Performance Measures Under Periodic Inspections

In this section we derive the long-run average performance measures $I$, $R$, $B$, and $\bar{B}$ (defined in §3.8) assuming that the firm employs the threshold disclosure policy described in Lemma 3 in response to the regulator’s periodic inspections. We assume that the disclosure window has an arbitrary size equal to $s \in [0, \tau]$. Recall from §3.4 that last-minute disclosure that occurs under the threshold disclosure policy is regarded as detection. In the proofs below, $U$ and $D$ refer to the exponential i.i.d. random variables with means $1/\lambda$ and $1/\mu$ representing the duration of each compliance and noncompliance episode, respectively.

The most convenient unit of analysis in evaluating the performance measures under periodic inspections is an inspection cycle. An inspection cycle is the time interval that contains at most one inspection performed, beginning in the compliance state and concluding with three possible endings that leave the firm back in the compliance state: (i) the cycle ends $\tau$ time units after the start when the regulator arrives for an inspection and finds the firm in compliance; (ii) the cycle ends when compliance is restored following last-minute disclosure (detection) at $\tau$ time units since the start; (iii) the cycle ends when compliance is restored following the firm’s disclosure at the onset of noncompliance.

Note that these are the only possible endings of an inspection cycle. Because of memorylessness of the exponential compliance and noncompliance durations, the start and end times of an inspection cycle mark regeneration epochs. Therefore, an inspection cycle forms a renewal (Heyman and Sobel 1982, p. 184; Tijms 2003, p. 40). Note also that an inspection cycle may end at, before, or after $\tau$ time units since the start, because it takes a random amount of time for compliance to be restored after detection or disclosure. Thus the length of an inspection cycle—denoted by $X$—is random, and it is to be distinguished from the constant inspection interval $T = \tau$.

Depending on whether the firm is in compliance at the beginning of the disclosure window (at time $\tau - s$ since the cycle start), under the threshold disclosure policy the three outcomes above are further divided into five cases, referred to as Case 1a, Case 1b, etc. See Figure A.1 that illustrates these cases.

1. The firm is in compliance at time $\tau - s$ since the cycle start. Moreover: (a) compliance lasts until or after time $\tau$, at which the current inspection cycle ends with an inspection (but no detection); (b) noncompliance starts before time $\tau$ and at that moment the firm discloses the state, and subsequently the current inspection cycle ends when compliance is restored.

2. The firm is in noncompliance at time $\tau - s$ since the cycle start. Moreover: (a) noncompliance lasts until or after time $\tau$, at which detection occurs and subsequently the current inspection cycle ends when compliance is restored; (b) compliance is restored before time $\tau$ and it lasts until or after time $\tau$, at which the current inspection cycle ends with an inspection (but no detection); (c) compliance is restored before time $\tau$ but it is followed by a transition to noncompliance before time $\tau$ which is disclosed by the firm, and subsequently the current inspection cycle ends when compliance is restored again.

Note that in all cases at most one new noncompliance episode occurs within the disclosure window because, under the threshold disclosure policy, the firm always discloses the first of such occurrences (see Cases 1b and 2c in Figure A.1) and subsequently the current inspection cycle ends as soon as compliance is restored. As a result, the five cases described above form a complete list of categories.
Figure A.1: Five possible inspection/detection/disclosure outcomes that may arise in an inspection cycle under the threshold disclosure policy combined with periodic inspections. Noncompliance episodes are denoted by thick horizontal lines. Note that any realization of state transitions is possible before time $\tau - s$, provided that the state starts with compliance at time 0 and ends with either compliance (Cases 1a and 1b) or noncompliance (Cases 2a, 2b, and 2c) at time $\tau - s$. (For brevity, only one noncompliance episode appears in the figures.)

for all possible inspection/detection/disclosure outcomes that may arise in an inspection cycle. In addition, as illustrated in Figure A.1, production lasts either until the firm discloses noncompliance or for exactly $\tau$ time units in case no disclosure is made.

With the complete list of stochastic outcome categories specified, we are now in a position to evaluate the long-run average performance measures. As an intermediate step, we first compute the probabilities of each case listed in Figure A.1 and then evaluate the following quantities: (i) $E[X]$, the expected length of an inspection cycle; (ii) $E[I]$, the expected number of inspections performed in an inspection cycle; (iii) $E[R]$, the expected duration of suspended noncompliance in an inspection cycle; (iv) $E[B]$, the expected cumulative duration of unsuspended noncompliance in an inspection cycle; (v) $E[\Psi]$, the firm’s expected cost in an inspection cycle. The results are summarized as follows.

- $E[X] = \tau - s + \left( \frac{1}{\lambda} + \frac{1}{\mu} \right) (1 - e^{-\lambda s}) + \left( \frac{1}{\mu} - \frac{\lambda + \mu}{\mu(\mu - \lambda)} (e^{-\lambda s} - e^{-\mu s}) \right) \theta(\tau - s)$,
- $E[I] = e^{-\lambda s} + \frac{\lambda(e^{-\lambda s} - e^{-\mu s})}{\mu - \lambda} \theta(\tau - s)$,
- $E[R] = \frac{1 - e^{-\lambda s}}{\mu} + \frac{\mu e^{-\mu s} - \lambda e^{-\lambda s}}{\mu(\mu - \lambda)} \theta(\tau - s)$,
- $E[B] = \frac{\lambda}{\lambda + \mu} (\tau - s) + \left( \frac{1 - e^{-\mu s}}{\mu} - \frac{1}{\lambda + \mu} \right) \theta(\tau - s)$,
- $E[\Psi] = \left( \frac{\tau}{\mu} + \kappa_e \right) (1 - e^{-\lambda s}) + \left( \frac{\tau \mu e^{-\mu s} - \lambda e^{-\lambda s}}{\mu(\mu - \lambda)} + \kappa_d e^{-\mu s} - \kappa_d \frac{\lambda(e^{-\lambda s} - e^{-\mu s})}{\mu - \lambda} \right) \theta(\tau - s)$.

All proofs are found in the following lemmas. Note that the function $\theta(t)$ appearing in the above expressions is defined in (1).
Lemma A.1 The probability of each case in an inspection cycle is as follows: (i) $\Pr (1a) = e^{-\lambda s} (1 - \theta (t - s))$; (ii) $\Pr (1b) = (1 - e^{-\lambda s}) (1 - \theta (t - s))$; (iii) $\Pr (2a) = e^{-\mu s} \theta (t - s)$; (iv) $\Pr (2b) = \frac{\mu}{\mu - \lambda} (e^{-\lambda s} - e^{-\mu s}) \theta (t - s)$; (v) $\Pr (2c) = \left( 1 - \frac{\mu}{\mu - \lambda} e^{-\lambda s} + \frac{\lambda}{\mu - \lambda} e^{-\mu s} \right) \theta (t - s)$, where $\theta(t)$ is defined in (1).

Proof. In Cases 1a and 1b the firm is in compliance at both time zero and time $t - s$, an event that occurs with probability $1 - \theta (t - s)$. On the other hand, in Cases 2a, 2b, and 2c the firm is in compliance at time zero but is in noncompliance at time $t - s$, an event that occurs with probability $\theta (t - s)$. Using the memoryless properties of these random variables, from Figure A.1 we see that the probabilities for the five cases are: (i) $\Pr (1a) = \Pr (U > s) (1 - \theta (t - s))$; (ii) $\Pr (1b) = \Pr (U \leq s) (1 - \theta (t - s))$; (iii) $\Pr (2a) = \Pr (D > s) \theta (t - s)$; (iv) $\Pr (2b) = \Pr (D \leq s < D + U) \theta (t - s)$; (v) $\Pr (2c) = \Pr (D + U \leq s) \theta (t - s)$.

It remains to evaluate the conditional probabilities. Along with $\Pr (U > s) = e^{-\lambda s}$ and $\Pr (D > s) = e^{-\mu s}$, we have $\Pr (D \leq s < D + U) = \frac{\mu}{\mu - \lambda} (e^{-\lambda s} - e^{-\mu s})$ and $\Pr (D + U \leq s) = 1 - \frac{\mu}{\mu - \lambda} e^{-\lambda s} + \frac{\lambda}{\mu - \lambda} e^{-\mu s}$.

Lemma A.2 The conditional expected length of an inspection cycle for each case in an inspection cycle is as follows: (i) $E [X|1a] = \tau$; (ii) $E [X|1b] = \tau - \frac{s}{1-e^{-\lambda s}} + \frac{1}{\mu} + \frac{1}{\mu}$; (iii) $E [X|2a] = \tau + \frac{1}{\mu}$; (iv) $E [X|2b] = \tau - \frac{(\mu - \lambda) s (e^{-\lambda s} - e^{-\mu s})}{\mu (1-e^{-\lambda s}) - \lambda (1-e^{-\mu s})} + \frac{1}{\mu} + \frac{2}{\mu}$. Unconditioning, the expected length of an inspection cycle is $E [X] = \tau - s + \left( \frac{1}{\lambda} + \frac{1}{\mu} \right) (1 - e^{-\lambda s}) + \left( \frac{\mu}{\mu - \lambda} (e^{-\lambda s} - e^{-\mu s}) \right) \theta (t - s)$.

Proof. In Case 1a and Case 2b a cycle takes $\tau$ time units as no disclosure or detection occurs by then. In Case 2a the cycle starts at time zero and ends after noncompliance is detected at time $t$ and until compliance is restored, which in total takes $\tau + 1/\mu$ time units in expectation. (The residual noncompliance duration after $\tau$ is exponentially distributed with mean $1/\mu$ due to memorylessness of $D$.) Consider Case 1b. The cycle lasts at least $\tau - s$ time units, after which it lasts until the noncompliance episode that starts between times $\tau - s$ and $\tau$ concludes. Let $Z$ be the residual compliance duration starting from time $\tau - s$. Then $Z$ is exponentially distributed with mean $1/\lambda$ because of memorylessness of $U$. The expected duration from time $\tau - s$ until compliance restoration conditioning on $Z \leq s$ is $E [Z|Z \leq s] + E [D] = \int_0^\infty zf_Z (z) dz + \frac{1}{\mu} = \int_0^s z \frac{e^{-\lambda s}}{1-e^{-\lambda s}} dz + \frac{1}{\mu} = \frac{1}{\lambda} - \frac{s e^{-\lambda s}}{1-e^{-\lambda s}} + \frac{1}{\mu}$. The truncated distribution of $Z$ is defined on the support $[0, s]$ is the truncated distribution of $Z$ (Mood et al. 1973, p. 124). Hence, the expected cycle length of Case 1b is $\tau - s + E [Z|Z \leq s] + E [D] = \tau - \frac{s}{1-e^{-\lambda s}} + \frac{1}{\lambda} + \frac{1}{\mu}$. Finally, consider Case 2c. The cycle lasts at least $\tau - s$ time units, after which it lasts until the state makes three transitions: to compliance and then to noncompliance between times $\tau - s$ and $\tau$, and then back to compliance again. Let $Y$ be the residual noncompliance duration starting from time $\tau - s$ which is exponentially distributed with mean $1/\mu$. Noting that the pdf and the cdf of the random variable $Y + U$ are $f_{Y+U} (z) = \frac{\lambda \mu}{\mu - \lambda} (e^{-\lambda z} - e^{-\mu z})$ and $F_{Y+U} (z) = 1 - \frac{\mu e^{-\lambda s} - \lambda e^{-\mu s}}{\mu - \lambda}$ (Ebeling 2009, p. 235), the expected duration from the truncated random variable $Y + U \leq s$ is $E [Y + U|Y + U \leq s] = \int_0^s z f_{Y+U} (z) dz = \int_0^s f_{Y+U} (z) dz = e^{-\lambda s} - e^{-\mu s} - s (\mu e^{-\lambda s} - \lambda e^{-\mu s}) + \frac{1}{\lambda} + \frac{1}{\mu}$. Hence, the conditional expected length of a renewal in this case is $\tau - s + E [Y + U|Y + U \leq s] + E [D] = \tau - \frac{(\mu - \lambda) s (e^{-\lambda s} - e^{-\mu s})}{\mu (1-e^{-\lambda s}) - \lambda (1-e^{-\mu s})} + \frac{1}{\lambda} + \frac{2}{\mu}$. Finally, $E [X]$ is obtained by unconditioning the conditional expectations using the probabilities computed in Lemma A.1 and collecting terms.

Lemma A.3 The expected number of inspections performed in an inspection cycle is $E [I] = e^{-\lambda s} + \frac{\lambda (e^{-\lambda s} - e^{-\mu s}) \theta (t - s)}{\mu - \lambda}$.

Proof. Since exactly one inspection is performed in Cases 1a, 2a, and 2b whereas none in Cases 1b and 2c (as a disclosure suspends inspections), the expected number of inspections performed per cycle is $\Pr (1a) + \Pr (2a) + \Pr (2b)$. The result follows from Lemma A.1.
Lemma A.4 The expected duration of suspended noncompliance in an inspection cycle is

\[ E[R] = \frac{1-e^{-\lambda \mu}}{\mu} + \frac{\mu e^{-\mu s} - \lambda e^{-\lambda s}}{\mu(\mu - \lambda)} \theta (\tau - s). \]

Proof. Noncompliance is reported either by a detection or a disclosure. Thus, only Cases 1b, 2a, and 2c are relevant. Because of memorylessness of \( D \), in each case the expected duration of a residual noncompliance after a report is equal to 1/\( \mu \). Accounting for all cases, \( E[R] = \frac{1}{\mu} \left[ \Pr(1b) + \Pr(2a) + \Pr(2c) \right] \). Then the result follows from Lemma A.1. 

Lemma A.5 The expected duration of unsuspended noncompliance in an inspection cycle is

\[ E[B] = \frac{\lambda}{\lambda + \mu} (\tau - s) + \left( \frac{1-e^{-\mu s}}{\mu} - \frac{1}{\lambda + \mu} \right) \theta (\tau - s). \]

Proof. The expected duration of noncompliance between time 0 and time \( \tau - s \) is (Nakagawa 2005, p. 45)

\[ E \left[ \int_0^{\tau-s} 1 \text{(noncompliance at } t) \, dt \right] = \int_0^{\tau-s} \Pr(\text{noncompliance at } t) \, dt = \int_0^{\tau-s} \theta(t) \, dt = \frac{\lambda}{\lambda + \mu} (\tau - s) - \frac{1}{\lambda + \mu} \theta (\tau - s). \]

Since the noncompliance episodes that start before time \( \tau - s \) are unreported, this expected duration is common in all five cases of Figure A.1. Now consider the disclosure window \([\tau - s, \tau]\). In Cases 1a, no noncompliance exists in the window, contributing zero to \( E[B] \). In Case 1b, any noncompliance episode that starts within the window is reported; hence, this case does not contribute to \( E[B] \). In Case 2a, noncompliance duration is greater than the window; hence, this case contributes \( s \) to \( E[B] \). In Cases 2b and 2c, the conditional expected duration of noncompliance in \([\tau - s, \tau]\) is equal to \( E[D|D < s] = \int_0^s \frac{\mu e^{-\mu x}}{1-e^{-\mu}} \, dx = \frac{1}{\mu} - \frac{se^{-\mu s}}{1-e^{-\mu}} \). Unconditioning using Lemma A.1 and adding the first result above yields

\[ E[B] = \int_0^{\tau-s} \theta(t) \, dt + s \Pr(2a) + E[D|D < s] (\Pr(2b) + \Pr(2c)) = \frac{\lambda}{\lambda + \mu} (\tau - s) + \left( \frac{1-e^{-\mu s}}{\mu} - \frac{1}{\lambda + \mu} \right) \theta (\tau - s). \]

Lemma A.6 The firm’s conditional expected profit per inspection cycle for each case is as follows: (i) \( E[\Pi|1a] = r \tau \); (ii) \( E[\Pi|1b] = r \left( \tau - s \frac{s}{1-e^{-s}} + \frac{1}{\lambda} \right) - \kappa_e \); (iii) \( E[\Pi|2a] = r \tau - \kappa_d \); (iv) \( E[\Pi|2b] = r \tau \); (v) \( E[\Pi|2c] = r \left( \tau - (\frac{s}{1-e^{-s}}) \frac{e^{-\lambda s}}{\mu(1-e^{-\mu})} - \lambda \frac{e^{-\mu s}}{\mu - \lambda} + \frac{1}{\lambda} + \frac{1}{\mu} \right) - \kappa_e \). Unconditioning, the firm’s expected cost per inspection cycle (including the penalty and the opportunity costs) is \( E[\Psi] = rE[X] - E[\Pi] = \left( \frac{r}{\mu} + \kappa_e \right) (1-e^{-\lambda s}) + \left( r \frac{\mu e^{-\mu s} - \lambda e^{-\lambda s}}{\mu(\mu - \lambda)} + \kappa_d e^{-\mu s} - \kappa_e e^{-\lambda s} - \kappa_e \frac{\lambda e^{-\lambda s} - e^{-\mu s}}{\mu - \lambda} \right) \theta (\tau - s). \]

Proof. In Case 1a and Case 2b production continues until time \( \tau \), when the inspection cycle ends with no disclosure or detection. Hence the expected profit is \( r \tau \). In Case 1b production continues until the firm discloses noncompliance at its onset, incurring the early disclosure penalty \( \kappa_e \), after which noncompliance lasts \( D \) additional amount of time before the cycle ends. Therefore, production expects to last \( E[X|1b] = \frac{1}{\mu} \) and consequently the expected profit is \( r \left( E[X|1b] - \frac{1}{\mu} \right) - \kappa_e \). Similarly, in Case 2c the expected profit is \( r \left( E[X|2c] - \frac{1}{\mu} \right) - \kappa_e \). In Case 2a production continues until time \( \tau \), when noncompliance is detected. Upon detection, the firm pays the fixed penalty \( \kappa_d \). Therefore, the expected profit in Case 2a is \( r \tau - \kappa_d \). The expressions for Cases 1b and 2c are obtained using the results in Lemma A.2. Finally, \( E[\Psi] = rE[X] - E[\Pi] \) is obtained by unconditioning the conditional expectations using the probabilities computed in Lemma A.1 and collecting terms. 

Since each inspection cycle forms a renewal, we apply the renewal-reward theorem (Tijms 2003, p. 41) to compute the long-run averages by forming a ratio between the expectations \( E[I] \), \( E[R] \), \( E[B] \), \( E[\Psi] \) and \( E[X] \) in (i). For instance, the long-run average number of inspections performed is evaluated as \( \bar{T} = E[I] / E[X] \). The remaining measures \( \bar{R}, \bar{B}, \bar{\Psi} \) and \( \bar{\Psi} \) are similarly defined. The approximate measures in Lemma 4 are obtained by expanding these exact measures with respect to the ratio \( \lambda/\mu \) and retaining up to the first-order terms. (The approximations are verified using the Series function of Mathematica®.)
B  Additional Results and Proofs

Lemma B.1 (For §4.2) Suppose that the condition \( \lambda \ll \mu \) allows the compliance-noncompliance cycle of length \( U + D \) with \( U \sim \exp (\lambda) \) and \( D \sim \exp (\mu) \) to be approximated as the cycle of length \( U \sim \exp (\lambda) \). Then, under periodic inspections, the firm that maximizes its expected profit for the duration of a single noncompliance episode also maximizes his long-run average profit.

Proof. Consider the interval of length \( \tau \) between two successive scheduled inspections under periodic inspections. With the approximation described in the lemma, noncompliance occurs as a Poisson process. Then by the order statistics argument (Ross 1996, pp. 66-67), the arrival time of each of these occurrences is uniformly distributed within the interval of length \( \tau \). Hence, the probability that noncompliance occurs with \( y \in (0, \tau) \) time units remaining until the next scheduled inspection is equal to \( f (y) = \frac{1}{\tau} \). Recall from Lemma 3 that the firm with the myopic objective employs the threshold disclosure policy, i.e., disclose a noncompliance episode at its onset if and only if it occurs within the disclosure window of size \( s \leq \tau \). Then, conditional on noncompliance occurrence with \( y \in (0, \tau) \) time units remaining until the next scheduled inspection, the firm’s expected profit from a single compliance-noncompliance cycle (“cycle”) in the considered interval of length \( \tau \) is equal to \( (r \cdot (\tau - y) - \kappa_e) \cdot \min \{D, y\} + \kappa_d \cdot \Pr (D \geq y) \cdot \tau \); the first term corresponds to the case where noncompliance starts inside the disclosure window and the second term corresponds to the case where noncompliance starts outside the window. Unconditioning this using \( f (y) = \frac{1}{\tau} \), the expected profit per interval of length \( \tau \) for a single cycle is \( V (s) = \int_0^s (r \cdot (\tau - y) - \kappa_e) \frac{1}{\tau} dy + \int_s^\tau (r \cdot (\tau - y) + r \cdot \Pr (D \geq y)) \frac{1}{\tau} dy \). Then the expected profit from all possible cycles within the interval of length \( \tau \) is \( E [\Pi] = V (s) \cdot \lambda \tau \), where \( \lambda \tau \) is the expected number of cycles within the interval. The firm’s long-run average profit is approximated as \( \frac{E [\Pi]}{\tau} \), using the fact that \( \lambda \ll \mu \) implies the suspension time after detection is negligible. Since \( \frac{E [\Pi]}{\tau} = \lambda V (s) \), the firm’s long-run average profit is proportional to the myopic profit by a constant; they share the same maximum. Differentiating \( V (s) \) using Leibniz’s rule and setting it to zero yields the equation \( 0 = -\kappa_e - r \cdot \Pr (\min \{D, s\}) + \kappa_d \cdot \Pr (D \geq s) = -\kappa_e - \frac{\lambda}{\mu} (1 - e^{-\mu s}) + \kappa_d e^{-\mu s} \), from which we get \( s^* = \frac{1}{\mu} \ln \frac{r \cdot \kappa_d}{\kappa_e + \kappa_d} \), the same value as the one appearing in Lemma 3.

Lemma B.2 (For the proof of Proposition 2) The equilibrium exists in the region \( \tau \geq \sigma \) and satisfies \( \kappa_e = 0 \) and \( \kappa_d = K \). Consequently, the firm sets \( s^* = \sigma \) in equilibrium.

Proof. Let \( \phi \equiv \frac{1}{\mu} \ln \frac{r + \kappa_d \mu}{r + \kappa_e \mu} \) and \( \sigma \equiv \frac{1}{\mu} \ln \left( 1 + \frac{K \mu}{\lambda} \right) \). Recall from Lemma 3 that the firm chooses \( s^* = \min \{\phi, \tau\} \). Since \( \phi \) increases in \( \kappa_d \) while it decreases in \( \kappa_e \) with \( 0 \leq \kappa_e \leq \kappa_d \leq K \), the lower bound on \( \phi \) is found by setting \( \kappa_e = \kappa_d \) while the upper bound is found by setting \( \kappa_e = 0 \) and \( \kappa_d = K \), resulting in \( 0 \leq \phi \leq \sigma \). We first prove that the equilibrium does not exist for \( \tau < \sigma \). Suppose \( \tau < \sigma \) and divide it into two regions: \( \tau < \phi \leq \sigma \) and \( \phi \leq \tau < \sigma \). In the first region, \( s^* = \min \{\phi, \tau\} = \tau \) and therefore the social cost (4) reduces to \( C^* = \frac{\lambda}{\mu} (r - \frac{\lambda \mu}{2}) + \left( 1 - \frac{\lambda}{\mu} \right) \frac{\lambda}{\tau} \), which decreases in \( \tau \). Since \( C^* \) keeps decreasing in \( \tau \) in the considered region \( \tau < \phi \), the minimum of \( C^* \), if it exists in \( \tau < \sigma \), should be found in the second region \( \phi \leq \tau < \sigma \). In this region, \( s^* = \min \{\phi, \tau\} = \phi \). Since \( C \) decreases in \( s \) (which is straightforward to prove using the expression in (4)), higher \( s^* = \phi \) results in lower \( C^* \). Hence, it is optimal to set \( s^* = \phi = \tau \) in the second region. Again, we have \( s^* = \tau \) and therefore the social cost is reduced to \( C^* = \frac{\lambda}{\mu} (r - \frac{\lambda \mu}{2}) + \left( 1 - \frac{\lambda}{\mu} \right) \frac{\lambda}{\tau} \). Since this function decreases in \( \tau \) for \( \tau < \sigma \), by the similar argument as above the minimum does not exist in \( \tau < \sigma \), thus confirming the statement we set out to prove. Now assume \( \tau \geq \sigma \). Since \( 0 \leq \phi \leq \sigma \), we have \( \phi \leq \sigma \leq \tau \) in this case. The latter condition implies \( s^* = \min \{\phi, \tau\} = \phi \). Since \( C^* \) decreases in \( s^* = \phi \) as we observed above, \( C^* \) is minimized at the upper bound of \( \phi \), i.e., \( s^* = \phi = \sigma \). Hence, the equilibrium exists in the region \( \tau \geq \sigma \) and it satisfies \( s^* = \phi = \sigma \), where \( \kappa_e = 0 \) and \( \kappa_d = K \). ■
Lemma B.3 (For the proof of Proposition 2) Let \( \varphi(x) = \frac{1}{x^2} \left( (2 - \beta)e^x - 2 - 2x - x^2 \right) (3 - x) \) where \( 0 \leq \beta \leq 2 \). Then \( \varphi(x) < 1 \) for all \( x > 0 \).

**Proof.** Suppose \( \beta = 0 \). Repeated application of l'Hopital's rule yields \( \lim_{x \to 0} \varphi(x) = 1 \) and \( \lim_{x \to -\infty} \varphi(x) = -\infty \). It can also be proved that \( \varphi'(x) < 0 \). Since \( \varphi(x) \) starts from one at \( x = 0 \) and decreases to \( -\infty \) as \( x \to \infty \), we conclude \( \varphi(x) < 1 \) for all \( x > 0 \) if \( \beta = 0 \). Now suppose \( 0 < \beta \leq 2 \). Observe that the numerator of \( \varphi(x) \) is equal to \(-3\beta < 0 \) at \( x = 0 \). Hence, \( \lim_{x \to 0} \varphi(x) = -\infty \). Applying l'Hopital's rule repeatedly, we get \( \lim_{x \to -\infty} \varphi(x) = -\infty \). Since \( \varphi'(x) = \frac{1}{x^3} \left( 18 + 8x + x^2 - (2 - \beta) \left( 9 - 5x + x^2 \right) e^x \right) \), we see that \( \varphi'(x) = 0 \) is equivalent to \( \omega(x) = 2 - \beta \) where \( \omega(x) = \frac{x^2 + 8x + 18}{x^2 - 5x + 9} e^{-x} \). Note \( \omega(0) = 2 \), \( \lim_{x \to -\infty} \omega(x) = 0 \), and \( \omega'(x) = -\frac{3(x+3)e^{-x}}{(x^2 - 5x + 9)^2} < 0 \). Since \( \omega(x) \) decreases from \( 2 \) to zero as \( x \) goes from zero to infinity and \( 0 < \beta \leq 2 \), there is exactly one solution to \( \omega(x) = 2 - \beta \). This implies that there is exactly one \( x \) that solves \( \varphi'(x) = 0 \), i.e., exactly one critical point of \( \varphi(x) \) exists. Given that \( \lim_{x \to 0} \varphi(x) = \lim_{x \to -\infty} \varphi(x) = -\infty \), this critical point is a maximizer, which we denote as \( \hat{x} \). Evaluating \( \varphi(x) \) at this point yields \( \varphi(\hat{x}) = \frac{1}{\hat{x}^2} \left( \frac{25x + 8x^2 + 18}{x^2 - 5x + 9} - 2 \hat{x} - \hat{x}^2 \right) (3 - \hat{x}) = \frac{9(2 - 3\hat{x})}{(2 - 3\hat{x})^2 + 3} < 1 \). Since the maximum of \( \varphi(x) \) is smaller than one, we conclude \( \varphi(x) < 1 \) for all \( x > 0 \) if \( 0 < \beta \leq 2 \). \( \blacksquare \)

**Proposition B.1** (For §6.1) With detection replaced by late disclosure, changes are made under periodic inspections but not under random inspections. Specifically, \( \overline{C} \) in (4) under periodic inspections is replaced by \( \frac{\lambda e^{\mu s} - e^{\mu \tau}}{\tau} \), and as a result, the equilibrium condition for \( \tau \) in Proposition 2 is modified to \( G(\tau) + e^{-\mu \tau} - (1 + \mu \tau) e^{-\mu \tau} = \frac{\xi}{\lambda} - 1 \). In equilibrium, the regulator sets \( \kappa_e = 0 \) and \( \kappa_l = \kappa_d = K \). None of the other statements in Propositions 2-4 are changed.

**Proof.** Recall from Lemma 1 that the firm never discloses noncompliance late under random inspections; hence, no change is made in this case. Consider periodic inspections. With detection replaced by late disclosure, two changes are made: the firm incurs the late disclosure penalty \( \kappa_d \) instead of the detection penalty \( \kappa_l \), and the event is not counted as an inspection. The first change does not alter the firm’s disclosure behavior except that \( \kappa_d \) appearing in \( s^* \) is replaced by \( \kappa_l \); hence, \( s^* = \min \left\{ \frac{1}{\mu} \left[ \ln \frac{r + \kappa_l}{r + \kappa_d \mu}, \tau \right] \right\} \). To account for the second, we observe from the analysis of §A in the Technical Appendix that the expected number of inspections performed in an inspection cycle is changed to \( E[I] = \Pr(1a) + \Pr(2b) = e^{-\lambda s} + \frac{\lambda e^{-\lambda s} - \mu e^{-\mu s}}{\mu - \lambda} \) and \( H(\tau) = e^{-\mu \tau} - (1 + \mu \tau) e^{-\mu \tau} = \frac{\xi}{\lambda} - 1 \). Note \( H' = e^{-\mu \tau} - (1 + \mu \tau) e^{-\mu \tau} \) and \( H'' = e^{-\mu \tau} - (1 + \mu \tau) e^{-\mu \tau} \). Note \( H'(\tau) = e^{-\mu \tau} - (1 + \mu \tau) e^{-\mu \tau} > 0 \) with \( H(0) = e^{-\mu \tau} - 1 \leq 0 \) and \( \lim_{\tau \to -\infty} H(\tau) = e^{-\mu \tau} > 0 \). Combining these properties of \( H(\tau) \) with that of \( G(\tau) \) shown in the proof of Proposition 2, we find that the left-hand side of the first-order condition, \( G(\tau) + H(\tau) \), crosses the right-hand side \( \frac{\xi}{\lambda} - 1 \) exactly once from below. Following the same argument as that in the proof of
Proposition 3, we also find that the statement in Proposition 3 is not altered by the addition of $H(\tau)$ in the first-order condition. Finally, since $\overline{C}$ with $s^p = \sigma$ (see Proposition 2) is lowered at each $\tau > 0$ with the additional term $-\frac{\lambda e^{-\mu \tau} - e^{-\mu \tau}}{\tau} < 0$ identified above, the equilibrium value $\overline{C}^p$ is also lowered. This implies that the first statement in Proposition 4, i.e., $\overline{C}^p < \overline{C}^r$ for sufficiently small $K$, remains true. The second statement, i.e., $\overline{C}^p > \overline{C}^r$ for sufficiently large $K$, also remains true because once full disclosure is induced under both inspection policies with a large $K$, no late disclosure occurs; in this case, the analysis does not change because there is no replacement of detection with late disclosure.

References