0.180	0.233
	16	-	0.201	0.233	0.298
	1	-	0.134	0.161	0.219
	4	-	0.438	0.504	0.628
0.5	8	-	0.684	0.754	0.861
	12	-	0.822	0.878	0.948
	16	-	0.900	0.940	0.981
	1	-	0.194	0.232	0.311
	4	-	0.579	0.652	0.774
0.75	8	-	0.822	0.879	0.949
	12	-	0.925	0.958	0.988
	16	-	0.968	0.985	0.997

Table 4

Autocorrelation of $P - D/r$

The table reports the model-implied value of the autocorrelation of $P - D/r$ at various lags k (in quarters) and for several pairs of values of the parameters μ and β. The calculations make use of Proposition 4 in the main text.

		μ			
β	k	1	0.75	0.5	0.25
0.05	1	-	0.986	0.984	0.978
	4	-	0.945	0.936	0.915
	8	-	0.894	0.876	0.838
	12	-	0.845	0.820	0.767
	16	-	0.799	0.767	0.702
0.5	1	-	0.866	0.839	0.781
	4	-	0.562	0.496	0.372
	8	-	0.316	0.246	0.139
	12	-	0.178	0.122	0.052
	16	-	0.100	0.060	0.019
0.75	1	-	0.806	0.768	0.689
	4	-	0.421	0.348	0.226
	8	-	0.178	0.121	0.051
	12	-	0.075	0.042	0.012
	16	-	0.032	0.015	0.003

Table 5

Volatility of price changes and of $P - D/r$

Panel A reports the model-implied value of the standard deviation of annual stock price changes for several pairs of values of the parameters μ and β; Panel B reports the standard deviation of $P - D/r$ for several pairs of μ and β. The calculations make use of Proposition 5 in the main text.

Panel A: Standard deviation of annual price changes

		μ		
β	1	0.75	0.5	0.25
0.05	10	11.20	13.15	17.43
0.5	10	11.17	13.03	16.86
0.75	10	11.04	12.67	15.90

Panel B: Standard deviation of $P - D/r$

		μ		
β	1	0.75	0.5	0.25
0.05	0	3.66	8.92	18.29
0.5	0	1.41	3.41	6.94
0.75	0	1.16	2.80	5.70

Table 6
Autocorrelation of price changes

The table reports the model-implied value of the autocorrelation of quarterly stock price changes at various lags *k* (in quarters) and for several pairs of values of the parameters μ and β. The calculations make use of Proposition 6 in the main text.

		Autocorrelation at lag *k*			
		μ			
β	*k*	1	0.75	0.5	0.25
0.05	1	0	-0.001	-0.004	-0.007
	2	0	-0.001	-0.003	-0.007
	3	0	-0.001	-0.003	-0.007
	4	0	-0.001	-0.003	-0.007
	8	0	-0.001	-0.003	-0.006
	12	0	-0.001	-0.003	-0.006
0.5	1	0	-0.016	-0.038	-0.079
	2	0	-0.013	-0.032	-0.062
	3	0	-0.012	-0.027	-0.048
	4	0	-0.010	-0.022	-0.038
	8	0	-0.006	-0.011	-0.014
	12	0	-0.003	-0.006	-0.005
0.75	1	0	-0.022	-0.054	-0.110
	2	0	-0.018	-0.041	-0.076
	3	0	-0.014	-0.032	-0.052
	4	0	-0.012	-0.024	-0.036
	8	0	-0.005	-0.008	-0.008
	12	0	-0.002	-0.003	-0.002

Table 7

Correlation of consumption changes, price changes, and $P - D/r$

Panel A reports the model-implied value of the correlation between quarterly changes in aggregate consumption and quarterly changes in the stock price for several pairs of values of the parameters μ and β; Panel B reports the correlation between annual changes in aggregate consumption and annual changes in price; Panel C reports the correlation between quarterly changes in aggregate consumption and quarter-end $P - D/r$. The calculations make use of Propositions 7 and 8 in the main text.

Panel A: Correlation between quarterly consumption changes and quarterly price changes

	μ			
β	1	0.75	0.5	0.25
0.05	1	0.994	0.985	0.984
0.5	1	0.929	0.842	0.840
0.75	1	0.903	0.794	0.793

Panel B: Correlation between annual consumption changes and annual price changes

	μ			
β	1	0.75	0.5	0.25
0.05	1	0.994	0.985	0.984
0.5	1	0.948	0.880	0.880
0.75	1	0.936	0.856	0.856

Panel C: Correlation between quarterly consumption changes and $P - D/r$

	μ			
β	1	0.75	0.5	0.25
0.05	-	0.152	0.148	0.148
0.5	-	0.436	0.398	0.409
0.75	-	0.504	0.446	0.457

Table 8
Predictive power of changes in consumption for future price changes

The table reports the model-implied value of the coefficient b in a regression of the stock price change from time t to time $t+k$ (in quarters) on the change in aggregate consumption over the most recent quarter,

$$P_{t+k} - P_t = a + b(C_t - C_{t-1}) + \varepsilon_{t+k} ,$$

for k = 1, 4, 8, 12, 16, and for several pairs of values of the parameters μ and β. The calculations make use of Proposition 9 in the main text.

		μ			
β	*k*	1	0.75	0.5	0.25
0.05	1	0	-0.011	-0.026	-0.053
	4	0	-0.043	-0.101	-0.205
	8	0	-0.084	-0.195	-0.393
	12	0	-0.123	-0.284	-0.565
	16	0	-0.159	-0.366	-0.722
0.5	1	0	-0.107	-0.214	-0.442
	4	0	-0.350	-0.671	-1.268
	8	0	-0.547	-1.004	-1.740
	12	0	-0.658	-1.169	-1.916
	16	0	-0.720	-1.250	-1.981
0.75	1	0	-0.144	-0.270	-0.553
	4	0	-0.429	-0.759	-1.378
	8	0	-0.610	-1.024	-1.689
	12	0	-0.686	-1.116	-1.760
	16	0	-0.718	-1.148	-1.776

Table 9
The equity premium and Sharpe ratio

Panel A reports the model-implied value of the equity premium for several pairs of values of the parameters μ and β; Panel B reports the Sharpe ratio. The calculations make use of Proposition 10 in the main text.

Panel A: Equity premium

		μ		
β	1	0.75	0.5	0.25
0.05	1.25	1.58	2.19	3.91
0.5	1.25	1.66	2.46	4.88
0.75	1.25	1.66	2.48	4.92

Panel B: Sharpe ratio

		μ		
β	1	0.75	0.5	0.25
0.05	0.125	0.140	0.166	0.221
0.5	0.125	0.144	0.176	0.247
0.75	0.125	0.144	0.176	0.248

Table 10

Model predictions for ratio-based quantities

The table summarizes the model's predictions for the ratio-based quantities in the left column. A full description of these quantities can be found in Section 5.2 of the main text. The values of the basic model parameters are listed in Table 2; μ, the fraction of rational traders, is 0.25. For β = 0.05, 0.5, and 0.75, we report the average value of each quantity across 10,000 simulated paths. In rows (1), (8), and (9), we report both a regression coefficient and, in parentheses, an R-squared. The right-most column presents the empirical estimates for the post-war period from 1947–2011 (1952–2011 for consumption-related quantities because nondurable consumption data are available only from 1952).

Quantity of interest	β			Post-war U.S. stock market data
	0.05	0.5	0.75	
(1) Predictive power of log(D/P)	0.29 (0.20)	0.46 (0.22)	0.45 (0.21)	0.11 (0.08)
(2) Autocorrelation of P/D	0.93	0.84	0.85	0.94
(3) Excess volatility of returns	2.31	2.57	2.46	-
(4) Excess volatility of P/D	7.21	4.85	4.55	-
(5) Autocorrelation of log excess return (k = 1)	-0.01	-0.09	-0.14	0.11
Autocorrelation of log excess return (k = 8)	-0.01	-0.01	-0.00	-0.02
(6) Correlation of log excess returns and log consumption growth	0.72	0.54	0.47	0.32
(7) Correlation of surplus consumption and P/D	0.24	0.32	0.26	0.10
(8) Predictive power of surplus consumption	-0.25 (0.15)	-0.88 (0.17)	-0.76 (0.17)	-0.77 (0.09)
(9) Predictive power of log(C/W)	0.51 (0.15)	0.12 (0.15)	0.01 (0.15)	0.33 (0.05)
(10) Equity premium	1.19%	1.62%	1.62%	7.97%
Sharpe ratio	0.25	0.30	0.32	0.44