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Stocks as Lotteries: The Implications of Probability Weighting for Security Prices

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We study the asset pricing implications of Tversky and Kahneman's (1992) cumulative prospect theory, with a particular focus on its probability weighting component. Our main result, derived from a novel equilibrium with nonunique global optima, is that, in contrast to the prediction of a standard expected utility model, a security's own skewness can be priced: a positively skewed security can be "overpriced" and can earn a negative average excess return. We argue that our analysis offers a unifying way of thinking about a number of seemingly unrelated financial phenomena. (*JEL* D81, G11, G12)

Over the past few decades, economists and psychologists have accumulated a large body of experimental evidence on attitudes to risk. This evidence reveals that, when people make decisions under risk, they often depart from the predictions of expected utility. In an effort to capture the experimental data more accurately, researchers have developed a number of so-called nonexpected utility models. Perhaps the most prominent of these is Amos Tversky and Daniel Kahneman's (1992) "cumulative prospect theory."

In this paper, we study the pricing of financial securities when investors make decisions according to cumulative prospect theory. Our goal is to see if a model like cumulative prospect theory, which captures attitudes to risk in experimental settings very well, can also help us understand investor behavior in financial markets. Of course, there is no guarantee that this will be the case. Nonetheless, given the difficulties the expected utility framework has encountered in addressing a number of financial phenomena, it may be useful to document the pricing predictions of nonexpected utility models and to see if these predictions shed any light on puzzling aspects of the data.

Cumulative prospect theory is a modified version of "prospect theory" (Kahneman and Tversky 1979). Under cumulative prospect theory, people evaluate risk using a value function that is defined over gains and losses, that is concave over gains and convex over losses, and that is kinked at the origin; and using *transformed* rather than objective probabilities, where the transformed probabilities are obtained from objective probabilities by applying a weighting function. The main effect of the weighting function is to overweight the tails of the distribution it is applied to. The overweighting of tails does not represent a bias in beliefs; it is simply a modeling device that captures the common preference for a lottery-like, or positively skewed, wealth distribution.

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Previous research on the pricing implications of prospect theory has focused mainly on the implications of the kink in the value function (Shlomo Benartzi and Richard Thaler 1995; Barberis, Huang, and Tano Santos 2001); or on the implications of the convex portion of the value function (Barberis and Wei Xiong, forthcoming). Here, we turn our attention to a less-studied aspect of cumulative prospect theory, namely, the probability weighting function.

First, we show that, in a one-period equilibrium setting with multivariate Normal security payoffs and homogeneous investors, the capital asset pricing model (CAPM) can hold even when investors evaluate risk according to cumulative prospect theory. Under multivariate Normality, then, the pricing implications of cumulative prospect theory are no different from those of expected utility.

Our second and principal result is that, as soon as we relax the assumption of Normality, cumulative prospect theory can have novel pricing predictions. We demonstrate this in the most parsimonious model possible. Specifically, we introduce a small, independent, positively skewed security into the economy. In a representative agent expected utility model with concave utility, this security would earn an average return slightly greater than zero in excess of the risk-free rate. We show that, in an economy with cumulative prospect theory investors, the skewed security can become overpriced relative to the prediction of the expected utility model, and can earn a *negative* average excess return. To be clear, it would not be surprising to learn that, in equilibrium, investors who overweight the tails of a portfolio return distribution value a positively skewed *portfolio* highly; what is surprising is that, in equilibrium, these investors value a positively skewed *security* highly, even if this security is in small supply.

Our unusual result emerges from an equilibrium structure that, to our knowledge, is new to the finance literature. In an economy with cumulative prospect theory investors and a skewed security, there can be nonunique global optima, so that even though investors have *homogeneous* preferences and beliefs, they can hold *different* portfolios. In particular, some investors take a large, undiversified position in the skewed security, because by doing so, they make the distribution of their overall wealth more lottery-like, which, as people who overweight tails, they find highly desirable. The skewed security is therefore very useful to these investors; as a result, they are willing to pay a high price for it and to accept a negative average excess return on it. We show that this effect persists even if there are a number of skewed securities in the economy. We also argue that it is not easy for expected utility investors to remove the effect: while expected utility investors can try to exploit the overpricing by taking short positions in skewed securities, there are risks and costs to doing so, and this limits the impact of their trading.

Our results suggest a unifying way of thinking about a number of seemingly unrelated facts. Consider, for example, the low long-term average return on IPO stocks (Jay Ritter 1991). IPOs have positively skewed returns, probably because they are issued by young firms, a large fraction of whose value is in the form of growth options. Our analysis implies that, in an economy with cumulative prospect theory investors, IPOs can become overpriced and earn a low average return. Under cumulative prospect theory, then, the poor historical performance of IPOs may not be so puzzling. We discuss several other applications, including the low average return on private equity and on distressed stocks, the diversification discount, the low valuations of certain equity stubs, the pricing of out-of-the-money options, and the underdiversification in many household portfolios.

Through the probability weighting function, cumulative prospect theory investors exhibit a preference for skewness. There are already a number of papers that analyze the implications of skewness-loving preferences. We note, however, that the pricing effects we demonstrate here are new to the skewness literature. Earlier papers have shown that a security's coskewness with the market portfolio can be priced (Alan Kraus and Robert Litzenberger 1976). We show that it is not just coskewness with the market that can be priced, but also a security's own skewness. For example, in our framework, a skewed security can earn a negative average excess return even if it is small and independent of other risks; in other words, even if its coskewness with the market is zero. We discuss the related literature in more detail in Section IIIH.

Our first result—that, under cumulative prospect theory, the CAPM can still hold—was originally proved by Enrico De Giorgi, Thorsten Hens, and Haim Levy (2003). We include this result here for two reasons. First, it provides a very useful springboard for our main contribution, namely the analysis of how skewed securities are priced. Second, we are able to offer a somewhat different proof of the CAPM result. As part of our proof, we show that, within certain classes of distributions, cumulative prospect theory preferences satisfy *second*-order stochastic dominance—a result that is interesting in its own right and that is new to the literature.

In Section I, we discuss cumulative prospect theory and its probability weighting feature in more detail. We then examine how cumulative prospect theory investors price multivariate Normal securities (Section II) and positively skewed securities (Section III). Section IV considers applications of our results and Section V concludes.

I. Cumulative Prospect Theory

Cumulative prospect theory is arguably the most prominent of all nonexpected utility theories. We introduce it by first reviewing the original version of prospect theory, laid out by Kahneman and Tversky (1979), on which it is based.

Consider the gamble

$$(x, p; y, q), \tag{1}$$

to be read as "gain x with probability p and y with probability q, independent of other risks," where $x \leq 0 \leq y$ or $y \leq 0 \leq x$, and where $p + q = 1$. In the expected utility framework, an agent with utility function $u(\cdot)$ evaluates this risk by computing

$$pu(W + x) + qu(W + y), \tag{2}$$

where W is his current wealth. In the original version of prospect theory, the agent assigns the gamble the value

$$\pi(p)v(x) + \pi(q)v(y), \tag{3}$$

where $v(\cdot)$ and $\pi(\cdot)$ are known as the value function and the probability weighting function, respectively. Figure 1 shows the forms of $v(\cdot)$ and $\pi(\cdot)$ suggested by Kahneman and Tversky (1979). These functions satisfy $v(0) = 0$, $\pi(0) = 0$, and $\pi(1) = 1$.

There are four important differences between (2) and (3). First, the carriers of value in prospect theory are gains and losses, not final wealth levels: the argument of $v(\cdot)$ in (3) is x, not $W + x$. Second, while $u(\cdot)$ is typically concave everywhere, $v(\cdot)$ is concave only over gains; over losses, it is convex. This captures the experimental finding that people are risk averse over moderate probability gains—they prefer a certain gain of \$500 to (\$1,000, 1/2)—but risk-*seeking* over moderate probability losses, in that they prefer (−\$1,000, 1/2) to a certain loss of \$500.[1]

Third, while $u(\cdot)$ is typically differentiable everywhere, the value function $v(\cdot)$ is kinked at the origin, so that the agent is more sensitive to losses—even small losses—than to gains of the

[1] We abbreviate $(x, p; 0, q)$ to (x, p).

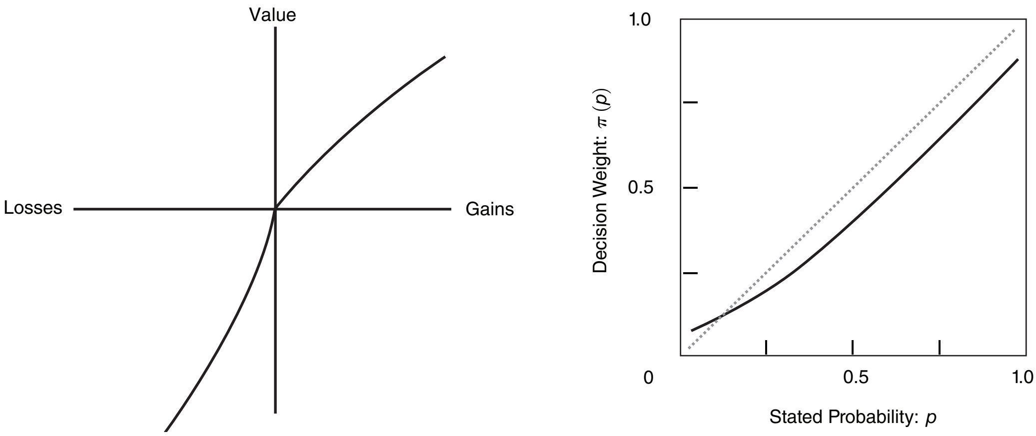

FIGURE 1. THE KAHNEMAN-TVERSKY (1979) VALUE FUNCTION AND PROBABILITY WEIGHTING FUNCTION

same magnitude. This element of prospect theory is known as loss aversion. Kahneman and Tversky (1979) infer it from the widespread aversion to bets such as ($110, 1/2; −$100, 1/2). Such aversion is hard to explain with differentiable utility functions, whether expected utility or nonexpected utility, because the very high local risk aversion required to do so typically predicts implausibly high aversion to large-scale gambles (Larry Epstein and Stanley Zin 1990; Matthew Rabin 2000; Barberis, Huang, and Thaler 2006).

Finally, under prospect theory, the agent does not use objective probabilities when evaluating the gamble but, rather, transformed probabilities obtained from objective probabilities via the probability weighting function $\pi(\cdot)$. The most important feature of this function is that low probabilities are overweighted: in the right panel of Figure 1, the solid line lies above the 45-degree dotted line for low *p*. This is inferred from subjects' preference for ($5,000, 0.001) over a certain $5, and from their preference for a certain loss of $5 over (−$5,000, 0.001); in other words, it is inferred from their simultaneous demand for both lotteries and insurance. Spelling this out in more detail,

$$(4) \qquad (\$5, 1) \prec (\$5000, 0.001)$$
$$\Rightarrow v(5)\pi(1) < v(5000)\pi(0.001) < 1000\, v(5)\pi(0.001)$$
$$\Rightarrow \pi(0.001) > 0.001,$$

so that low probabilities are overweighted. A similar calculation in the case of the (−$5,000, 0.001) gamble, using the fact that $v(\cdot)$ is convex over losses, produces the same result.[2]

The transformed probabilities $\pi(p)$ and $\pi(q)$ in (3) should not be thought of as beliefs, but as decision weights which help us capture the experimental evidence on risk attitudes. In Kahneman

[2] Another way of expressing the overweighting of low probabilities is to say that the agent is acutely sensitive to the difference between an outcome that is unlikely and an outcome that is impossible. A second feature of Kahneman and Tversky's (1979) weighting function, one that is less important for our analysis, is that the agent is also acutely sensitive to the difference between an outcome that is highly probable and one that is certain, and puts much more weight on the latter.

and Tversky's (1979) framework, an agent evaluating the lottery-like (\$5,000, 0.001) gamble understands that he will receive the \$5,000 only with probability 0.001. The overweighting of 0.001 introduced by prospect theory is simply a modeling device which captures the agent's preference for the lottery over a certain \$5.

In this paper, we do not work with the original prospect theory, but with a modified version, cumulative prospect theory, proposed by Tversky and Kahneman (1992). In this modified version, Tversky and Kahneman (1992) suggest explicit functional forms for $v(\cdot)$ and $\pi(\cdot)$. Moreover, they apply the probability weighting function to the *cumulative* probability distribution, not to the probability density function. This ensures that cumulative prospect theory does not violate first-order stochastic dominance—a weakness of the original prospect theory—and also that it can be applied to gambles with any number of outcomes, not just two. Finally, Tversky and Kahneman (1992) allow the probability weighting functions for gains and losses to differ.

Formally, under cumulative prospect theory, the agent evaluates the gamble

$$(x_{-m}, p_{-m}; \ldots; x_{-1}, p_{-1}; x_0, p_0; x_1, p_1; \ldots; x_n, p_n), \tag{5}$$

where $x_i < x_j$ for $i < j$ and $x_0 = 0$, by assigning it the value

$$\sum_{i=-m}^{n} \pi_i v(x_i), \tag{6}$$

where[3]

$$\pi_i = \begin{cases} w^+(p_i + \ldots + p_n) - w^+(p_{i+1} + \ldots + p_n) & \text{for} \quad 0 \le i \le n \\ w^-(p_{-m} + \ldots + p_i) - w^-(p_{-m} + \ldots + p_{i-1}) & \phantom{\text{for}} \quad -m \le i < 0 \end{cases} \tag{7}$$

and where $w^+(\cdot)$ and $w^-(\cdot)$ are the probability weighting functions for gains and losses, respectively. Tversky and Kahneman (1992) propose the functional forms

$$v(x) = \begin{cases} x^{\alpha} & \text{for} \quad x \ge 0 \\ -\lambda(-x)^{\beta} & \phantom{\text{for}} \quad x < 0 \end{cases} \tag{8}$$

and

$$w^+(P) = \frac{P^{\gamma}}{(P^{\gamma} + (1-P)^{\gamma})^{1/\gamma}}, \qquad w^-(P) = \frac{P^{\delta}}{(P^{\delta} + (1-P)^{\delta})^{1/\delta}}. \tag{9}$$

For $\alpha \in (0, 1)$, $\beta \in (0, 1)$, and $\lambda > 1$, the value function $v(\cdot)$ in (8) captures the features highlighted earlier: it is concave over gains, convex over losses, and exhibits a greater sensitivity to losses than to gains. The degree of sensitivity to losses is determined by λ, the coefficient of loss aversion. For $\gamma \in (0, 1)$ and $\delta \in (0, 1)$, the weighting functions $w^+(\cdot)$ and $w^-(\cdot)$ in (9) capture the overweighting of low probabilities described earlier: for low, positive P, $w(P) > P$.

Equation (7) shows that, under cumulative prospect theory, the weighting function is applied to the cumulative probability distribution. If it were instead applied to the probability density function, as in the original prospect theory, the probability weight π_i, for $i < 0$ say, would be $w^-(p_i)$. Instead, equation (7) shows that, under cumulative prospect theory, the probability weight π_i is

[3] When $i = n$ and $i = -m$, equation (7) reduces to $\pi_n = w^+(p_n)$ and $\pi_{-m} = w^-(p_{-m})$, respectively.

obtained by taking the total probability of all outcomes equal to or worse than x_i, namely $p_{-m} + \ldots + p_i$, the total probability of all outcomes strictly worse than x_i, namely $p_{-m} + \ldots + p_{i-1}$, applying the weighting function to each, and computing the difference.

The effect of applying the weighting function to a *cumulative* probability distribution is to make the agent overweight the *tails* of that distribution. In equations (6)–(7), the most extreme outcomes, x_{-m} and x_n, are assigned the probability weights $w^-(p_{-m})$ and $w^+(p_n)$, respectively. If p_{-m} and p_n are small, we then have $w^-(p_{-m}) > p_{-m}$ and $w^+(p_n) > p_n$. The most extreme outcomes—the outcomes in the tails—are therefore overweighted. Just as in the original prospect theory, then, a cumulative prospect theory agent likes positively skewed, or lottery-like, wealth distributions. This will play an important role in our analysis.

Using experimental data, Tversky and Kahneman (1992) estimate $\alpha = \beta = 0.88$, $\lambda = 2.25$, $\gamma = 0.61$, and $\delta = 0.69$. Figure 2 plots the weighting function $w^-(\cdot)$ for $\delta = 0.65$ (the dashed line), for $\delta = 0.4$ (the dash-dot line), and for $\delta = 1$, which corresponds to no probability weighting at all (the solid line). The overweighting of low probabilities is clearly visible for $\delta < 1$.

In our subsequent analysis, we work with the specification of cumulative prospect theory laid out in equations (6)–(9), adjusted only to allow for continuous probability distributions.

II. The Pricing of Multivariate Normal Securities

In Sections II and III, we study security prices in economies where investors evaluate risk using cumulative prospect theory. We pay particular attention to the implications of the probability weighting function. In this section, we start with the case of multivariate Normal security payoffs.

Suppose that an investor uses cumulative prospect theory to evaluate risk and that his beginning-of-period and end-of-period wealth are W_0 and $\tilde{W} = W_0\tilde{R}$, respectively. In prospect theory, utility is defined over gains and losses, which we interpret as final wealth $\tilde{W}$ minus a reference wealth level W_z. In symbols, the gain or loss in wealth, $\hat{W}$, is

$$\hat{W} = \tilde{W} - W_z. \tag{10}$$

One possible reference level is initial wealth W_0. In this paper, we use another reference level, namely W_0R_f, where R_f is the gross risk-free rate for the period, so that

$$\hat{W} = \tilde{W} - W_0R_f. \tag{11}$$

This specification is more tractable and potentially more plausible: the investor thinks of the change in his wealth as a gain only if it exceeds the change he would have experienced by investing at the risk-free rate. We also assume:

ASSUMPTION 1: $|E(\hat{W})| < \infty$ and $\mathrm{Var}(\hat{W}) < \infty$.

In the economies we study in this paper, each investor has cumulative prospect theory preferences. Specifically, each investor has the goal function

$$U(\tilde{W}) \equiv V(\hat{W}) = V(\hat{W}^+) + V(\hat{W}^-), \tag{12}$$

where $\hat{W}^+ = \max(\hat{W}, 0)$, $\hat{W}^- = \min(\hat{W}, 0)$, and

$$V(\hat{W}^+) = -\int_0^\infty v(W)\ dw^+(1 - P(W)), \tag{13}$$

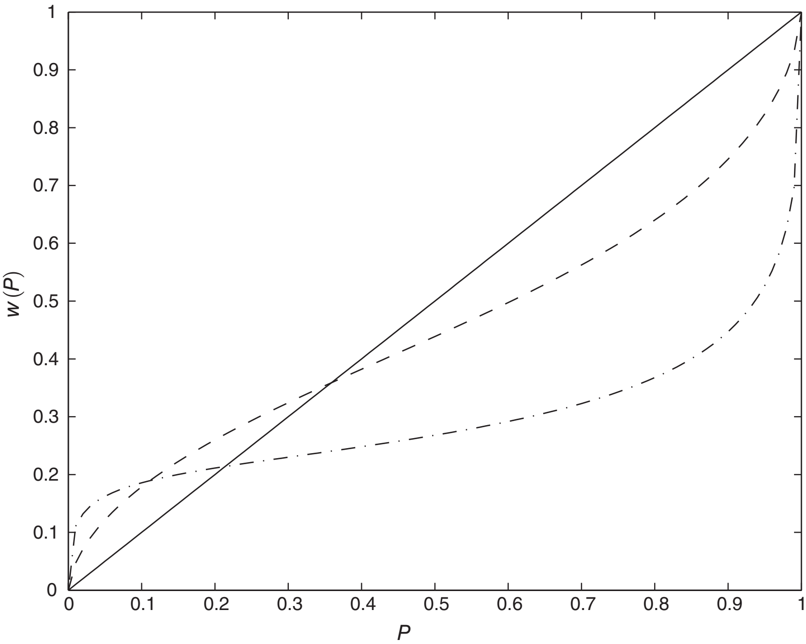

FIGURE 2. THE TVERSKY-KAHNEMAN (1992) PROBABILITY WEIGHTING FUNCTION

Notes: The figure shows the form of the probability weighting function proposed by Tversky and Kahneman (1992), namely, $w(P) = P^\delta/(P^\delta + (1 - P)^\delta)^{1/\delta}$. The dashed line corresponds to $\delta = 0.65$, the dash-dot line to $\delta = 0.4$, and the solid line to $\delta = 1$.

$$V(\hat{W}^-) = \int_{-\infty}^{0} v(W)\ dw^-(P(W)), \tag{14}$$

and where $P(\cdot)$ is the cumulative distribution function of $\hat{W}$. Equations (12)–(14) are equivalent to equations (6)–(7), modified to allow for continuous probability distributions. We assume:

ASSUMPTION 2: The probability weighting functions $w^+(\cdot)$ and $w^-(\cdot)$ take the form in (9).

ASSUMPTION 3: Since the experimentally estimated values of γ and δ in (9) are similar, we set $\gamma = \delta$, so that the probability weighting functions for gains and losses are the same, and define $w^+(\cdot) = w^-(\cdot) \equiv w(\cdot)$. We require $\delta \in (0.28, 1)$. The lower bound of 0.28 ensures that $w(P)$ is strictly increasing for $P \in (0, 1)$. As mentioned above, experimental evidence suggests $\delta \approx 0.65$.

ASSUMPTION 4: $v(\cdot)$ takes the form in (8). Since the experimentally estimated values of α and β are very similar, we set $\alpha = \beta$. We require $\alpha \in (0, 1)$ and $\lambda > 1$. As noted earlier, experimental evidence suggests $\alpha \approx 0.88$ and $\lambda \approx 2.25$.

ASSUMPTION 5: $\alpha < 2\delta$.

Taken together with Assumptions 1–4, Assumption 5 ensures that the goal function $V(\cdot)$ in (12) is well-defined. The values of α and δ estimated by Tversky and Kahneman (1992) satisfy this condition. If $\hat{W}$ has a Normal or Lognormal distribution, Assumption 5 is not needed.

In the Appendix, we prove the following useful lemma. In informal terms, the lemma shows that we can reverse the order of $v(\cdot)$ and $w(\cdot)$ in each of equations (13)–(14).

LEMMA 1: *Under Assumptions 1–5,*

$$V(\widehat{W}^{+}) = \int_{0}^{\infty} w(1 - P(x))\ dv(x), \tag{15}$$

$$V(\widehat{W}^{-}) = -\int_{-\infty}^{0} w(P(x))\ dv(x). \tag{16}$$

We now show that, in a one-period equilibrium model with multivariate Normal security payoffs, the CAPM can hold. Under multivariate Normality, then, the pricing implications of cumulative prospect theory are no different from those of expected utility. To derive the CAPM result, we make the following assumptions:

ASSUMPTION 6: We study a one-period economy with two dates, $t = 0$ and $t = 1$.

ASSUMPTION 7: *Asset supply.* The economy contains a risk-free asset, which is in perfectly elastic supply and has a gross return of R_f. There are also J risky assets. Risky asset j has $n_j > 0$ shares outstanding, a per-share payoff of $\widetilde{X}_j$ at time 1, and a gross return of $\widetilde{R}_j$. The random payoffs $\{\widetilde{X}_1, \ldots, \widetilde{X}_J\}$ have a positive-definite variance-covariance matrix, so that no linear combination of the J payoffs is a constant.

ASSUMPTION 8: *Distribution of payoffs.* The joint distribution of the time 1 payoffs on the J risky assets is multivariate Normal.

ASSUMPTION 9: *Investor preferences.* The economy contains a large number of price-taking investors who derive utility from the time 1 gain or loss in wealth, $\widehat{W}$, defined in (11). All investors have the *same* preferences, namely those described in equations (12)–(14) and Assumptions 2–5. In particular, the parameters α, δ, and λ are the same for all investors.

ASSUMPTION 10: *Investor beliefs.* All investors assign the same probability distribution to future payoffs and security returns.

ASSUMPTION 11: *Investor endowments.* Each investor is endowed with positive net wealth in the form of traded securities.

ASSUMPTION 12: There are no trading frictions or constraints.

In the Appendix, we prove:

PROPOSITION 1: *Under Assumptions 6–12, there exists an equilibrium in which the CAPM holds, so that*

$$E(\widetilde{R}_j) - R_f = \beta_j(E(\widetilde{R}_M) - R_f), \quad j = 1, \ldots, J, \tag{17}$$

where

$$\beta_j \equiv \frac{\mathrm{Cov}(\tilde{R}_j, \tilde{R}_M)}{\mathrm{Var}(\tilde{R}_M)} \tag{18}$$

and where $\tilde{R}_M$ is the market return. The excess market return, $\hat{R}_M \equiv \tilde{R}_M - R_f$, satisfies

$$V(\hat{R}_M) \equiv -\int_{-\infty}^{0} w(P(\hat{R}_M))dv(\hat{R}_M) + \int_{0}^{\infty} w(1 - P(\hat{R}_M))dv(\hat{R}_M) = 0 \tag{19}$$

and the market risk premium is positive:

$$E(\hat{R}_M) > 0. \tag{20}$$

As part of our proof of Proposition 1, we show that cumulative prospect theory preferences satisfy first-order stochastic dominance and, within certain classes of distributions, second-order stochastic dominance as well. That they satisfy first-order stochastic dominance is not surprising: Tversky and Kahneman (1992) themselves point out that their discrete-distribution formulation in (6)–(9) satisfies this property. The result that, under certain conditions, cumulative prospect theory preferences can also satisfy second-order stochastic dominance *is* surprising and is new to the literature. We explain this result in the Appendix.

The intuition behind Proposition 1 is straightforward. When security payoffs are multivariate Normal, the goal function in (12) becomes a function of the mean and standard deviation of the distribution of wealth. Since cumulative prospect theory satisfies first-order stochastic dominance, all investors choose a portfolio on the mean-variance efficient frontier, in other words, a portfolio that combines the risk-free asset and the tangency portfolio. Market clearing means that the tangency portfolio is the market portfolio. The CAPM then follows in the usual way.

The previous paragraph explains why, if there *is* an equilibrium, that equilibrium must be a CAPM equilibrium. In our proof of Proposition 1, we also show that a CAPM equilibrium satisfying conditions (17), (19), and (20) does indeed exist. It is in this part of the argument that we make use of the second-order stochastic dominance result.[4]

III. The Pricing of Skewed Securities

Under multivariate Normality, then, the pricing implications of cumulative prospect theory are similar to those of expected utility. We now show that, as soon as we relax this assumption, cumulative prospect theory can have novel pricing predictions. We demonstrate this in the most parsimonious model possible, one with the minimum amount of additional structure. Specifically, we study an economy in which Assumptions 6–12 still apply, but which, in addition to the risk-free asset and the J Normally distributed risky assets, also contains a positively skewed security. We make the following simplifying assumptions:

ASSUMPTION 13: *Independence.* The return on the skewed security is independent of the returns on the J original risky securities.

[4] An interesting direction for future research is to understand the conditions under which the CAPM holds when investors have cumulative prospect theory preferences with *heterogeneous* preference parameters. De Giorgi, Hens, and Levy (2003) point out a problem that can arise in such an economy, namely that some investors may want to take an infinite position in the market portfolio. One approach to dealing with this difficulty is to add a second term—a concave utility of consumption term—to investor preferences; another, suggested by De Giorgi, Hens, and Levy (2003) themselves, is to slightly modify Tversky and Kahneman's (1992) specification.

ASSUMPTION 14: *Supply*. The payoff of the skewed security is infinitesimal relative to the total payoff of the J original risky securities.[5]

In a representative agent economy with concave, expected utility preferences, a small, independent, positively skewed security earns an average return slightly above the risk-free rate; in other words, an average *excess* return slightly above zero. We now show that, when investors have the cumulative prospect theory preferences in (12)–(14), we obtain a very different prediction: a small, independent, positively skewed security can earn a *negative* average excess return. To simplify the exposition, our initial analysis assumes that the skewed security cannot be sold short. Our results are not driven by this constraint, however: in Section IIIE, we show that our main conclusions are valid even when investors *can* sell the skewed security short.

We first note that, in any equilibrium, all investors must hold portfolios that are some combination of the risk-free asset, the skewed security, and the tangency portfolio T formed in the mean/standard deviation plane from the J original risky assets. To see this, suppose that an investor allocates a fraction $1 - \theta$ of his wealth to a portfolio P that is some combination of the risk-free asset and the J original risky assets, and a fraction θ of his wealth to the skewed security. If the gross returns of portfolio P and of the skewed security are $\tilde{R}_p$ and $\tilde{R}_n$, respectively, the expected return E and variance V of the overall allocation strategy are

$$E = (1 - \theta)E(\tilde{R}_p) + \theta E(\tilde{R}_n), \tag{21}$$

$$V = (1 - \theta)^2 \operatorname{Var}(\tilde{R}_p) + \theta^2 \operatorname{Var}(\tilde{R}_n). \tag{22}$$

Now recall that cumulative prospect theory satisfies first-order stochastic dominance. The investor is therefore interested in portfolios that, for given variance in (22), maximize expected return in (21). For a fixed position in the skewed security, these are portfolios that maximize $E(\tilde{R}_p)$ for given Var $(\tilde{R}_p)$, in other words, as claimed above, portfolios that combine the risk-free asset with the tangency portfolio T formed in the mean/standard deviation plane from the J original risky assets. Market clearing further implies that the tangency portfolio T must be the market portfolio formed from the J original risky assets alone, excluding the skewed asset. If we call the latter portfolio the "J-market portfolio" for short, we conclude that all investors hold portfolios that are some combination of the risk-free asset, the J-market portfolio, and the skewed security.

The simplest candidate equilibrium is a homogeneous holdings equilibrium: an equilibrium in which all investors hold the same portfolio. In Section IIIA, however, we show that, for a wide range of parameter values, no such equilibrium exists. We therefore consider the next simplest candidate equilibrium: a heterogeneous holdings equilibrium with two groups of investors, where all investors in the same group hold the same portfolio. Specifically, we look for an equilibrium with the following structure: all investors in the first group hold a portfolio that combines the risk-free asset and the J-market portfolio but takes no position at all in the skewed security; and all investors in the second group hold a portfolio that combines the risk-free asset, the J-market portfolio, *and* a long position in the skewed security.

[5] We assume an infinitesimal payoff for expositional convenience. In practice, our main result requires only that the payoff of the skewed security be small, relative to the total payoff of the J original risky securities. We discuss this in more detail later in the section.

The heterogeneous holdings in our proposed equilibrium do not stem from heterogeneous preferences: as specified in Assumptions 9 and 10, all investors have identical preferences and beliefs. Rather, they stem from the existence of nonunique optimal portfolios. By assigning each investor to one of the two proposed optimal portfolios, we can clear the market in the small, skewed security.

Let $\hat{R}_M$ and $\hat{R}_n \equiv \tilde{R}_n - R_f$ be the excess returns of the J-market portfolio and of the skewed security, respectively. The conditions for our proposed equilibrium are then

$$V(\hat{R}_M) = V(\hat{R}_M + x^* \hat{R}_n) = 0, \tag{23}$$

$$V(\hat{R}_M + x\hat{R}_n) < 0 \quad \text{for} \quad 0 < x \neq x^*, \tag{24}$$

$$V(\hat{R}_n) < 0, \tag{25}$$

where

$$V(\hat{R}_M + x\hat{R}_n) = -\int_{-\infty}^{0} w(P_x(R))dv(R) + \int_{0}^{\infty} w(1 - P_x(R))dv(R) \tag{26}$$

and

$$P_x(R) = \Pr(\hat{R}_M + x\hat{R}_n \leq R). \tag{27}$$

Here, x^* is the fraction of wealth allocated to the skewed security relative to the fraction allocated to the J-market portfolio for those investors who allocate a positive amount to the skewed security.

Why are these the appropriate equilibrium conditions? First, recall that, in the proposed equilibrium, each investor in the first group holds a portfolio with return $(1 - \theta)R_f + \theta\tilde{R}_M$, with $\theta > 0$. Since, for $\theta \geq 0$,

$$U(W_0((1 - \theta)R_f + \theta\tilde{R}_M)) = V(W_0\theta\hat{R}_M) = W_0^{\alpha}\theta^{\alpha}V(\hat{R}_M), \tag{28}$$

an investor will only choose a finite and positive θ if $V(\hat{R}_M) = 0$. Each investor in the second group holds a portfolio with return $(1 - \phi_1 - \phi_2) R_f + \phi_1\tilde{R}_M + \phi_2\tilde{R}_n$, with $\phi_1 > 0$ and $\phi_2 > 0$. If this portfolio is to be a second global optimum, it must also offer utility of zero, so that, if $x^* = \phi_2/\phi_1$,

$$U(W_0((1 - \phi_1 - \phi_2)R_f + \phi_1\tilde{R}_M + \phi_2\tilde{R}_n)) = W_0^{\alpha}\phi_1^{\alpha}V(\hat{R}_M + x^*\hat{R}_n) = 0. \tag{29}$$

This explains condition (23). Conditions (24) and (25) ensure that these two portfolios are the *only* global optima: condition (24) checks that they offer higher utility than any other feasible combination of the risk-free asset, the J-market portfolio, and the skewed security; and condition (25) checks that they offer higher utility than any feasible combination of just the risk-free asset and the skewed security.

In general, when a new security is introduced into an economy, the prices of existing securities are affected. An interesting feature of our proposed equilibrium, which we derive formally in the Appendix, is that, if the skewed security cannot be sold short, its introduction does *not* affect the prices of the J original risky assets: their prices in the heterogeneous holdings equilibrium are identical to what they were in the economy of Section II, where there was no skewed security.

A. *An Example*

We now show that an equilibrium satisfying conditions (23)–(25) actually exists. To do this, we construct an explicit example.

From Assumption 8, the excess J-market return—the excess return on the market portfolio excluding the skewed security—is Normally distributed:

$$\hat{R}_M \sim N(\mu_M, \sigma_M^2). \tag{30}$$

We model the skewed security in the simplest possible way, using a binomial distribution: with some low probability q, the security pays out a large "jackpot" L, and with probability $1 - q$, it pays out nothing. Using our earlier notation, the payoff distribution is therefore

$$(L, q; 0, 1 - q). \tag{31}$$

For *very* large L and *very* low q, this resembles the payoff distribution of a lottery ticket. If the price of this security is p_n, its gross return $\tilde{R}_n$ and excess return $\hat{R}_n = \tilde{R}_n - R_f$ are distributed as

$$\tilde{R}_n \sim \left(\frac{L}{p_n}, q; 0, 1 - q\right), \tag{32}$$

$$\hat{R}_n \sim \left(\frac{L}{p_n} - R_f, q; -R_f, 1 - q\right). \tag{33}$$

We now specify the preference parameters $(\alpha, \delta, \lambda)$, the skewed security payoff parameters (L, q), the risk-free rate R_f, and the standard deviation of the J-market return σ_M, and search for a mean excess return on the J-market, μ_M, and a price p_n for the skewed security such that conditions (23)–(25) hold. Specifically, we take the unit of time to be a year and set the annual stock market standard deviation to $\sigma_M = 0.15$ and the annual gross risk-free rate to $R_f = 1.02$. We set $L = 10$ and $q = 0.09$, which imply substantial positive skewness in the new security's payoff. Finally, we set $(\alpha, \delta, \lambda) = (0.88, 0.65, 2.25)$, the values estimated by Tversky and Kahneman (1992).

We use numerical integration to compute $V(\hat{R}_M)$ in (23). For the parameter values above, we find that the condition $V(\hat{R}_M) = 0$ implies $\mu_M = 0.075$. This is consistent with Benartzi and Thaler (1995) and Barberis, Huang, and Santos (2001), who show that, in an economy with prospect theory investors who derive utility from annual fluctuations in the value of their stock market holdings, the equity premium can be very substantial. The intuition is that, since investors are much more sensitive to stock market losses than to stock market gains, they perceive the stock market to be very risky and charge a high average return as compensation.

We now search for a price p_n of the skewed security such that conditions (23)–(25) hold. To do this, we need to compute $P_x(R)$, defined in (27). Given our assumptions about the distribution of security returns,

$$\begin{aligned} P_x(R) &= \Pr(\hat{R}_M + x\hat{R}_n \leq R) \\ &= \Pr\left(\hat{R}_n = \frac{L}{p_n} - R_f\right)\Pr\left(\hat{R}_M \leq R - x\left(\frac{L}{p_n} - R_f\right)\right) + \Pr(\hat{R}_n = -R_f)\Pr(\hat{R}_M \leq R + xR_f) \\ &= qN\left(\frac{R - x(\frac{L}{p_n} - R_f) - \mu_M}{\sigma_M}\right) + (1-q)\ N\left(\frac{R + xR_f - \mu_M}{\sigma_M}\right), \end{aligned} \tag{34}$$

where $N(\cdot)$ is the cumulative Normal distribution. With $P_x(R)$ in hand, we use numerical integration to compute the goal function in (26).

We find that the price level $p_n = 0.925$ satisfies conditions (23)–(25). Figure 3 provides a graphical illustration. For this value of p_n, the solid line plots the goal function $V(\hat{R}_M + x\hat{R}_n)$ for a range of values of x, where x is the amount allocated to the skewed security relative to the amount allocated to the J-market portfolio. The two global optima are clearly visible: one at $x = 0$ and the other at $x = 0.085$. Since the skewed security is in infinitesimal supply, we can clear the market for it by assigning each investor to one of the two global optima. Given the return distribution in (33), the equilibrium average excess return on the skewed security is

$$E(\hat{R}_n) = \frac{qL}{p_n} - R_f = \frac{(0.09)(10)}{0.925} - 1.02 = -0.047, \tag{35}$$

so that the average net return is $E(\tilde{R}_n) - 1 = E(\hat{R}_n) + R_f - 1 = -0.027$.

The shape of the solid line can be understood as follows. Adding a small position in the skewed security to an existing position in the J-market portfolio initially lowers utility because of the security's negative average excess return and because of the lack of diversification the strategy entails. As we increase x further, however, the security starts to add significant skewness to the return on the investor's portfolio. Since the investor overweights the tails of his wealth distribution, he values this highly and his utility increases. At a price level of $p_n = 0.925$, the skewness effect offsets the underdiversification and negative excess return effects in a way that produces two global optima at $x = 0$ and $x = 0.085$. As x increases beyond 0.085, the investor's wealth retains a lottery-like structure but the size of the lottery jackpot increases. The benefit of a larger jackpot is too small, however, to compensate for the lack of diversification, and utility falls.

Figure 3 also explains *why* the skewed security earns a negative average excess return. By taking a large position in this security, some investors can add skewness to their portfolio return; they value this highly and are therefore willing to accept a low average return on the security. We emphasize that this result is by no means an obvious one. It would not be surprising to learn that, in equilibrium, investors who overweight the tails of a portfolio return distribution value a positively skewed *portfolio* highly; what is surprising is that, in equilibrium, these investors value a positively skewed *security* highly, even if this security is in infinitesimal supply.

Do our results go through when the supply of the skewed security is merely small, rather than infinitesimal? Note that, whatever the skewed security's supply, optimal portfolios must still be some combination of the risk-free asset, the J-market portfolio, and the skewed security. Also, whatever the skewed security's supply, the conditions for our proposed equilibrium are still those in (23)–(25). And whatever the supply of the skewed security, its introduction does not affect the prices of the J original risky assets. The only restriction we need on the supply of the skewed security comes from the size of the nonzero optimum in the heterogeneous holdings equilibrium. In our example, the two optima are $x = 0$ and $x = 0.085$. To ensure market clearing, the value of the skewed security can therefore be at most 8.5 percent of the value of the J-market portfolio. If this condition holds, our proposed equilibrium exists and the skewed security earns a negative average excess return.

Do the parameter values in our example, namely $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.09)$, admit any equilibria other than the heterogeneous holdings equilibrium described above? While it is difficult to give a definite answer, we can at least show that, for these parameter values, there is no *homogeneous* holdings equilibrium, in other words, no equilibrium in which all investors hold the same portfolio.

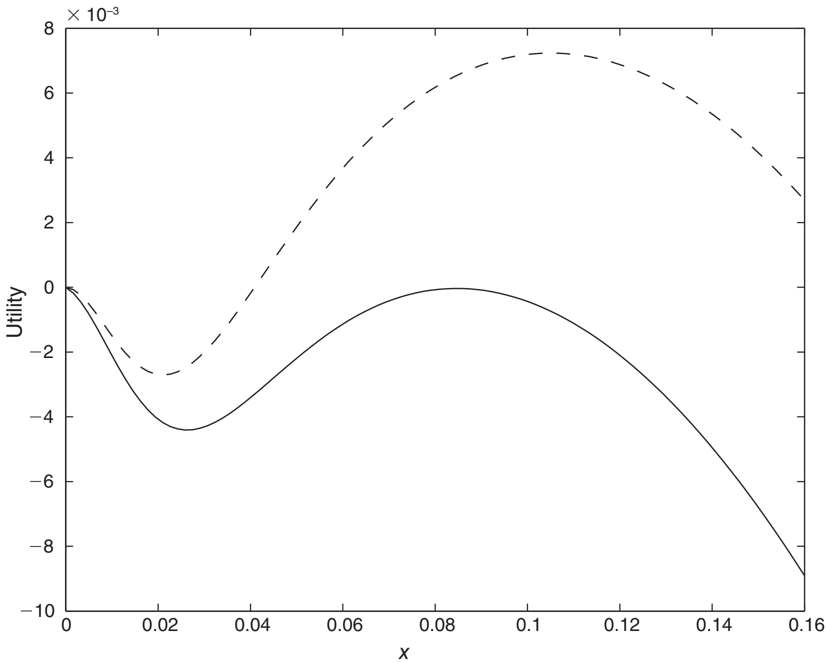

FIGURE 3. A HETEROGENEOUS HOLDINGS EQUILIBRIUM

Notes: The figure shows the utility that an investor with cumulative prospect theory preferences derives from adding a position in a positively skewed security to his current holdings of a Normally distributed market portfolio. The skewed security is highly skewed. The variable x is the fraction of wealth allocated to the skewed security relative to the fraction of wealth allocated to the market portfolio. The two lines correspond to different mean returns on the skewed security.

In any homogeneous holdings equilibrium, each investor would need to be happy to hold an infinitesimal amount ε^* of the skewed security. The equilibrium conditions are therefore

(36) $$V(\hat{R}_M + \varepsilon^* \hat{R}_n) = 0,$$

(37) $$V(\hat{R}_M + \varepsilon \hat{R}_n) < 0, \quad 0 \leq \varepsilon \neq \varepsilon^*,$$

(38) $$V(\hat{R}_n) < 0.$$

Using the same reasoning as for condition (23), we need condition (36) to ensure that investors will optimally choose positive but *finite* allocations to the J-market portfolio and the skewed security. Conditions (37) and (38) ensure that a portfolio that combines the J-market with an allocation ε^* to the skewed security is a global optimum. Since this global optimum is also a local optimum, a necessary condition for equilibrium is

(39) $$\frac{dV(\hat{R}_M + \varepsilon \hat{R}_n)}{d\varepsilon}\Big|_{\varepsilon=\varepsilon^*} = 0.$$

If a homogeneous holdings equilibrium exists, we can approximate it by studying the limiting case as $\varepsilon^* \rightarrow 0$. We therefore search for a mean excess return on the J-market, μ_M, and a price p_n of the skewed security such that $V(\hat{R}_M) = 0$ and[6]

$$\frac{dV(\hat{R}_M + \varepsilon \hat{R}_n)}{d\varepsilon} \Big|_{\varepsilon=0} = 0. \tag{40}$$

As before, $V(\hat{R}_M) = 0$ implies $\mu_M = 0.075$. We find that condition (40) is satisfied for $p_n = 0.882$. The dashed line in Figure 3 plots the goal function $V(\hat{R}_M + x\hat{R}_n)$ for this case. We immediately see that $p_n = 0.882$ does *not* support an equilibrium as it violates condition (37): all investors would prefer a substantial positive position in the skewed security to an infinitesimal one, making it impossible to clear the market. There is therefore no homogeneous holdings equilibrium for these preference and payoff parameters.

B. *How Does Expected Return Vary with Skewness?*

The skewness of the new security's excess return in (33) depends primarily on q, the probability of the large payoff: a low value of q corresponds to a high degree of skewness. In this section, we examine how the equilibrium average excess return on the new security changes as we vary q, and hence the level of skewness.[7]

Our main finding, obtained by searching across many different values of q, is that the results in Section IIIA for the case of $q = 0.09$ are typical of those for all low values of q. Specifically, for $q \leq 0.1035$, a heterogeneous holdings equilibrium can be constructed but a homogeneous holdings equilibrium cannot. For q in this range, the expected excess return on the new security is negative and becomes more negative as q falls. The intuition is that, when q is low, the skewed security is highly skewed and can add a large amount of skewness to investors' portfolios; as a result, it is more valuable to them and they lower the expected return they require on it.

For q above 0.1035, however—in other words, for a skewed security that is only mildly skewed—the opposite is true: a homogeneous holdings equilibrium can be constructed but a heterogeneous holdings equilibrium cannot. The reason the heterogeneous holdings equilibrium breaks down for higher values of q is that, if the new security is not sufficiently skewed, no position in it, however large, adds enough skewness to investors' portfolios to compensate for the lack of diversification the position entails.

To see this last point, suppose that, as before, $(\sigma_M, R_f, L) = (0.15, 1.02, 10)$, so that, once again, $\mu_M = 0.075$, but that q is set to 0.2 rather than to 0.09. Figure 4 plots the goal function $V(\hat{R}_M + x\hat{R}_n)$ for various values of p_n, namely, $p_n = 2.5$ (dashed line), $p_n = 1.96$ (solid line), and $p_n = 1.35$ (dash-dot line). While these lines correspond to only three values of p_n, they hint at the outcome of a more extensive search, namely, *no* value of p_n can deliver two global optima. In other words, no value of p_n can satisfy conditions (23)–(25) for a heterogeneous holdings equilibrium.

For the parameter values $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.2)$, we can only obtain a homogeneous holdings equilibrium, one that satisfies conditions (36)–(38). As before, we study this equilibrium in the limiting case as $\varepsilon^* \rightarrow 0$ by searching for a price p_n of the skewed security that

[6] Here, we make use of the fact that the left-hand side of (39) is continuous in ε^*.

[7] The skewness of the excess return in (33) is $(L/p_n)(1 - 2q)$. We can approximate the price of the new security by $p_n \approx qL/R_f$, its price in a representative agent expected utility model with concave utility, where it earns an average excess return slightly above zero. The skewness of the new security is therefore approximately $R_f(1/q - 2)$.

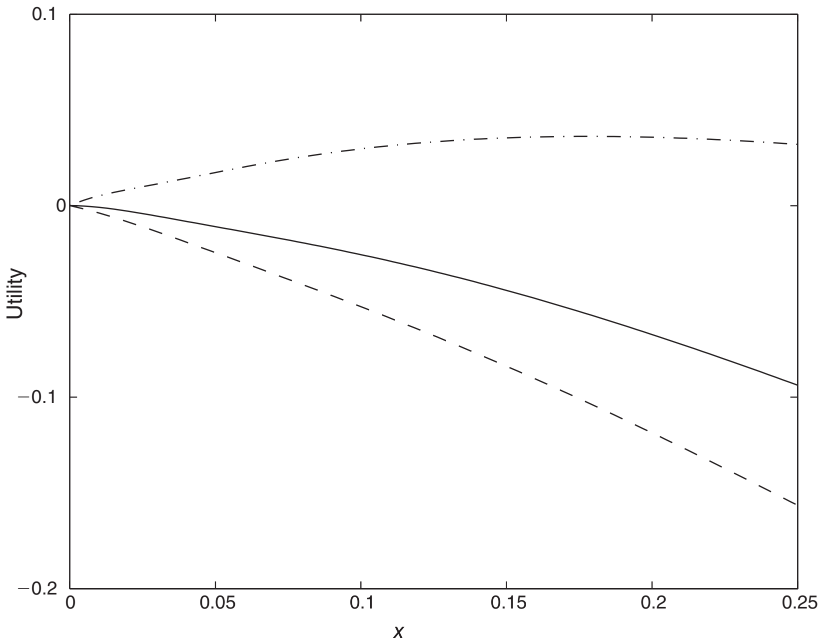

Figure 4. A Homogeneous Holdings Equilibrium

Notes: The figure shows the utility that an investor with cumulative prospect theory preferences derives from adding a position in a positively skewed security to his current holdings of a Normally distributed market portfolio. The skewed security is only moderately skewed. The variable x is the fraction of wealth allocated to the skewed security relative to the fraction of wealth allocated to the market portfolio. The three lines correspond to different mean returns on the skewed security.

satisfies condition (40). We find that $p_n = 1.96$ satisfies this condition. The solid line in Figure 4 plots the goal function for this value of p_n. It shows that $x = \varepsilon^*$ is not only a local optimum but also a global optimum. We have therefore identified a homogeneous holdings equilibrium. In this equilibrium, the expected excess return of the skewed security is

$$E(\hat{R}_n) = \frac{qL}{p_n} - R_f = \frac{(0.2)(10)}{1.96} - 1.02 = 0. \tag{41}$$

It is no coincidence that the skewed security earns an expected excess return of zero. The following proposition, which we prove in the Appendix, shows that, whenever a homogeneous holdings equilibrium exists, the expected excess return on the skewed security is *always* zero, or, more precisely, infinitesimally greater than zero. In other words, in a homogeneous holdings equilibrium, cumulative prospect theory assigns the skewed security the same average return that a concave expected utility specification would.

PROPOSITION 2: *Consider a one-period economy with a risk-free asset and J risky assets that satisfy Assumptions 7–8, a skewed security that satisfies Assumptions 13–14, and investors who satisfy Assumptions 9–12. If this economy supports a homogeneous holdings equilibrium—one in which all investors hold the same portfolio—then, in this equilibrium, the expected excess return of the skewed security is infinitesimally greater than zero.*

Figure 5 summarizes the findings of this section by plotting the expected excess return of the skewed security as a function of q when $(\alpha, \delta, \lambda) = (0.88, 0.65, 2.25)$ and $(\sigma_M, R_f, L) = (0.15, 1.02, 10)$. For $q \leq 0.1035$, we obtain heterogeneous holdings equilibria in which the expected excess return is negative and falls as q falls. For $q > 0.1035$, a heterogenous holdings equilibrium can no longer be constructed, but a homogeneous holdings equilibrium can, and here, the skewed security earns an expected excess return of zero.

C. *How Does Expected Return Vary with the Preference Parameters?*

In this section, we fix the return and payoff parameters at their benchmark values, $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.09)$ and examine the effect of varying the preference parameters α, δ, and λ. The three panels in Figure 6 plot the equilibrium average excess return of the skewed security against each of these three parameters in turn, holding the other two constant.

The top-left panel shows that, as δ increases, the expected return of the skewed security also rises. A low value of δ means that investors weight the tails of a probability distribution particularly heavily and therefore that they are strongly interested in a positively skewed portfolio. Since the skewed security offers a way of constructing such a portfolio, it is very valuable and investors are willing to hold it in exchange for a very low average return. Once δ rises above 0.677, however, a heterogeneous holdings equilibrium is no longer possible: by this point, investors do not care enough about having a positively skewed portfolio for them to want to take on an undiversified position in the skewed security. Only a homogeneous holdings equilibrium is possible, and, in such an equilibrium, the skewed security earns an average excess return of zero.

The top-right panel shows that, as λ increases, the expected return on the skewed security also goes up. The parameter λ governs investors' aversion to losses. In order to add skewness to their portfolios, investors need to hold a large, undiversified position in the skewed security. As λ increases, they find it harder to tolerate the volatility of this undiversified position and are therefore willing to hold the skewed security only if it offers a high expected return. Once λ rises above 2.48, no position in the skewed security, however large, contributes enough skewness to offset the painful lack of diversification in the overall portfolio. In this range, only a homogeneous holdings equilibrium is possible.

Finally, the lower panel shows that, as α falls, the expected return on the skewed security goes up. A lower α means that, in the region of gains, the value function is more concave. This, in turn, means that investors derive less utility from a positively skewed portfolio. The skewed security is therefore less useful to them and they are willing to hold it only in exchange for a higher average return. For $\alpha \geq 0.845$, the equilibrium involves heterogeneous holdings; and for $\alpha < 0.845$, homogeneous holdings.

Do we need *all* the elements of cumulative prospect theory in order to predict the overpricing of positive skewness? Probability weighting clearly plays a crucial role—we cannot do without it. We also cannot do without some degree of loss aversion: if λ falls to a level that is too close to 1, no equilibrium exists. For example, if we maintain $(\alpha, \delta) = (0.88, 0.65)$ but set $\lambda = 1.01$, then, as we vary the price of the skewed security, investors take either a zero or an infinite position in it, making it impossible to clear the market. This issue arises only for very low values of λ. For $\lambda \in (1.15, 2.48)$, a heterogeneous holdings equilibrium does exist and the skewed security earns a negative average excess return.

The concave-convex shape of the value function is *not* needed for our results. For example, if we maintain $(\delta, \lambda) = (0.65, 2.25)$ but set $\alpha = 1$, so that the value function is piecewise linear, a heterogeneous holdings equilibrium continues to exist and the positively skewed security continues to earn a negative average excess return.

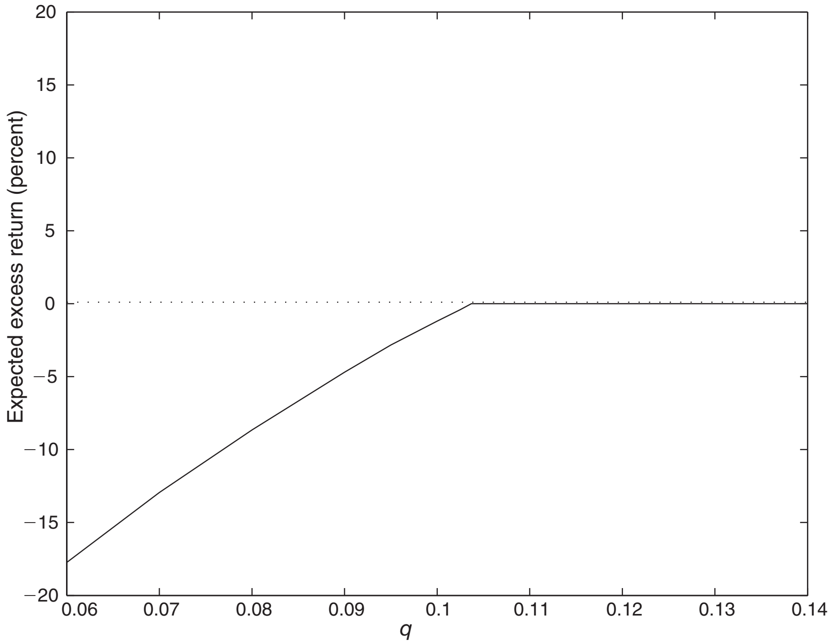

FIGURE 5. SKEWNESS AND EXPECTED RETURN

Notes: The figure shows the expected return in excess of the risk-free rate earned by a small, independent, positively skewed security in an economy populated by cumulative prospect theory investors, plotted against a parameter of the security's return distribution, q, which determines the security's skewness. A low value of q corresponds to a high degree of skewness.

D. *Additional Skewed Securities*

Our main result—that, under cumulative prospect theory, a small, independent, positively skewed security can earn a negative average excess return—continues to hold even when we introduce additional skewed securities into the economy.

To see this, suppose that Assumptions 6–12 apply as before, but that, in addition to the risk-free asset and the J multivariate Normal securities described there, the economy now contains *two* skewed securities, each in infinitesimal supply, independent of other securities, and with the payoff distribution in (31). We assume, for now, that the payoff parameters L and q are the same for both skewed securities. The excess return and price of the j'th skewed security are $\hat{R}_{n,j}$ and $p_{n,j}$, respectively.

As before, optimizing investors hold portfolios that combine the risk-free asset, the J-market portfolio, and the skewed securities. Each investor's goal function is therefore

$$(42) \qquad V(\hat{R}_M + x_1\hat{R}_{n,1} + x_2\hat{R}_{n,2}) = -\int_{-\infty}^{0} w(P_{x_1,x_2}(R))dv(R) + \int_{0}^{\infty} w(1 - P_{x_1,x_2}(R))dv(R),$$

where

$$(43) \qquad P_{x_1,x_2}(R) = \Pr(\hat{R}_M + x_1\hat{R}_{n,1} + x_2\hat{R}_{n,2} \leq R)$$

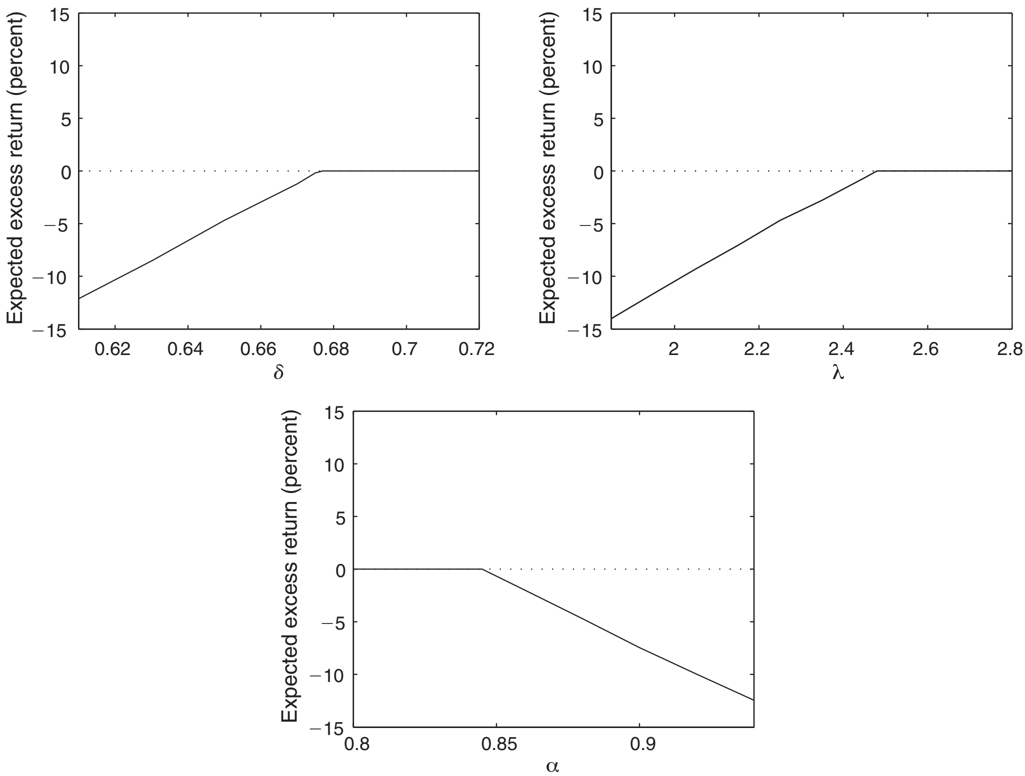

Figure 6. Expected Return as a Function of the Preference Parameters

Notes: The figure shows the expected return in excess of the risk-free rate earned by a small, independent, positively skewed security in an economy populated by cumulative prospect theory investors, plotted against parameters of investors' utility functions. As δ falls, investors overweight tails of distributions more heavily; as λ increases, they become more sensitive to losses; and as α falls, their marginal utility from gains diminishes more rapidly.

and where x_j is the amount allocated to skewed security j relative to the amount allocated to the J-market portfolio.

Now that there are two skewed securities, we propose an equilibrium with *three* global optima: a portfolio that combines the risk-free asset and the J-market portfolio with a position $x^* > 0$ in just the first skewed security; a portfolio that combines the risk-free asset and the J-market portfolio with a position x^* in just the second skewed security; and a portfolio that holds only the risk-free asset and the J-market portfolio and takes no position at all in either of the skewed securities. In short, the three proposed optima are

$$(x_1, x_2) = (x^*, 0), (0, x^*), \text{ and } (0, 0). \tag{44}$$

By assigning each investor to one of these optima, we can clear markets in all securities.

Earlier, we saw that if $(x_1, x_2) = (0, 0)$ is an optimum, we must have $V(\hat{R}_M) = 0$. The goal function must therefore take the value 0 at all three optima. This leads to the equilibrium conditions:

$$V(\hat{R}_M) = V(\hat{R}_M + x^*\hat{R}_{n,j}) = 0, \quad j = 1, 2, \tag{45}$$

$$V(\hat{R}_M + x_1\hat{R}_{n,1} + x_2\hat{R}_{n,2}) < 0, \quad \forall\,(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \setminus \{(x^*, 0), (0, x^*), (0, 0)\}, \tag{46}$$

$$V(y_1\hat{R}_{n,1} + y_2\hat{R}_{n,2}) < 0, \quad \forall\,(y_1, y_2) \in \mathbb{R}^2_{\geq 0} \setminus \{(0, 0)\}. \tag{47}$$

We now check that our proposed equilibrium exists. Suppose that, as in the example of Section IIIA, $(\alpha, \delta, \lambda) = (0.88, 0.65, 2.25)$ and $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.09)$. The condition $V(\hat{R}_M) = 0$ again implies $\mu_M = 0.075$. All that remains is to find $p_{n,1}$ and $p_{n,2}$ that satisfy conditions (45)–(47).

We find that the values $p_{n,1} = p_{n,2} = 0.925$ satisfy these conditions with $x^* = 0.085$. In particular, the two skewed securities have the same price as the skewed security in the original one-skewed-security economy of Section IIIA. Their average excess return is therefore also the same, namely −4.7 percent.

An important feature of this heterogeneous holdings equilibrium is that investors prefer an undiversified position in just one skewed security to a diversified position in two of them. The intuition is that, by diversifying, an investor lowers the volatility of his overall portfolio—which is good—but also lowers its skewness, which is bad. Which of the two forces dominates depends on the distribution of security returns. For the binomial distribution in (32), skewness falls faster than volatility as the investor diversifies; diversification is therefore unattractive.

The prices $p_{n,1} = p_{n,2} = 0.925$ continue to support a heterogeneous holdings equilibrium even if the two skewed securities are in small rather than infinitesimal supply; and even if they *differ* in their supply. Specifically, so long as the supply of each skewed security is less than 8.5 percent of the value of the J-market portfolio, the same equilibrium holds. By assigning the appropriate number of investors to each of the three optima, we can still clear markets.

A heterogeneous holdings equilibrium can exist even if the two skewed securities differ in their *skewness*. For example, if they both offer a payoff of either 0 or 10, but one of them pays the jackpot with probability $q_1 = 0.09$ and the other pays it with probability $q_2 = 0.06$, then there is again a heterogeneous holdings equilibrium with three global optima: a portfolio that combines the risk-free asset and the J-market portfolio with a position in just the first skewed security; a portfolio that combines the risk-free asset and the J-market portfolio with a position in just the second skewed security; and a portfolio that holds only the risk-free asset and the J-market portfolio and takes no position at all in either of the skewed securities. The prices of the two skewed securities are $p_{n,1} = 0.925$ and $p_{n,2} = 0.712$, which are also the prices that each skewed security would have in a one-skewed-security economy. At these prices, no investor wants to diversify across both skewed securities: doing so would remove valuable positive skewness.

What happens when there are more than two skewed securities? Suppose that there are N skewed securities in the economy, each in infinitesimal supply, independent of other securities, and with the payoff distribution in (31). Suppose also that the parameters L and q are the same for all the skewed securities. In this case, we propose an equilibrium with $N + 1$ optima: a portfolio that combines the risk-free asset and the J-market portfolio with a position $x^* > 0$ in just the first skewed security; a portfolio that combines the risk-free asset and the J-market portfolio with a position x^* in just the second skewed security, and so on for each of the skewed securities; and, finally, a portfolio that holds only the risk-free asset and the J-market portfolio and takes no position at all in any of the skewed securities. By assigning each investor to one of these optima, we can clear markets for all securities.

Following the usual reasoning, the equilibrium conditions are that the goal function takes the value zero at each of the conjectured optima and that it takes a value less than zero elsewhere. For the benchmark parameter values $(\alpha, \delta, \lambda) = (0.88, 0.65, 2.25)$ and $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.09)$, we search for a mean excess return μ_M on the J-market and prices for the N skewed securities that satisfy these conditions.

We have analyzed economies with as many skewed securities as computational limits will allow: specifically, economies with $N \leq 10$. For N in this range, we find that we can construct a heterogeneous holdings equilibrium of the form conjectured. In this equilibrium, the average excess return on each of the skewed securities is the same as in the original one-skewed-security economy of Section IIIA. Even as we add more skewed securities, then, investors show no interest in diversifying across them. Diversification lowers volatility, but also lowers skewness, and the latter effect dominates.

E. *Relaxing the Short-Sales Constraint*

We now show that the novel pricing implications of cumulative prospect theory can continue to hold even when all securities can be sold short.

We start with the simplest case: that of one skewed security. Suppose that Assumptions 6–12 apply as before, but that, in addition to the risk-free asset and the J multivariate Normal securities described there, the economy also contains an independent, skewed security in infinitesimal supply and with the payoff distribution in (31). The excess return and price of the skewed security are $\hat{R}_n$ and p_n, respectively.

Now that the skewed security can be sold short, we propose an equilibrium with two optima: a portfolio that combines the risk-free asset and the J-market portfolio with a positive position x^* in the skewed security; and a portfolio that combines the risk-free asset and the J-market portfolio with a short position x^{**} in the skewed security. By assigning the appropriate number of investors to each optimum, we can clear markets.

The equilibrium we have just described requires that the goal function $V(\hat{R}_M + x\hat{R}_n)$ is maximized at $x = x^*$ and $x = -x^{**}$. As before, $V(\cdot)$ must take the value zero at both optima; otherwise, investors would prefer to hold only the risk-free asset or to take an infinite position in the risky assets. The equilibrium conditions are therefore

$$\text{(48)} \qquad \mathrm{V}(\hat{R}_M + x^*\hat{R}_n) = V(\hat{R}_M - x^{**}\hat{R}_n) = 0,$$

$$\text{(49)} \qquad V(\hat{R}_M + x\hat{R}_n) < 0, \qquad \forall x \notin \{x^*, -x^{**}\},$$

$$\text{(50)} \qquad V(\hat{R}_n) < 0, \quad V(-\hat{R}_n) < 0.$$

We now check that this equilibrium exists. For our benchmark parameter values, namely $(\alpha, \delta, \lambda) = (0.88, 0.65, 2.25)$ and $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.09)$, we search for μ_M and p_n such that conditions (48)–(50) hold.

We find that $\mu_M = 0.075$ and $p_n = 0.9245$ satisfy these conditions with $x^* = 0.085$ and $x^{**} = 0.0013$. In this equilibrium, the skewed security earns an average excess return of −4.65 percent. Our main result—that, under cumulative prospect theory, a small, independent, positively skewed security can earn a negative average excess return—therefore continues to hold even when the security can be sold short.[8]

The intuition behind the two optima is straightforward. Some investors take a long position in the skewed security. This gives them a small chance of a large wealth payoff. Since they overweight the tails of the wealth distribution, they find this very attractive and are willing to hold the skewed security even if it offers a low average return. Other investors take a short position

[8] The value of μ_M implied by conditions (48)–(50) differs, very slightly, from its value in previous sections, where it was determined by $V(\hat{R}_M) = 0$. This, in turn, means that, when the skewed security can be sold short, its introduction *does* affect the prices of the J original risky securities.

in the skewed security. This exposes them to the possibility of a large drop in wealth. Since they overweight the tails of the wealth distribution, they find this unattractive, but, because they are compensated by a high average return, they are nonetheless willing to take the position. For $p_n = 0.9245$, the two strategies are equally attractive. The expected excess return of the skewed security, -4.65 percent, is slightly higher than in the economy of Section IIIA in which the security could not be sold short. Shorting allows some investors to exploit the "overpricing" of the skewed security. This exerts some downward pressure on the price and some upward pressure on the expected return.

Now suppose that the economy contains $N > 1$ skewed securities, each in infinitesimal supply, independent of other securities, and with the payoff distribution in (31), where L and q are the same for all N securities. The excess return and price of the j'th skewed security are $\hat{R}_{n,j}$ and $p_{n,j}$, respectively. What form does the equilibrium take now?

We propose a heterogeneous holdings equilibrium with $N + 1$ optima: the N portfolios that combine the risk-free asset and the J-market portfolio with a long position x^* in any one of the N skewed securities and a short position $x^{**}/(N - 1)$ in the remaining $N - 1$ skewed securities, where $0 < x^{**} < x^*$; and a portfolio that combines the risk-free asset and the J-market portfolio with a short position x^{***} in each of the N skewed securities. Mathematically, our conjecture is that the goal function

$$V(\hat{R}_M + x_1\hat{R}_{n,1} + \ldots + x_N\hat{R}_{n,N}), \tag{51}$$

where x_j is the allocation to the j'th skewed security relative to the allocation to the J-market portfolio, is maximized at the $N + 1$ points

$$(x_1, x_2, \ldots, x_N) = \left(x^*, \frac{-x^{**}}{N-1}, \ldots, \frac{-x^{**}}{N-1}\right), \ldots, \left(\frac{-x^{**}}{N-1}, \frac{-x^{**}}{N-1}, \ldots, x^*\right) \tag{52}$$

and

$$(x_1, x_2, \ldots, x_N) = (-x^{***}, -x^{***}, \ldots, -x^{***}), \tag{53}$$

where

$$x^*, x^{**}, x^{***} > 0 \text{ and } x^{**} < x^*. \tag{54}$$

Since $x^{**} < x^*$, we can clear markets for all securities by assigning each investor to one of these optima.

Why do we propose that the optimal portfolios involve a long position in one skewed security but a short position in many of them? As noted in Section IIID, investors prefer to go long in just one skewed security because, while diversifying across several long positions reduces volatility, it also reduces desirable positive skewness; since the latter effect dominates, investors prefer that their long positions remain undiversified. On the short side, however, diversification across skewed securities is very valuable: it reduces both volatility and undesirable negative skewness.

Following the usual reasoning, the equilibrium conditions are that the goal function takes the value zero at each of the optima in (52) and (53) and that it takes a value less than zero for all other portfolios. For the parameter values of Section IIIA, namely $(\alpha, \delta, \lambda) = (0.88, 0.65,$

2.25) and $(\sigma_M, R_f, L, q) = (0.15, 1.02, 10, 0.09)$, we search for μ_M and $\{p_{n,j}\}_{j=1}^{N}$ that satisfy these conditions.

We have again analyzed economies with as many skewed securities as computational limits will allow: specifically, economies with $N \leq 10$. For N in this range, we find that the conjectured heterogeneous holdings equilibrium does indeed exist; and that, for all N in this range, the average excess return of each skewed security remains below -4 percent. In other words, even though, when $N = 10$, say, all investors take a short position in many skewed securities, the shorting activity does not remove the overpricing. A strategy that shorts ten positively skewed securities has significant negative skewness—ten securities are not enough to diversify the negative skewness away. The strategy is therefore risky. In light of this risk, investors limit the size of their short positions. This, in turn, means that significant overpricing remains.

Of course, if there were *many* skewed securities in the economy—if N were as high as 200, say—the overpricing of the skewed securities would likely be significantly reduced. A strategy that shorts 200 positively skewed securities exhibits a more tolerable level of negative skewness. As a result, investors would be willing to take larger short positions, and this would attenuate the overpricing.[9]

The analysis in this section also allows us to make predictions as to how cumulative prospect theory investors would price *negatively* skewed securities. Since a long (short) position in a positively skewed security is equivalent to a short (long) position in a negatively skewed security, our results imply that, in an economy with negatively skewed securities, these securities will earn a significantly positive average excess return.

F. *Can Arbitrageurs Correct the Mispricing?*

The economies we study in this paper do not contain riskless arbitrage opportunities. Since cumulative prospect theory satisfies first-order stochastic dominance, investors with these preferences would immediately exploit a riskless arbitrage opportunity: in equilibrium, then, there are no such opportunities.

At the same time, it is reasonable to ask how overpriced skewed securities would be in an economy with both cumulative prospect theory investors and more standard, risk-averse expected utility investors. It is hard to give a definite answer because constructing such a model poses significant technical challenges. However, there is good reason to think that expected utility investors would not fully reverse the overpricing. While they could try to exploit the overpricing by taking a short position in a large number of skewed securities, such a strategy entails significant risks and costs and these may blunt its impact.

One way to see this is to think about the model of Section IIIE. The cumulative prospect theory investors in that economy are already trying to exploit the overpricing: they are all shorting many of the skewed securities. Interestingly, however, this does not remove the overpricing: unless there are many skewed securities in the economy, the strategy of shorting skewed securities retains significant negative skewness; investors do not, therefore, short very aggressively, and the overpricing remains. For many utility specifications, expected utility investors are also averse to negative skewness. They will therefore also refuse to short aggressively, making it likely that they, too, will fail to remove the overpricing.

[9] We conjecture that, for very high values of N, the equilibrium takes a different form, namely one in which *every* investor holds a portfolio that combines the risk-free asset and the J-market portfolio with a long position in one skewed security and a short position in the remaining $N - 1$ skewed securities. Since we lack the computational power to do accurate numerical analysis for values of N much higher than 10, we must leave this as a conjecture.

Investors who short overpriced securities also face other risks and costs. They may have to pay significant short-selling fees. They run the risk that some of the borrowed securities are recalled before the strategy pays off, as well as the risk that the strategy performs poorly in the short run, triggering an early liquidation. Taken together, these factors suggest that investors may be unwilling to trade very aggressively against the overpricing of skewed securities, thereby allowing it to persist.

An interesting prediction of our model is that, since cumulative prospect theory investors value positively skewed securities so highly, we should see the creation of *new* skewed securities that cater to this demand. For example, there is an incentive to raise some capital and to issue N lottery tickets, each offering a $1/N$ chance of winning the capital. A firm with a subsidiary whose business model has positively skewed fundamentals has an incentive to spin that subsidiary off. And there is an incentive to issue out-of-the-money options; in the context of our model, for example, there may be an incentive to issue options on the J original risky securities.

In practice, we *do* see the creation of new, positively skewed securities. New riskless lotteries do get initiated; firms do spin off subsidiaries with positively skewed fundamentals—subsidiaries working with cutting-edge technologies, say; and out-of-the-money options are actively traded. One interpretation of this activity is that it represents a response to a preference for skewness like that captured by cumulative prospect theory.

One concern is that the supply of new skewed securities may be so large as to dampen the premium that investors pay for them. We note, however, that there are important limits to the creation of skewed securities. The creation of riskless lotteries, for example, is limited by state regulation. And the issuance of securities backed by positively skewed fundamentals is limited by the supply of businesses that have such fundamentals.

Option issuance may, to some extent, attenuate the effects we describe in this paper. At the same time, there are risks to writing options. A market maker who issues options will typically try to hedge his exposure by taking a position in the underlying assets. In practice, however, it is very difficult to maintain a perfect hedge at all times. The market maker will therefore often be exposed to potentially significant losses.

G. *Alternative Framing Assumptions*

In our analysis, we assume that investors apply cumulative prospect theory to gains and losses in overall wealth. A simpler way to derive the pricing of idiosyncratic skewness is to assume that investors apply cumulative prospect theory at the level of individual stocks: if investors overweight the tails of an individual stock's return distribution, it is intuitive that a positively skewed security will be overpriced and will earn a negative average excess return. When an agent gets utility directly from the outcome of a specific risk he is facing, even if it is just one of many that determine his overall wealth risk, he is said to exhibit "narrow framing" (Kahneman 2003; Barberis, Huang, and Thaler 2006).

In this paper, we retain the traditional assumption of portfolio-level framing in order to show that we do not need to appeal to narrow framing to draw interesting implications out of cumulative prospect theory. At the same time, a framework that allows for both portfolio-level and stock-level framing might fit the data better. For example, our current model predicts that some investors hold a nontrivial fraction of their wealth in one positively skewed security. A model that allows for narrow framing would likely preserve the pricing of idiosyncratic skewness while also predicting a lower allocation to a skewed security. Such a model poses significant technical hurdles, however, and is beyond the scope of our current analysis.

H. *Relation to Other Research on Skewness*

Our analysis of economies with cumulative prospect theory investors has led us to the novel prediction that idiosyncratic skewness can be priced. Our motivation for working with cumulative prospect theory is that, given its status as a leading model of how people evaluate risk, it is interesting to document its implications for security prices. At the same time, it is reasonable to ask whether the pricing of idiosyncratic skewness can also be derived in other frameworks.

Expected utility models based on concave, skewness-loving utility functions do *not* predict the pricing of idiosyncratic skewness. In such models, only a security's coskewness with the market portfolio is priced; the security's own skewness is not (Kraus and Litzenberger 1976). An infinitesimal, independent, positively skewed security therefore earns a zero risk premium: its coskewness with the market is zero. It does not earn the negative risk premium we observe under cumulative prospect theory.

One way to think about this point is to note that, in our model, the pricing of idiosyncratic skewness traces back to the undiversified positions some investors hold in the skewed security. By contrast, investors with concave, skewness-loving, expected utility preferences always hold diversified portfolios. As a result, only coskewness with the market is priced; idiosyncratic skewness is not.

Can idiosyncratic skewness be priced when investors have expected utility preferences based on skewness-loving utility functions that are in part *convex*, such as cubic utility functions? It is hard to give a definite answer, because the pricing implications of these preferences have not been fully analyzed yet. One well-known difficulty with such preferences, however, is that, given a skewed security as an investment option, the optimal portfolio may involve an infinite position in that security, a phenomenon known as "plunging" (Alex Kane 1982; Valery Polkovnichenko 2005).

One framework that does predict the pricing of idiosyncratic skewness is the optimal expectations framework analyzed by Markus Brunnermeier and Jonathan Parker (2005) and Brunnermeier, Christian Gollier, and Parker (2007). In this framework, the agent *chooses* his beliefs to maximize his well-being. These beliefs can differ from objective probabilities because, at each date, the agent derives utility not only from consumption at that time but also from *anticipating* future consumption. When choosing beliefs, the agent trades off the benefit of holding pleasurable beliefs against the cost that these beliefs may lead him to take unwise actions.

It is straightforward to see how the optimal expectations framework can predict the pricing of idiosyncratic skewness. An agent holding a positively skewed security can improve his well-being by choosing to believe that a big payoff is more likely than it really is. He is therefore willing to pay a high price for the security. Since this price is too high, the security earns a low average return. The overpricing of positive skewness can even manifest itself in a heterogeneous holdings equilibrium in which some agents overestimate the chance of a big payoff and take a long position, while others underestimate the chance of a big payoff and take a short position. On this last point, we emphasize that, while both the optimal expectations and cumulative prospect theory frameworks can generate heterogeneous holdings equilibria, the economic interpretation is different. Under optimal expectations, the heterogeneous holdings are driven by heterogeneous beliefs. Under cumulative prospect theory, we obtain heterogeneous holdings even when agents have *identical* preferences and beliefs.

While the optimal expectations and cumulative prospect theory frameworks make similar predictions in some circumstances, they can also differ in their predictions. Consider, for example, the case of a *negatively* skewed asset. Cumulative prospect theory predicts that such an asset will earn a *high* average return: since cumulative prospect theory agents overweight tails, they find the asset unattractive and require a high average return on it. The optimal expectations framework, on the other hand, can predict that a negatively skewed asset will earn a *low* average return. An agent holding a negatively skewed asset can improve his well-being by choosing to believe that a large negative payoff is less likely than it really is. He is therefore willing to pay a high price for the asset. Since this price is too high, the asset earns a low average return.

IV. Applications

The cumulative prospect theory models of Section III predict that a positively skewed security in small supply will earn a low average return. This result offers a unifying way of thinking about a number of pricing phenomena which, at first sight, may appear unrelated.

Our first application is to the low average return on IPO securities (Ritter 1991). The distribution of IPO returns in the three years after issue is highly positively skewed, probably because IPOs are conducted by young firms, a large fraction of whose value is in the form of growth options. Our analysis therefore implies that, in an economy with cumulative prospect theory investors, IPOs can be overpriced and earn a low average return. By taking a substantial position in an IPO, an investor gives himself a chance, albeit a small chance, of a very large return on wealth. He values this highly and is willing to hold the IPO even if it offers a low average return. Under cumulative prospect theory, then, the historical performance of IPOs may not be so puzzling.

Our model may also be relevant to the "private equity premium puzzle" documented by Tobias Moskowitz and Annette Vissing-Jorgensen (2002): the fact that the average return on private business holdings is low despite the high idiosyncratic risk these holdings entail. In their analysis, the authors find that the returns on private equity are highly positively skewed. Under cumulative prospect theory, then, a low average return is exactly what we would expect to see.

John Y. Campbell, Jens Hilscher, and Jan Szilagyi (forthcoming) suggest that our framework may shed light on the average return of "distressed" stocks: stocks of firms with a high predicted probability of bankruptcy. Some theories of distress risk predict that such stocks will earn a high return, on average; but Campbell, Hilscher, and Szilagyi (forthcoming) show that their average return is, in fact, very low. While investigating this puzzle, they also find that distressed stocks have high idiosyncratic skewness. In an economy with cumulative prospect theory investors, then, such stocks should indeed earn a low average return.

Todd Mitton and Keith Vorkink (2006) point out that the pricing of idiosyncratic skewness predicted by our model may be relevant to the diversification discount: the fact that conglomerate firms trade at a discount to a matched portfolio of single segment firms (Larry Lang and René Stulz 1994; Philip Berger and Eli Ofek 1995). The idea is that investors may pay a premium for single segment firms if the returns of these firms are more positively skewed than those of conglomerates. Mitton and Vorkink (2006) confirm that the returns of single segment firms *are* more positively skewed and that the diversification discount is most pronounced when the difference between the return skewness of a conglomerate and its matched single segment firms is particularly large.

A related line of reasoning suggests a link between our results and the recently documented examples of "equity stubs" with remarkably low valuations (Mark Mitchell, Todd Pulvino, and Erik Stafford 2002; Owen Lamont and Thaler 2003). These are cases of firms with publicly traded subsidiaries in which the subsidiaries make up a surprisingly large fraction of the value of their parent company, in extreme cases, more than 100 percent of the value of their parent company, so that the equity stub—the claim to the parent company's businesses outside of the subsidiary—has negative value.

Our model cannot explain negative stub values, but it may explain stub values that are surprisingly low, albeit positive. If a subsidiary is valued mainly for its growth options, its returns may be positively skewed, leading investors to overprice it relative to its parent and thereby generating a low stub value. Consistent with this, in most of the examples listed by Mitchell, Pulvino, and Stafford (2002), the subsidiary's business activities involve newer technologies—and, therefore, in all likelihood, more growth options—than do the parent company's.[10]

Our analysis may also shed light on a recent finding of Mark Grinblatt and Moskowitz (2004): that, in the cross section, and after controlling for the past year's return, stocks with more "consistent" momentum—defined as eight or more up months over the previous year—have a higher subsequent return, on average. A stock that had a high return last year *without* momentum consistency is more likely to have had a positively skewed return distribution over that year. Our analysis suggests that it may therefore have become overpriced, explaining its lower subsequent return.

Our results may also be relevant for understanding option prices. Deep out-of-the-money options have positively skewed returns and so, according to our analysis, may be overpriced. By put-call parity, deep *in*-the-money options will then also be overpriced. Our model therefore predicts a "smile" in the implied volatility curve.[11]

Options on individual stocks do indeed exhibit a smile (Bollen and Whaley 2004). For index options, however, the implied volatility curve is downward-sloping rather than U-shaped. Why does our model fit individual stock option data better? One possible reason is that, for index options, a larger fraction of trading volume comes from institutional investors (Jun Pan and Allen Poteshman 2006). We suspect that cumulative prospect theory is not as good a description of institutional investor preferences as it is of individual investor preferences.[12]

Our analysis also points to a possible cause for the lack of diversification in many household portfolios. Under cumulative prospect theory, an investor may willingly take an undiversified position in a positively skewed stock in order to add skewness to his portfolio.[13] William Goetzmann and Alok Kumar (forthcoming) and Mitton and Vorkink (2007) present some relevant evidence. Using data on the portfolios of individual investors, they show that the stocks held by undiversified investors have greater return skewness than those held by diversified investors.

Our framework suggests a link, then, between a number of seemingly unrelated phenomena. At the same time, it also offers a new empirical prediction: that positively skewed securities will earn lower average returns. Unfortunately, this prediction is not easy to test because it is hard to forecast a security's future skewness: past skewness, the most obvious potential predictor, has little actual predictive power. We therefore need to find a better way of forecasting skewness.

Yijie Zhang (2006) suggests one way forward. He groups stocks into industries, and, for each industry in turn, records the return, over the past month, of each stock in the industry. For an industry with 100 firms, say, he therefore has 100 return observations. He then computes the

[10] Of course, since the subsidiary forms part of the parent company, its growth options will also give the returns of the parent company a positively skewed distribution. The parent company's returns will be less skewed than the subsidiary's returns, however, and, as we saw in Section IIIB, it is only high levels of skewness that are overpriced; more moderate skewness is fairly valued. The subsidiary can therefore be overpriced even relative to its parent.

[11] Implicit in this argument are two assumptions: first, that, as discussed by Nicolas Bollen and Robert Whaley (2004) and others, market makers face risks when they attempt to implement the dynamic strategy needed to enforce Black-Scholes pricing and that option prices can therefore deviate from those predicted by that formula; and, second, that put-call parity, based as it is on a static strategy, does hold.

[12] Joost Driessen and Pascal Maenhout (forthcoming) offer a portfolio choice view of this argument. In a partial equilibrium setting, they show that, across several expected utility and nonexpected utility specifications, the only preferences for which an investor would *not* short out-of-the-money puts on the S&P 500 index are cumulative prospect theory preferences. This suggests, as does our analysis, that such preferences may offer one way of understanding the high prices of out-of-the-money options.

[13] This point is also noted by Polkovnichenko (2005) who, in a portfolio choice setting, shows that an investor with cumulative prospect theory preferences may take an undiversified position in a positively skewed security.

cross-sectional skewness of those 100 returns and uses it as a measure of the skewness of *all* stocks in that industry. The logic is that, if one stock in the industry did very well last month, leading to a high cross-sectional skewness measure for that industry, then, since stocks in the same industry are similar, we can conjecture that *all* stocks in that industry could potentially earn a high return in the coming months. Zhang (2006) shows that his measure of skewness does predict future skewness, but also, in line with the models in this paper, that it predicts returns, negatively, in the cross section.

V. Conclusion

Many financial phenomena are hard to understand in the context of the expected utility paradigm. It may therefore be useful to document the implications of *non*expected utility theories and to see if they offer novel predictions which can help us make sense of financial markets.

In this paper, we study the pricing implications of cumulative prospect theory—perhaps the most prominent nonexpected utility theory of all—paying particular attention to its probability weighting component. We find that cumulative prospect theory does indeed have a novel prediction, namely that an asset's *own* skewness can be priced. We argue that this prediction offers a unifying way of thinking about a range of seemingly unrelated facts.

Our analysis suggests a number of directions for future research. First, we can think more deeply about the applications discussed in Section IV: the challenge is to provide new evidence that a preference for skewness really is an underlying driver of the phenomena discussed there. Second, we can try to test one of our model's most basic predictions: that assets that investors perceive to be positively skewed will earn low average returns (see Brian Boyer, Mitton, and Vorkink (2008) for a very recent study on this topic). And third, on the theoretical front, we can continue our analysis of nonexpected utility models in order to see if they offer other novel predictions that might further our understanding of financial markets.

Appendix

PROOF OF LEMMA 1:

Since $w(1 - P(\cdot))$ is right-continuous and $v(\cdot)$ is continuous, we can integrate (13) by parts to obtain

$$V(\hat{W}^{+}) = [-v(x)w(1 - P(x))]_{x=0}^{x=\infty} + \int_0^\infty w(1 - P(x))dv(x). \tag{55}$$

From Chebychev's inequality, we know that, for $Z > 0$,

$$\Pr[|\hat{W} - E(\hat{W})| \geq Z] \leq \frac{\mathrm{Var}(\hat{W})}{Z^2},$$

which, in turn, implies that, for $x > E(\hat{W})$,

$$1 - P(x) \leq \Pr[|\hat{W} - E(\hat{W})| \geq x - E(\hat{W})] \leq \frac{\mathrm{Var}(\hat{W})}{(x - E(\hat{W}))^2}.$$

Assumptions 1–5 then imply that there exists $A > 0$ such that

$$v(x)w(1 - P(x)) \leq Ax^{\alpha - 2\delta} \to 0, \quad \text{as } x \to \infty.$$

The first term on the right-hand side of equation (55) is therefore zero, and so equation (15) is valid. A similar argument leads to equation (16).

PROOF OF PROPOSITION 1:

Our proof builds on the following three results, labelled Proposition A1, Proposition A2, and Proposition A3.

PROPOSITION A1: *Under Assumptions 1–5, the preferences in (12)–(14) satisfy the first-order stochastic dominance property. That is, if* $\hat{W}_1$ *first-order stochastically dominates* $\hat{W}_2$*, then* $V(\hat{W}_1) \geq V(\hat{W}_2)$*. Moreover, if* $\hat{W}_1$ *strictly first-order stochastically dominates* $\hat{W}_2$*, then* $V(\hat{W}_1) > V(\hat{W}_2)$.

PROOF:

Since $\hat{W}_1$ first-order stochastically dominates $\hat{W}_2$, $P_1(x) \leq P_2(x)$ for all $x \in \mathbb{R}$, where $P_i(\cdot)$ is the cumulative distribution function for $\hat{W}_i$. Equations (15) and (16) imply $V(\hat{W}_1^+) \geq V(\hat{W}_2^+)$ and $V(\hat{W}_1^-) \geq V(\hat{W}_2^-)$, and therefore that $V(\hat{W}_1) \geq V(\hat{W}_2)$. If, moreover, $\hat{W}_1$ strictly first-order stochastically dominates $\hat{W}_2$, then $P_1(x) < P_2(x)$ for some $x \in \mathbb{R}$. Given that cumulative distribution functions are right-continuous, we have $V(\hat{W}_1) > V(\hat{W}_2)$.[14]

PROPOSITION A2: *Suppose that Assumptions 1–5 hold. Take two distributions,* $\hat{W}_1$ *and* $\hat{W}_2$*, and suppose that:*

(*i*) $E(\hat{W}_1) = E(\hat{W}_2) \geq 0$;

(*ii*) $\hat{W}_1$ *and* $\hat{W}_2$ *are both symmetrically distributed;*

(*iii*) $\hat{W}_1$ *and* $\hat{W}_2$ *satisfy a single-crossing property, so that if* $P_i(\cdot)$ *is the cumulative distribution function for* $\hat{W}_i$*, there exists* z *such that* $P_1(x) \leq P_2(x)$ for $x < z$ and $P_1(x) \geq P_2(x)$ for $x > z$.

Then, $V(\hat{W}_1) \geq V(\hat{W}_2)$*. If, furthermore, the inequalities in condition (iii) hold strictly for some* x*, then* $V(\hat{W}_1) > V(\hat{W}_2)$.

PROOF:

If we define $g(\cdot) : [0, \infty) \to \mathbb{R}$ to be $g(x) = x^\alpha$, we can write

$$v(x) = \begin{cases} g(x) & \text{for } x \geq 0 \\ -\lambda g(|x|) & \text{for } x < 0. \end{cases}$$

For $\hat{W}_i$, $i = 1, 2$, with the same mean $\mu \geq 0$, we have

$$V(\hat{W}_i) = V(\hat{W}_i^-) + V_A(\hat{W}_i^+) + V_B(\hat{W}_i^+) + V_C(\hat{W}_i^+), \quad i = 1, 2,$$

[14] Not all of Assumptions 1–5 are needed for this result. Assumptions 2–4 can be replaced by "$w^+(\cdot)$, $w^-(\cdot)$, and $v(\cdot)$ are strictly increasing and continuous," and Assumption 5 can be replaced by "the integrals in equations (13) and (14) are finite."

where

$$V(\hat{W}_i^-) = -\lambda \int_{-\infty}^{0} w(P_i(x))g'(|x|)dx, \quad i = 1, 2, \tag{56}$$

$$V_A(\hat{W}_i^+) = \int_{0}^{\mu} w(1 - P_i(x))g'(x)dx, \quad i = 1, 2, \tag{57}$$

$$V_B(\hat{W}_i^+) = \int_{\mu}^{2\mu} w(1 - P_i(x))g'(x)dx, \quad i = 1, 2, \tag{58}$$

$$V_C(\hat{W}_i^+) = \int_{2\mu}^{\infty} w(1 - P_i(x))g'(x)dx, \quad i = 1, 2. \tag{59}$$

Applying the change of variable $x = 2\mu - x'$ to equations (58) and (59) and noting that, since the distributions are symmetric, $1 - P_i(x) = P_i(2\mu - x) = P_i(x')$, we have

$$V_B(\hat{W}_i^+) = \int_{0}^{\mu} w(P_i(x))g'(2\mu - x)dx, \quad i = 1, 2, \tag{60}$$

$$V_C(\hat{W}_i^+) = \int_{-\infty}^{0} w(P_i(x))g'(2\mu + |x|)dx, \quad i = 1, 2. \tag{61}$$

Summing up equations (56), (57), (60), and (61), we have

$$V(\hat{W}_i) = -\int_{-\infty}^{0} w(P_i(x))[\lambda g'(|x|) - g'(2\mu + |x|)]\,dx + \int_{0}^{\mu} [w(1 - P_i(x))g'(x) + w(P_i(x))g'(2\mu - x)]\,dx, \quad i = 1, 2$$

and

$$\begin{aligned} V(\hat{W}_1) - V(\hat{W}_2) &= \int_{-\infty}^{0} [w(P_2(x)) - w(P_1(x))][\lambda g'(|x|) - g'(2\mu + |x|)]\,dx \\ &+ \int_{0}^{\mu} \{[w(1 - P_1(x)) - w(1 - P_2(x))]g'(x) - [w(P_2(x)) - w(P_1(x))]g'(2\mu - x)\}\,dx. \end{aligned} \tag{62}$$

Since $\hat{W}_1$ and $\hat{W}_2$ are symmetric, if condition (iii) holds at all, it must hold for $z = \mu$. This means that $P_1(x) \leq P_2(x)$ for $x < 0 \leq \mu$. Using this fact, as well as the fact that $w(\cdot)$ is increasing and that $\lambda g'(|x|) - g'(2\mu + |x|) > 0$, we see that the first term on the right-hand side of equation (62) is nonnegative. Below, we show that

$$w(1 - P_1(x)) - w(1 - P_2(x)) \geq w(P_2(x)) - w(P_1(x)) \geq 0 \quad \text{for } x \in (0, \mu), \tag{63}$$

and that this holds strictly if $P_1(x) < P_2(x)$. Combining this with the fact that $g'(x) > g'(2\mu - x) > 0$ for $x \in (0, \mu)$, we see that the second term on the right-hand side of equation (62) is also nonnegative. This implies $V(\hat{W}_1) \geq V(\hat{W}_2)$. Furthermore, given that all distribution functions are right-continuous, it is straightforward to show that, if the inequalities in the single-crossing property hold strictly for some $x \in \mathbb{R}$, then $V(\hat{W}_1) > V(\hat{W}_2)$.

To finish the proof, we need to show that (63) holds, and that it holds strictly if $P_1(x) < P_2(x)$. To do this, consider the function

$$h(p) \equiv w(p) + w(1-p) = (p^\delta + (1-p)^\delta)^{-(1-\delta)/\delta}.$$

Since

$$h'(p) = -(1-\delta)(p^\delta + (1-p)^\delta)^{-1/\delta}\left(\frac{1}{p^{1-\delta}} - \frac{1}{(1-p)^{1-\delta}}\right) < 0 \quad \text{for } p \in (0, 1/2),$$

the function $h(p)$ is strictly decreasing for $p \in (0, 1/2)$. Since, for any $x \in (0, \mu)$, $P_1(x) \leq P_2(x) \leq 1/2$, we have $h(P_1(x)) \geq h(P_2(x))$, which then implies the first inequality in (63). The second inequality in (63) follows from the monotonicity of $w(\cdot)$. Finally, if $P_1(x) < P_2(x)$, then (63) holds strictly because $h(\cdot)$ is strictly decreasing in $(0, 1/2)$ and $w(\cdot)$ is strictly increasing.

Proposition A2 immediately implies that, for certain classes of distributions—specifically, for any set of symmetric distributions that have the same nonnegative mean and that, pairwise, satisfy a single-crossing property—cumulative prospect theory preferences satisfy second-order stochastic dominance. To see this, take any two distributions in the set, $\hat{W}_1$ and $\hat{W}_2$, say. The single-crossing property means that we can rank $\hat{W}_1$ and $\hat{W}_2$ according to the second-order stochastic dominance criterion: we can obtain one distribution from the other by adding a mean-preserving spread. Proposition A2 then shows that the distribution that dominates is preferred by a cumulative prospect theory investor.

We now discuss the intuition behind this result. Specifically, we explain why, within any class of symmetric distributions that have the same nonnegative mean and that satisfy the single-crossing property, a mean-preserving spread is undesirable for an agent with the preferences in (12)–(14), in spite of the risk-seeking induced by the probability weighting function $w(\cdot)$ and the convexity of $v(\cdot)$ in the region of losses.

Note first that, for a distribution with positive mean $\mu > 0$ that satisfies conditions (i)–(iii) in the statement of Proposition A2, a mean-preserving spread means one of two things: either (a), taking density around μ and spreading it symmetrically outward toward losses and toward larger gains; or (b), taking density around μ and spreading it symmetrically outward toward smaller gains and toward larger gains.

For spreads of type (a), the convexity of $v(\cdot)$ in the region of losses is irrelevant, precisely because it applies to gambles involving only losses. Moreover, the probability weighting function $w(\cdot)$ is neutral to such spreads: while they do add mass to the right tail, which is attractive to an agent who overweights the tails of distributions, they also add mass to the *left* tail, which is unattractive. The agent's attitude to type (a) spreads is therefore determined by the kink in $v(\cdot)$ at the origin, which, of course, generates aversion to these spreads.

For type (b) spreads, the convexity of $v(\cdot)$ in the region of losses is again irrelevant. The probability weighting function $w(\cdot)$ induces *aversion* to these spreads: since $w'(p) < w'(1-p)$ for $p \in (0, 1/2)$, the agent is more sensitive to shifts in mass from μ to below μ than to shifts in mass from μ to above μ. The concavity of $v(\cdot)$ in the region of gains compounds this aversion.

PROPOSITION A3: *Consider the preferences in (12)–(14) and suppose that Assumptions 2–5 hold. If $\hat{W}$ is Normally distributed with mean μ_W and variance σ_W^2, then $V(\hat{W})$ can be written as a function of μ_W and σ_W^2, $F(\mu_W, \sigma_W^2)$. Moreover, for any σ_W^2, $F(\mu_W, \sigma_W^2)$ is strictly increasing in μ_W; and for any $\mu_W \geq 0$, $F(\mu_W, \sigma_W^2)$ is strictly decreasing in σ_W^2.*

PROOF:

Since every Normal distribution is fully specified by its mean and variance, we can write $V(\hat{W}) = F(\mu_W, \sigma_W^2)$. Proposition A1 implies that $F(\mu_W, \sigma_W^2)$ is strictly increasing in μ_W. Now consider any pair of Normal wealth distributions, $\hat{W}_1$ and $\hat{W}_2$, with the same nonnegative mean but different variance. These two wealth distributions satisfy conditions (i)–(iii) in Proposition A2. That proposition therefore implies that, for any $\mu_W \geq 0$, $F(\mu_W, \sigma_W^2)$ is strictly decreasing in σ_W^2.

Our proof of Proposition 1 will now proceed in the following way. We first derive the conditions that characterize equilibrium, assuming that an equilibrium exists. We then show that an equilibrium does indeed exist.

We ignore the violation of limited liability and assume that all securities have positive prices in equilibrium. Consider the mean/standard deviation plane. For any set of positive prices for the J risky assets, Assumption 7 means that we can construct a hyperbola representing the mean-variance (MV) frontier for those assets. If we then introduce the risk-free asset, the MV frontier becomes the tangency line from the risk-free asset to the hyperbola plus the reflection of this tangency line off the vertical axis. The MV *efficient* frontier is the upper of these two lines. The tangency portfolio, composed only of the J risky securities, has return $\tilde{R}_T$.

By Proposition A3, each investor chooses a portfolio on the MV efficient frontier, in other words, a portfolio with return $\tilde{R} = R_f + \theta(\tilde{R}_T - R_f)$, where θ is the weight in the tangency portfolio. Since investors have identical preferences, they choose the same θ. Market clearing implies $\theta > 0$ and so the tangency portfolio has to be on the upper half of the hyperbola in equilibrium. This, in turn, implies $E(\hat{R}_T) > 0$: the risk-free rate R_f has to be lower than the expected return of the minimum-variance portfolio, the left-most point of the hyperbola.

At time 1, each investor's wealth is given by

$$\tilde{W} = W_0(R_f + \theta(\tilde{R}_T - R_f)),$$

$$\hat{W} = W_0\theta\hat{R}_T.$$

For $\theta \geq 0$, utility is therefore given by

$$U(\theta) = W_0^{\alpha}\theta^{\alpha}V(\hat{R}_T),$$

where

$$V(\hat{R}_T) = -\int_{-\infty}^{0} w(P(\hat{R}_T))dv(\hat{R}_T) + \int_{0}^{\infty} w(1 - P(\hat{R}_T))dv(\hat{R}_T).$$

The optimal strategy for an investor is therefore

$$\theta = \begin{cases} 0 & \text{for } V(\hat{R}_T) < 0 \\ \text{any } \theta \geq 0 & \text{for } V(\hat{R}_T) = 0 \\ \infty & \text{for } V(\hat{R}_T) > 0. \end{cases} \tag{64}$$

This means that, to clear markets, we need $V(\hat{R}_T) = 0$.

In equilibrium, the aggregate demand for risky assets, given by (64), must equal the aggregate supply of risky assets, namely the market portfolio. Therefore, $\hat{R}_T = \hat{R}_M$. Earlier, we saw that $E(\hat{R}_T) > 0$ and that $V(\hat{R}_T) = 0$. Since $\hat{R}_T = \hat{R}_M$, we immediately obtain $E(\hat{R}_M) > 0$ and $V(\hat{R}_M) = 0$, as claimed in (19) and (20). Finally, $\tilde{R}_T = \tilde{R}_M$ also implies equations (17)–(18). For example, this can be shown by noting that a portfolio with return $\tilde{R}_M + x(\tilde{R}_j - R_f)$ must attain its highest Sharpe ratio at $x = 0$.

So far, we have shown that, *if* an equilibrium exists, it is characterized by conditions (17), (19), and (20). We now show that an equilibrium does indeed exist, in other words, that we can find prices for the J risky assets such that conditions (17), (19), and (20) hold.

In conditions (17) and (19), we have J nonredundant equations in J nonredundant unknowns: the J nonredundant equations are condition (19) and any $J-1$ of the J equations in (17); the J nonredundant unknowns are the market price $p_M = \sum_{j=1}^{J} n_j p_j$ and any $J-1$ of the J prices $\{p_1, p_2, \ldots, p_J\}$. We can therefore solve the J nonredundant equations to obtain the risky asset prices.

It only remains to show that the risky asset prices also imply condition (20). To see this, note that

$$0 = F(0,0) > F(0, \sigma^2(\hat{R}_M)),$$

where the inequality follows from Proposition A3, which also introduces the function $F(\cdot,\cdot)$. If $E(\hat{R}_M) \leq 0$, Proposition A3 would then also imply $F(E(\hat{R}_M), \sigma^2(\hat{R}_M)) < 0$, contradicting condition (19), which says that $F(E(\hat{R}_M), \sigma^2(\hat{R}_M)) = 0$. We therefore have $E(\hat{R}_M) > 0$, as in condition (20). The intuition is straightforward. Under conditions that apply here, cumulative prospect theory satisfies second-order stochastic dominance. An investor with these preferences therefore dislikes the variance of the market portfolio and only holds it if compensated by a positive risk premium.

The Effect of Introducing a Skewed Security on the Prices of Existing Securities

In Section III, we noted an interesting feature of the heterogeneous holdings equilibrium: if the skewed security cannot be sold short, its introduction does not affect the prices of the J original risky assets. To see this, note that, after the introduction of the skewed security, the prices of the J original risky assets are determined by

$$E(\tilde{R}_j) = R_f + \beta_j(E(\tilde{R}_M) - R_f), \quad j = 1,\ldots,J,$$

or, equivalently,

$$\frac{E(\tilde{X}_j)}{p_j} = R_f + \frac{\mathrm{Cov}(\tilde{X}_j, \tilde{X}_M)}{\mathrm{Var}(\tilde{X}_M)}\frac{p_M}{p_j}\left(\frac{E(\tilde{X}_M)}{p_M} - R_f\right), \quad j = 1, \ldots, J, \tag{65}$$

where p_j is the price of asset j, $p_M = \sum_{j=1}^{J} n_j p_j$, and $\tilde{X}_M = \sum_{j=1}^{J} n_j \tilde{X}_j$.

In the economy of Section II, the prices $\{p_j'\}$ of the J original risky assets are given by

$$\frac{E(\tilde{X}_j)}{p_j'} = R_f + \frac{\mathrm{Cov}(\tilde{X}_j, \tilde{X}_M)}{\mathrm{Var}(\tilde{X}_M)}\frac{p_M'}{p_j'}\left(\frac{E(\tilde{X}_M)}{p_M'} - R_f\right), \quad j = 1, \ldots, J, \tag{66}$$

where $p_M' = \sum_{j=1}^{J} n_j p_j'$. From equations (19) and (23), we know that the return on the market portfolio formed from the J risky assets alone satisfies $V(\hat{R}_M) = 0$, whether or not the skewed security is present. This implies $p_M = p_M'$, which, in turn, means that the equations for $\{p_j\}$ in (65) are identical to the equations for $\{p_j'\}$ in (66). The prices of the J original risky assets are therefore the same, whether or not the skewed security is present.

PROOF OF PROPOSITION 2:

Consider an investor with the preferences in (12)–(14) who holds a portfolio with return $\tilde{R} \equiv \hat{R} + R_f$. Suppose that he adds a small amount of an independent security with excess return $\hat{R}_n$ to his portfolio; and that he finances this by borrowing, so that his excess portfolio return becomes $\hat{R} + \varepsilon\hat{R}_n$. If $\hat{R}$ has a probability density function that satisfies $\sigma(\hat{R}) > 0$, then, to the first order of ε,

$$(67)\quad V(\hat{R}+\varepsilon\hat{R}_n) - V(\hat{R}) \approx -\int_{-\infty}^{0} w'(P(R))\delta P(R)dv(R) - \int_{0}^{\infty} w'(1-P(R))\delta P(R)dv(R),$$

where $P(R) = \Pr(\hat{R} \leq R)$ and where, again to the first order of ε,

$$
\begin{aligned}
(68)\quad \delta P(R) &\equiv P(\hat{R} + \varepsilon\hat{R}_n \leq R) - P(\hat{R} \leq R) \\
&= E[1(\hat{R} + \varepsilon\hat{R}_n \leq R) - 1(\hat{R} \leq R)] \\
&= E[-1(\varepsilon\hat{R}_n > 0)\,1(R - \varepsilon\hat{R}_n < \hat{R} \leq R) + 1(\varepsilon\hat{R}_n < 0)\,1(R < \hat{R} \leq R - \varepsilon\hat{R}_n)] \\
&\approx E[-1(\varepsilon\hat{R}_n > 0) f(R\,|\,\hat{R}_n)(\varepsilon\hat{R}_n) + 1(\varepsilon\hat{R}_n < 0)\, f(R\,|\,\hat{R}_n)(-\varepsilon\hat{R}_n)] \\
&= E[f(R\,|\,\hat{R}_n)(-\varepsilon\hat{R}_n)] \\
&= -\varepsilon E(\hat{R}_n)\, f(R),
\end{aligned}
$$

where $f(R|\hat{R}_n)$ and $f(R)$ are the conditional and unconditional probability densities of $\hat{R}$ at R, respectively, and where the last equality in equation (68) follows from the independence assumption. Substituting equation (68) into equation (67), we obtain

$$
\begin{aligned}
(69)\quad &\lim_{\varepsilon \to 0} \frac{V(\hat{R} + \varepsilon\hat{R}_n) - V(\hat{R})}{\varepsilon} \\
&= E(\hat{R}_n)\left[\int_{-\infty}^{0} w'(P(R))P'(R)dv(R) + \int_{0}^{\infty} w'(1-P(R))P'(R)dv(R)\right],
\end{aligned}
$$

where the term within square brackets is strictly positive.

In any homogeneous holdings equilibrium, we need $V(\hat{R}_M + x\hat{R}_n)$ to have a local maximum at $x = \varepsilon^*$ for infinitesimal ε^*. Since $V(\hat{R}_M + x\hat{R}_n)$ is smooth in x, as $\varepsilon^* \downarrow 0$,

$$\left.\frac{dV(\hat{R}_M + x\hat{R}_n)}{dx}\right|_{x=\varepsilon^*} - \left.\frac{dV(\hat{R}_M + x\hat{R}_n)}{dx}\right|_{x=0} \uparrow 0.$$

The derivative of $V(\hat{R}_M + x\hat{R}_n)$ with respect to x at $x = 0$ must therefore be infinitesimally greater than zero. Equation (69) then implies that $E(\hat{R}_n)$ must also be infinitesimally greater than zero.

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