1 Maximizing Utility Without Constraints
Suppose I told you that my preference for Jamba Juice Smoothies (\(J\)) and Wine (\(W\)) could be characterized by the function

\[ u = \sqrt{J} \cdot \sqrt{W} \]

and I asked you to help me figure out how much of each good to buy. If that were all the information I gave you, the answer would be simple, and trivial. Without any kind of constraint on my ability to spend, I would purchase as much of each as I could get my hands on since each additional smoothie and bottle of wine raises my utility.

2 Maximizing Utility with Constraints
Now suppose I told you I had the same utility function as above, but that I could spend no more than \(Y\) dollars. In real life, of course, my income is a number. Nevertheless, let’s solve the problem without using a number up front. That will make it easier to see how my optimal decision changes as income varies later on.

To figure out how much to spend on each good, I need to maximize my utility subject to my income. If \(J\) and \(W\) are the only two goods I can buy, then the amount I spend on these goods has to be equal to my income. This can be written algebraically as

\[ p_J J + p_W W = Y, \]

where \(p_J\) and \(p_W\) are the prices of a Jamba Juice and a bottle of good wine, respectively. Here, again, we could substitute numbers for the prices of each good, but we’ll hold off doing that for a bit longer.

Given the utility function, prices and income, how should we proceed? In words, I want to figure out how to spend all of my income until the value that I place on the goods is the same as the value the market places on the goods. Geometrically, as illustrated in the figure below, this is equivalent to saying that utility is maximized when the budget line is tangent to the utility curve. The two conditions which much be satisfied for this to occur are first, that the slope of the utility curve equals that of the budget line, and second, that the utility curve and the budget line touch.
To solve the problem algebraically, we can take care of the first condition by equating minus the Marginal Rate of Substitution (i.e. minus the slope of the utility curve) to the Terms of Trade (i.e. the slope of the budget line). The MRS is defined as

$$MRS = \frac{\text{Marginal Utility}_W}{\text{Marginal Utility}_J} = \frac{\frac{du}{dW}}{\frac{du}{dJ}}.$$

To find the slope of the budget line, we need to manipulate it a bit from the version given above to get it into the form we’re more familiar with, $J=mW+b$. (See TA Handout 1 if this form is confusing.) This is done as follows

$$p_J J + p_W W = Y$$
$$p_J J = -p_W W + Y$$
$$J = -\frac{p_W}{p_J} W + \frac{Y}{p_J}$$

With this result, we can now equate the slopes. Notice the negative signs drop out in the third step.
We now have two equations containing the two unknowns for which we’ve been searching. The two equations are the budget constraint, \( p_J J + p_W W = Y \), and the equation resulting from the equality of slopes, \( p_J J = p_W W \), while the two unknowns are the amounts of \( W \) and \( J \) to purchase. Using the information in both equations guarantees the second condition referred to above, which is that the budget line and utility curves touch.

We can solve the two equations by focusing on one unknown at a time. Do this by substituting the second equation into the first and solving for \( W \)

\[
p_w W + p_w W = Y \\
2 p_w W = Y \\
W = \frac{Y}{2 p_w}
\]

Now that we know \( W \), we can use either of the two equations to solve for \( J \). Suppose we use the budget constraint

\[
p_J J + p_w W = Y \\
p_J J + p_w \frac{Y}{2 p_w} = Y \\
p_J J + \frac{Y}{2} = Y \\
J = \frac{Y}{2 p_J}
\]

Success! We have solved for the optimal quantities of \( J \) and \( W \) in terms of prices and income. It is now easy to substitute numbers for variables and see what these quantities are. If, for example, my income is $20,000 and the prices of a Jamba Juice and a bottle of wine are $2.50 and $10, respectively, I will buy 4000 smoothies and 1000 bottles of wine.
3 A Check of Your Understanding
To check your understanding, see if you can answer the following questions.

How much of each good do I buy if my income doubles?

How much of each good do I buy if the price of a Jamba Juice rises to $3.00?

What is my total expenditure on each good? Can you figure out which feature of the utility function causes this result?

Can you solve the system of equations by finding the optimal quantity of J first?

What information must be provided before you can help someone maximize their utility?