Supply Chain Contracting with Quality Choice

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Abstract

We analyze a firm designing and selling a seasonal product with demand uncertainty and a single ordering opportunity. Prior to the start of the selling season, product quality and inventory must be jointly determined; a higher quality product results in a greater selling price but also greater marginal production cost. We consider both a centralized supply chain, in which a single firm determines quality and inventory, and a decentralized supply chain, in which a manufacturer determines product quality and a retailer determines inventory. In a centralized supply chain, we provide a simple characterization of the optimal inventory and quality, and demonstrate that at the optimal inventory-quality pair the sales function elasticity is equal to the reciprocal of the cost function elasticity. In a decentralized supply chain, we demonstrate that standard wholesale price contracts cannot coordinate the supply chain in inventory or quality, and discuss the impact of these contracts on quality and inventory levels; interestingly, we show that quality is often higher under wholesale price contracts than in a centralized system due to substitutability with inventory, despite the fact that double marginalization reduces manufacturer incentives to invest in quality, all else being equal. We also show that, surprisingly, standard revenue sharing contracts can coordinate the supply chain in quality choice in some cases; however, this requires truthful sharing of the manufacturer’s cost structure, which we demonstrate is not in the manufacturer’s interests to reveal. We conclude that simultaneously coordinating product design and inventory in decentralized supply chains, while feasible, is likely to be challenging in practice, perhaps helping to explain the value of the recent surge of “direct-to-consumer” distribution that avoids coordination problems by having the same firm design and sell seasonal goods.

Keywords: supply chain coordination; contracting; product design; quality choice; revenue sharing; cost sharing.

1 Introduction

In recent years, a number of start-up firms have achieved significant success in the apparel industry by bypassing traditional distribution channels (such as department stores) and adopting a “direct-to-consumer” model. Examples include Bonobos, Everlane, Indochino, Lululemon, and Warby Parker (Mount 2013). While these firms sell a wide variety of goods ranging from apparel to

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prescription eyeglasses, they share a common strategy: they design and sell their own products. Indeed, while technological advances (e.g., the e-commerce capabilities of the internet) have, in recent years, lowered the fixed costs associated with operating a direct-to-consumer supply chain, this strategy is not limited to recent start-ups or small firms: it is also followed by the largest apparel retailer in the world, Zara, as well as many other successful retailers such as American Apparel, H&M, and Uniqlo (Rohwedder & Johnson 2008; Negishi et al. 2013).

Motivated by the success of these “direct-to-consumer” apparel firms, in this paper we seek to understand the implications of this practice. While there are numerous conjectures about the consequences and benefits of direct sales channels, ranging from manufacturers avoiding draconian rules and contracts with large retailers (Mount 2013) to maintaining a closer relationship with consumers and understanding their needs better (Miller & Clifford 2013), we focus on one particular effect: coordinating product design and operational decisions. Drawing from the preceding examples in the apparel and accessory industries, we study firms selling seasonal fashion goods with long manufacturing leadtimes relative to the selling season. With such products, supply-demand mismatch is a frequent problem, leading to stock-outs and unsold inventory. While significant attention has been paid in the literature to coordinating supply chain inventory decisions in such a setting (Cachon 2003), there has been far less focus on coordinating product design decisions in decentralized supply chains with seasonal or perishable items.

In this paper, we explore this topic by analyzing a stylized model in which a firm sells a seasonal good under newsvendor dynamics. We modify this classic inventory framework by including a key product design decision: value-enhancing product quality. All else being equal, consumers value quality and are willing to pay more for a higher quality product; however, quality comes at the expense of increased marginal manufacturing costs (e.g., the quality of fabric in an article of clothing). We thus merge a classic inventory model (the newsvendor) with a quality choice model in which a higher quality product leads to a higher selling price, but also a higher marginal production cost. Because quality directly impacts both cost and price, it clearly interacts with optimal inventory decisions that, in the newsvendor framework for seasonal goods, depend crucially on the cost of having too little inventory (the lost margin on a sale) and the cost of having too much inventory (the loss on each unsold unit). In what follows, we explore this interaction, focusing on three primary research questions:

1. How should a firm who sells a product in a centralized supply chain optimally choose product quality and inventory?

2. When using common wholesale price contracts, what are the equilibrium quality and inventory in a decentralized supply chain, in which a manufacturer determines product quality and a
retailer determines inventory? Do quality and inventory increase or decrease as a result of decentralization?

3. Is it possible to coordinate both product quality and inventory in a decentralized supply chain (i.e., to achieve the centralized system optimal values, maximizing total supply chain profit)?

We characterize the optimal centralized decisions via a simple system of two equations, discuss the behavior and features of the optimal joint inventory and quality levels, and offer a novel economic interpretation of these results based on how the sales and cost functions interact with each other. We demonstrate that, as one might expect, a decentralized supply chain using a simple wholesale price contract fails to achieve coordination along either the inventory or quality dimensions: supply chain profit could always be increased by adopting higher quality or higher inventory. Surprisingly, we show that simple revenue sharing contracts (and, by extension, buyback contracts)—well known to coordinate the supply chain when quality is exogenous—can coordinate the supply chain with endogenous quality under some conditions, despite the fact that an additional decision has been introduced to the supply chain; however, we demonstrate that this requires having the manufacturer truthfully reveal its cost information, which is not in the manufacturer’s best interests. Hence, we argue that coordination in this setting, while feasible, is challenging from a practical perspective, thus suggesting that the rise of “direct-to-consumer” firms has some inherent value in overcoming the quality-and-inventory coordination problem. Using extensive numerical experiments, we show that wholesale price contracts result in supply chain inefficiency (reduction in total profit) of, on average, roughly 23%, suggesting that the benefits of coordination are potentially significant.

2 Related Literature

We consider the simultaneous choice of product quality and inventory in centralized and decentralized supply chains. Consequently, there are three streams of research that bear particular relevance to this paper: the literature on vertical product quality, the literature on decentralized supply chain inventory coordination, and the literature that looks at the combination of these two decisions.

Product Quality Literature. There is extensive work in marketing, economics, and operations that examines the trade-off between product quality and production costs. Included in this stream is the literature on market segmentation, product differentiation, and product line design (e.g., Mussa & Rosen 1978, Moorthy 1984, Moorthy & Png 1992, Krishnan & Zhu 2006, Heese & Swaminathan 2006, Netessine & Taylor 2007, Chayet et al. 2011, and others; see also Tirole 2003). While much of this literature considers the optimal design of a product line to serve multiple heterogeneous customer segments, we focus instead on the quality decision for a single product intended to serve
a single, homogenous customer segment—however, in our model the product is seasonal and the size of the market is uncertain, necessitating a non-trivial inventory decision. In addition, we explore coordination of quality and inventory decisions in a decentralized supply chain while the vast majority of the preceding papers consider optimal quality decisions by a centralized firm.

Supply Chain Inventory Coordination Literature. This area of research focuses on coordinating inventory decisions when product design (or quality) has been exogenously determined. Cachon (2003) provides an extensive overview. Of particular relevance is the literature analyzing optimal decisions using the wholesale price contract, which is known to be suboptimal but is of research interest due to its simplicity and pervasiveness in practice. Lariviere & Porteus (2001) derive the optimal wholesale price contract (from the manufacturer’s point of view) when selling to a newsvendor retailer who assumes all inventory risk. We similarly derive the optimal wholesale price contract from the manufacturer’s perspective, but add quality to the decisions made by the upstream supply chain member. Other relevant works in this stream include Gerchak & Wang (2004) and Bernstein & DeCroix (2004). We also examine revenue sharing and buyback contracts, two contractual types that are well known to coordinate supply chains in the absence of product quality considerations; such contracts have been extensively explored by Pasternack (1985), Bernstein & Federgruen (2005), Cachon & Lariviere (2005), and many others.

Quality and Inventory Decisions in Decentralized Supply Chains. There is some existing work (Jeuland & Shugan 1983; Villas-Boas 1998; Economides 1999; Matsubayashi 2007) that has examined the impact of decentralization on vertical product quality choice, typically demonstrating that decentralization and the associated double marginalization reduces product quality. In contrast, Xu (2009) shows that the shape of the marginal revenue function, which depends on consumer valuations, is key to determining whether quality increases or decreases upon decentralization. Shi et al. (2013) demonstrate that if consumer valuations follow non-uniform distributions or if consumers are two-dimensionally heterogeneous, it’s possible for product quality to increase as a result of decentralization. The papers in this stream do not focus on seasonal goods and hence do not model supply-demand mismatch, a key feature of our model.

Recently, there has been growing interest in the topic of design and manufacturing outsourcing, in which a buyer (a brand like IBM) outsources the quality (or cost) decision on a product to a contract manufacturer (e.g., Flextronics)—see Zhu et al. (2007), Gray et al. (2009), Kaya & Özler (2009), Kim & Netessine (2013), Wang & Shin (2013a), Wang & Shin (2013b), and Druehl & Raz (2013). In some of these papers the term “quality” means vertical product quality leading to higher consumer willingness to pay, as we assume, while in some it means conformance quality, i.e., the number of products that are defect-free. Perhaps the most closely related of these papers to
our own are Bernstein & Kök (2009) and Bhaskaran & Krishnan (2009), both of whom emphasize the importance of cost sharing, which also plays a key role in our analysis, in achieving supply chain coordination; however, the former focuses on process improvement while the latter focuses on collaborative innovation and new product development, two very different settings than our own. There are two key differences between our model and this overall stream of work: first, in contrast to our model, in which a powerful manufacturer chooses quality and attempts to induce a retailer to choose an appropriate level of inventory, these papers typically focus on a powerful buyer attempting to induce its supplier to choose an appropriate level of quality. Hence, our model is more appropriate for a comparison of direct-to-consumer and retail distribution channels, while the outsourcing literature is more appropriate for analysis of in-house or outsourced manufacturing channels. Second, we explicitly model quality as increasing marginal production costs rather than increasing fixed costs, as many papers in this stream assume; hence, our model is more appropriate for product selection rather than product innovation.

3 Model

A firm (the “manufacturer”) designs and manufactures a single seasonal product. The product is either sold directly to consumers by the manufacturer (the “centralized” supply chain) or sold to a retailer who sells to consumers (the “decentralized” supply chain). The product is sold over a single selling season and, at the conclusion of the season, excess inventory is salvaged for zero revenue. Design and production leadtimes are sufficiently long that the manufacturer must design the product and determine inventory levels in advance of the revelation of demand information, and there is no opportunity to procure additional inventory after the selling season begins. Consequently, the model setup is that of a single period newsvendor with zero salvage value, and is reflective of seasonal or short lifecycle consumer products such as apparel or seasonal sporting goods.

Prior to the start of the selling season, two decisions must be made concerning this product:

1. The product quality level $\theta$.

2. The inventory level $q$.

Thus, we augment the classical newsvendor model with an additional decision, the choice of quality $\theta$. In the context of this paper, we define quality as an attribute of the product that increases the

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1Positive salvage value may be readily introduced to the model, although the resulting analysis is more cumbersome, and analytically proving that the equilibrium is unique in the decentralized supply chain is challenging (though we have numerically observed all results continue to hold with non-zero salvage value). Hence, for ease of exposition, we assume zero salvage value.
consumers’ willingness-to-pay. This may include: new product features, enhanced performance of existing features, improved product durability by upgrading raw materials or components, or improved quality of the manufacturing process (e.g., handcrafted items versus mass produced items). While increased willingness-to-pay allows the firm to charge a higher price, it also requires a higher manufacturing cost. As a result, when choosing the quality of a product, firms must strike a balance between consumer utility and marginal production cost (Mussa & Rosen 1978; Krishnan & Zhu 2006; Netessine & Taylor 2007).

We model the quality-induced price and product cost increases as follows. For simplicity, we assume that the price is linearly increasing in quality: the price at a given quality $\theta$ is equal to $p_0 \theta$, where $p_0$ denotes a constant price coefficient. This modeling choice is consistent with our definition of quality as equivalent to consumers’ willingness-to-pay. To be more precise, it represents the valuation of the product among a single consumer segment with homogenous preferences. The linearity assumption is consistent with much of the literature on quality choice (see, for instance, Moorthy 1984, Tirole 2003, and Netessine & Taylor 2007) and is without loss of generality, since one can always map $\theta$ to a nonlinear scale via a monotonic transformation with no material impact on any structural results; the only important consideration is how quality impacts the production cost relative to price.

The unit production cost is an increasing function of product quality, $c(\theta)$. It is assumed to be twice differentiable and its elasticity—the percentage change in unit production cost with respect to a percentage change in quality—is denoted by $\epsilon(\theta) \equiv \frac{\theta c'(\theta)}{c(\theta)}$. (1)

We make the following assumptions about the production cost curve:

**Assumption 1.** The unit production cost $c(\theta)$ and its elasticity $\epsilon(\theta)$ have the following properties:

(i) $c'(\theta) > 0$; $c''(\theta) > 0$; $\lim_{\theta \to 0} c(\theta) = \lim_{\theta \to 0} c'(\theta) = 0$; $\lim_{\theta \to \infty} c(\theta) = \lim_{\theta \to \infty} c'(\theta) = \infty$.

(ii) $\epsilon'(\theta) \geq 0$; $\lim_{\theta \to \infty} \epsilon(\theta) > 1$.

The first part of the assumption states that $c(\theta)$ is convex increasing and that zero quality can be had at no cost while attaining infinite quality is infinitely costly. Combined with the fact that the selling price linearly increases in $\theta$, these assumptions on the cost of quality capture the plausible scenario of decreasing marginal returns on quality, and hence, an infinite amount of investment in quality is never optimal. The assumptions on elasticity in the second part are made to facilitate the analysis, but they are not restrictive.\[^2\]

\[^2\]In fact, the last condition $\lim_{\theta \to \infty} \epsilon(\theta) > 1$ follows from other assumptions; by l’Hôpital’s rule, $\lim_{\theta \to \infty} \epsilon(\theta) = \infty$.
While many representative cost functions satisfy all of the conditions listed in Assumption 1, throughout the paper we pay special attention to the *isoelastic unit cost function* \( c(\theta) = c_0 \theta^n \), defined for \( n > 1 \). We refer to the constant \( c_0 \) appearing in this expression as the *unit cost coefficient*. This function brings both versatility (i.e., varying degrees of quality cost are represented by \( n \)) and simplicity, enabled by the fact that its elasticity is independent of \( \theta \), a decision variable: \( \epsilon(\theta) = n \). For similar reasons many authors in the literature have adopted this function in their analyses, especially the quadratic function \( (n = 2) \); see Moorthy & Png (1992) and Netessine & Taylor (2007), for example.

Because the unit cost function is convex in \( \theta \) while price is linearly increasing in \( \theta \), an excessive quality level will reduce the profit margin to zero or negative. To rule out such a scenario, we restrict attention to the range of \( \theta \) that guarantees a positive margin for the supply chain. This range is defined by the upper bound \( \bar{\theta} > 0 \), which is the unique solution to the equation \( c(\theta)/\theta = p_0 \). (It can be shown that such a solution exists under Assumption 1.) Any quality of value \( \theta > \bar{\theta} \) leads to a negative margin. In Lemma A1 found in the Appendix we provide a number of additional properties of \( c(\theta) \) in the range \( \theta \in (0, \bar{\theta}) \) that we use throughout the paper.

The total market size during the selling season is \( D \), a positive random variable with mean \( \mu = \mathbb{E}[D] \), distribution function \( F \), density \( f \), and support on \((0, \bar{D})\), where \( \bar{D} \) may be infinite. For most of our analysis, market size is assumed to be exogenous and independent of product quality. In fact, our model and results extend to the case where quality determines market size (under certain conditions); we discuss this extension and its implications in §7. Let \( \bar{F} \) denote the complement of \( F \). We assume that the demand distribution has an increasing generalized failure rate (IGFR, see Lariviere 2006), i.e., \( g'(x) > 0 \) where \( g(x) \equiv xf'(x)/F(x) \). Let \( S(x) \equiv \mathbb{E}[\min\{D, x\}] \) be the expected sales or the *sales function*, whose elasticity is denoted by \( \eta(x) \), i.e.,

\[
\eta(x) = \frac{xS'(x)}{S(x)} = \frac{xF(x)}{\int_0^x F(y) \, dy}.
\]

This function plays a central role in our analysis, and a number of its useful properties are derived in Lemma A2 found in the Appendix.

4 Centralized Supply Chain

We begin our analysis with a centralized supply chain in which the manufacturer both designs the product and sells to consumers. Consequently, the manufacturer has decision rights for both

\[
\lim_{\theta \to \infty} \frac{\theta c''(\theta)}{c'(\theta)} = \frac{c'(\theta) + \theta c''(\theta)}{c'(\theta)} = 1 + \frac{\theta c''(\theta)}{c'(\theta)} > 1.
\]
quality and inventory. Due to uncertainty in the market size $D$, demand-supply mismatch may occur, leading to one of two possible scenarios. If $q < D$, the firm sells out during the selling season. If $q > D$, all demand is satisfied during the selling season, and $q - D$ units are scrapped for zero salvage value at the end of the season. Hence, anticipating these events, the firm determines the quality $\theta$ and quantity $q$. Its expected profit is equal to

$$\Pi^c(\theta, q) = p_0 \theta S(q) - c(\theta)q,$$

where the superscript $c$ denotes a centralized supply chain. The following proposition identifies the unique optimal quality-inventory pair and establishes their relationship:

**Proposition 1.** The centralized supply chain’s optimal inventory-quality pair $(q^c, \theta^c)$ is the unique solution to the system of equations

$$F(q) = \frac{c(\theta)}{p_0},$$

$$\frac{S(q)}{q} = \frac{c'(\theta)}{p_0}.$$ 

Moreover, $q^c$ and $\theta^c$ are substitutes.

**Proof.** Differentiating $\Pi^c = p_0 \theta S(q) - c(\theta)q$ with respect to $q$ and $\theta$ yields $\frac{\partial \Pi^c}{\partial q} = p_0 \theta F(q) - c(\theta)$ and $\frac{\partial \Pi^c}{\partial \theta} = p_0 S(q) - c'(\theta)q$. In Assumption 1 and Lemma A1 we established $\lim_{\theta \to 0} c'(\theta) = 0$, $c(\theta) / \theta < p_0$ for all $\theta \in (0, \bar{\theta})$, and $\lim_{\theta \to \bar{\theta}} c'(\theta) > p_0$. Using these results together with $\lim_{q \to 0} F(q) = 1$ and $\lim_{q \to D} F(q) = 0$, we then have the following limiting values: (i) $\lim_{q \to 0} \frac{\partial \Pi^c}{\partial q} = p_0 \theta - c(\theta) > 0$; (ii) $\lim_{q \to D} \frac{\partial \Pi^c}{\partial q} = -c(\theta) < 0$; (iii) $\lim_{\theta \to 0} \frac{\partial \Pi^c}{\partial \theta} = p_0 S(q) > 0$; (iv) $\lim_{\theta \to \bar{\theta}} \frac{\partial \Pi^c}{\partial \theta} = p_0 S(q) - q \lim_{\theta \to \bar{\theta}} c'(\theta) < p_0 (E[\min\{D, q\}] - q) \leq 0$. These inequalities imply that a maximizer exists in the interior. Let $(\hat{q}, \hat{\theta})$ be the maximizer. These values should satisfy the first-order conditions $\frac{\partial \Pi}{\partial q} = 0$ and $\frac{\partial \Pi}{\partial \theta} = 0$ or

$$p_0 \hat{\theta} F(\hat{q}) = c(\hat{\theta}) \text{ and } p_0 S(\hat{q}) = c'(\hat{\theta}) \hat{q}.$$ 

Dividing the first equation by the second equation yields $\eta(\hat{q}) \epsilon(\hat{\theta}) = 1$. It is shown in Lemma A2 that $\eta(x) < 1$ for all $x \in (0, \bar{D})$; hence, $\epsilon(\hat{\theta}) > 1$. The Hessian $H$ of $\Pi^c$ has the following components (subscripts 1 and 2 correspond to $q$ and $\theta$, respectively): $H_{11} = -p_0 \theta f(q)$, $H_{12} = 0$. 

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$H_{21} = p_0 F(q) - c'(\theta)$, and $H_{22} = -c''(\theta) q$. Evaluating these at the maximizer $(\hat{q}, \hat{\theta})$ yields

$$
\hat{H} = \begin{bmatrix}
-c(\hat{\theta}) f(\hat{q}) / F(\hat{q}) & -c(\hat{\theta}) [\epsilon(\hat{\theta}) - 1] / \hat{\theta} \\
-c(\hat{\theta}) [\epsilon(\hat{\theta}) - 1] / \hat{\theta} & -c''(\hat{\theta}) \hat{q}
\end{bmatrix}.
$$

Its determinant is then $\det \hat{H} = c(\hat{\theta}) c''(\hat{\theta}) g(\hat{q}) - \frac{c(\hat{\theta})^2}{\theta^2} [\epsilon(\hat{\theta}) - 1]^2$. Note that, since $c'(\theta) = \frac{c(\theta) \epsilon(\theta)}{\theta}$,

$$
c''(\theta) = \frac{\theta c(\theta) c'(\theta) + \theta c'(\theta) \epsilon(\theta) - c(\theta) \epsilon(\theta)}{\theta^2} = \frac{\theta c(\theta) c'(\theta) + c(\theta) \epsilon(\theta)^2 - c(\theta) \epsilon(\theta)}{\theta^2} \geq \frac{c(\theta) \epsilon(\theta)}{\theta^2} [\epsilon(\theta) - 1],
$$

where we used the condition $\epsilon'(\theta) \geq 0$ given in Assumption 1. Using this inequality, we get

$$
\det \hat{H} = c(\hat{\theta}) c''(\hat{\theta}) g(\hat{q}) - \frac{c(\hat{\theta})^2}{\theta^2} [\epsilon(\hat{\theta}) - 1]^2 \geq \frac{c(\hat{\theta})^2}{\theta^2} \epsilon(\hat{\theta}) [\epsilon(\hat{\theta}) - 1] \left( g(\hat{q}) - \frac{\epsilon(\hat{\theta}) - 1}{\epsilon(\hat{\theta})} \right)
$$

$$
\geq \frac{c(\hat{\theta})^2}{\theta^2} \epsilon(\hat{\theta}) [\epsilon(\hat{\theta}) - 1] [g(\hat{q}) + \eta(\hat{q}) - 1] > 0,
$$

where the second inequality follows from $\epsilon(\hat{\theta}) > 1$ we found above and the inequality $g(x) + \eta(x) > 1$ proved in Lemma A2. Since $\hat{H}_{11} < 0$, $\hat{H}_{22} < 0$, and $\det \hat{H} > 0$, the maximizer is unique. Finally, to prove that $\hat{q}$ and $\hat{\theta}$ are substitutes, observe that, since $\epsilon(\hat{\theta}) > 1$, the cross partial $\hat{H}_{12}$ satisfies $\hat{H}_{12} = -\frac{c(\hat{\theta})}{\theta} [\epsilon(\hat{\theta}) - 1] < 0$. This implies that $\frac{\partial \hat{q}}{\partial \theta} = -\frac{\hat{H}_{12}}{\hat{H}_{11}} < 0$, i.e., $\hat{q}$ and $\hat{\theta}$ are substitutes. \hfill \Box

Notice that equation (4) determines the optimal inventory level for a given level of quality $\theta$ via the usual newsvendor fractile solution. Equation (5), on the other hand, determines the optimal quality level for a given level of inventory $q$. This equation states that at the optimum, the ratio $c'(\theta)/p_0$—the relative marginal increase in the unit cost against the marginal increase in price due to higher quality—should exactly match the ratio $S(q)/q$, i.e., the sales conversion rate (the fraction of inventory eventually sold). Together, these equations specify which combination of quality and inventory levels should be chosen by the centralized firm to maximize profit.

While the two equations (4) and (5) together provide a linkage between the optimal choices of quality and inventory levels, a cleaner relationship emerges after we combine the two. Observe that dividing (4) by (5) yields

$$
\eta(q^c) \epsilon(\theta^c) = 1.
$$

(6)

In other words, the sales function elasticity $\eta(q)$ should be equal to the reciprocal of the unit cost function elasticity $\epsilon(\theta)$ at the optimum.\footnote{That the magnitudes of these two values are equal originates from the mirror symmetry built in the problem. Examining the firm profit function (3), we see that a unit increase in quality $\theta$ leads to a linear increase in the revenue function while a unit increase in inventory $q$ leads to a linear reduction in the cost function. These changes...
and it suggests that the firm that simultaneously determines quality and inventory does so while maintaining a constant level of joint elasticity (i.e., the product of $\eta(q)$ and $\epsilon(\theta)$) embedded in its profit function.

Another interesting observation from the equation (6) is that the firm always chooses the quality level at an elastic part of the unit cost curve, i.e., $\epsilon(\theta^c) > 1$. This is true because the sales function, by definition, is always inelastic: $\eta(q) < 1$ for any $q > 0$. To see this, observe that $\eta(q)$ defined in (2) can be written as

$$\eta(q) = \frac{qF'(q)}{\int_0^q x f(x) \, dx + qF(q)},$$

which is clearly smaller than one. Written this way, we see that the sales function elasticity is equal to the fraction of expected sales that occur in the event of a stockout, i.e., $qF'(q)$. Therefore, optimal quality is always set at a level where a percentage increase in quality leads to an increase in the unit cost by more than a percent.

Proposition 1 also states that optimal quality and inventory levels are substitutes. This is a direct consequence of our last finding that optimal quality is set at an elastic part of the unit cost function. To see this, consider adjusting the newsvendor fractile in the equation (4) by increasing the quality level $\theta$:

$$\frac{d}{d\theta} \left( \frac{c(\theta)}{p_0 \theta} \right) = \frac{c(\theta)}{p_0 \theta^2} [\epsilon(\theta) - 1].$$

Since $\epsilon(\theta^c) > 1$, this implies that the critical ratio appearing in the newsvendor formula $F(q) = 1 - c(\theta)/p_0 \theta$ is decreasing in $\theta$. To match this change the firm has to reduce the in-stock probability $F(q)$, which is accomplished by lowering the inventory level $q$. Hence, higher quality results in lower inventory. Viewed this way, one can frame the firm’s joint optimization problem as a modified newsvendor model with endogenous determination of the optimal critical ratio.

The elasticity requirement $\epsilon(\theta^c) > 1$ is automatically satisfied by the isoelastic unit cost function $c(\theta) = c_0 \theta^n$ with $n > 1$. That is, this convex function is elastic everywhere. With this function, we can combine (4) and (6) to derive closed-form solutions for optimal inventory and quality levels:

$$q^c = \eta^{-1}\left(\frac{1}{n}\right),$$

$$\theta^c = \left(\frac{p_0}{c_0} F(q^c)\right)^{-\frac{1}{n-1}}.$$
We make several interesting observations about these optimal values, as summarized below.

**Corollary 1.** In a centralized supply chain with the isoelastic unit cost function $c(\theta) = c_0\theta^n$:

(i) The optimal inventory $q^c$ is independent of $p_0$ and $c_0$.

(ii) The optimal quality $\theta^c$ increases in the ratio $p_0/c_0$, and it does not exceed $\left(\frac{p_0}{nc_0}\right)^{1/(n-1)}$.

(iii) The stockout probability $\overline{F}(q^c)$ at the optimum does not exceed $1/n$.

(iv) If the demand distribution is symmetric or its median is greater than its mean, then $q^c > \mu$ for all $n \geq 2$.

**Proof.** In Lemma A2 we proved $\overline{F}(x) < \eta(x)$. Hence, $\overline{F}(q^c) < \eta(q^c) = 1/n$. Moreover, $\frac{c_0}{p_0} (\theta^c)^{n-1} = \overline{F}(q^c) < \eta(q^c) = \frac{1}{n}$ implies $\theta^c < \left(\frac{p_0}{nc_0}\right)^{1/(n-1)}$. (iv) follows directly from (iii). \qed

Part (i) of Corollary 1 states that the optimal inventory level is determined solely by the elasticities of the sales and unit cost functions, if the latter is isoelastic; the coefficients $p_0$ and $c_0$ do not impact the inventory decision. On the other hand, these coefficients determine the optimal quality, as part (ii) shows. Thus, the relative scale of price and unit cost—represented by the ratio $p_0/c_0$—impacts the quality choice but not the inventory choice. This is to be contrasted to the classical newsvendor model where this ratio appears in the firm’s optimal inventory decision. Adding a quality choice has the effect of separating the roles played by scale coefficients and the elasticities, the former impacting the quality choice more and the latter the inventory choice more.

Part (iii) reaffirms substitutability between optimal quality and inventory. The upper bound $1/n$ on the stockout probability suggests that higher cost of improving quality (greater $n$) leads the firm to adjust the quality-inventory combination so as to compensate the limited pricing ability caused by quality loss with more inventory, thereby increasing product availability. In effect, the firm resorts to “flood the market” strategy when it is unable to offer a high-quality product.

Lastly, part (iv) reveals how the firm’s quality choice determines the newsvendor critical ratio. Stating this result a different way, the optimal critical ratio (in-stock probability) is always greater than $(n-1)/n$. If $n \geq 2$, this means the optimal critical ratio is always greater than $1/2$; hence, it is optimal for a firm facing quadratic (or greater) quality costs to adjust the quality level such that the underage cost exceeds the overage cost. If the demand distribution is symmetric or right skewed, as many commonly used demand distributions are, this implies it is optimal for the firm to produce more units than the expected demand.

It is illustrative to compare the results in Corollary 1 with the special case of deterministic demand. If demand is perfectly known to be equal to $D = \mu$, the firm produces exactly the amount that satisfies demand: $\mu$. The firm’s profit function is then $(p_0\theta - c_0\theta^n) \mu$, and the resulting optimal

\footnote{More accurately, if the median is greater than the mean.}
quality is \((p_0/n c_0)^{1/(n - 1)}\). Notably, with deterministic demand, the optimal quality and inventory are independent of one another; inventory depends solely on the market size \(\mu\), whereas quality is a function of the production cost parameters \(c_0\) and \(n\), and the price coefficient \(p_0\). Combined with part (ii) of Corollary 1, this demonstrates that optimal quality with stochastic demand is less than the optimal value with deterministic demand. This is because the benefit of quality (an increase in price) is earned on each unit sold, whereas the cost of quality (manufacturing cost) is paid on each unit produced; with stochastic demand, sales are less than production, hence quality has less value than with deterministic demand.

Moreover, if \(n \geq 2\) and the demand distribution is symmetric or right skewed, it is guaranteed that the optimal inventory with stochastic demand is greater than the optimal inventory with deterministic demand—that is, the product is designed such that positive safety stock is optimal.\(^6\) Thus, interestingly, while it is well known that negative safety stock is possible in the classical newsvendor model even with symmetric demand distributions (such as normal), if quality costs are sufficiently elastic, such a product has been improperly designed: firm profit would be increased with a lower quality product and a higher inventory level.

In addition, we make the following observation:

**Corollary 2.** In a centralized supply chain with the isoelastic unit cost function \(c(\theta) = c_0 \theta^n\), if demand is stochastic but the firm (sub-optimally) chooses the deterministic optimal product quality, \((p_0/n c_0)^{1/(n - 1)}\), the product margin is maximized and the optimal in-stock probability is \(F(q) = (n - 1)/n\).

In other words, if the firm fails to take demand uncertainty into account when making product design decisions, then it selects a quality level that is too high and an inventory level that is too low. In addition, it will maximize the margin of the product; this necessarily implies that the optimal product quality with stochastic demand is not the quality level that maximizes the margin of the product; hence, shortsighted attempts by management to “maximize the margin” on each sale will result in suboptimal firm profits.

5 **Decentralized Supply Chain with Wholesale Price Contracts**

In the preceding section we assumed that the quality and inventory decisions were made by the same, centralized firm. Many supply chains, however, separate the design and operational decisions, with one firm making the quality decision (a manufacturer) while a different firm makes the inventory

\(^6\)Indeed, even if \(n < 2\) and the distribution is left skewed, it still may be the case that positive safety stock is optimal, although this depends on the problem parameters and is not assured.
decision (a retailer); examples include traditional apparel brands (e.g., Ralph Lauren, Hugo Boss, and Tommy Hilfiger) and department stores (e.g., Macy’s, Nordstrom, and Bloomingdale’s). In this section we consider such a decentralized supply chain and assume that the quality-setting manufacturer sells the products through a inventory-setting retailer via a wholesale price contract. Wholesale price contracting without quality choice has been thoroughly analyzed by Lariviere & Porteus (2001), and it is well-known that the wholesale price contract does not coordinate the supply chain. What is unknown, however, is how quality choice impacts the performance of this contract and how it distorts the inventory decision. In particular, we are interested in comparing the optimal decisions under the wholesale price contract against the results from the centralized supply chain case.

Given the two decisions made in this supply chain, there are multiple degrees of coordination that can occur; see Table 1. If the retailer chooses an inventory level that maximizes total supply chain profit conditional on the manufacturer’s chosen quality level, we say the supply chain achieves inventory coordination. If the manufacturer selects the quality that maximizes total supply chain profit conditional on the retailer’s inventory level, we say the supply chain achieves quality coordination. If both inventory and quality coordination are simultaneously achieved, the supply chain makes the decisions described in Proposition 1, and we say the supply chain achieves full coordination. If neither inventory nor quality coordination are achieved, we say the supply chain achieves no coordination.

Here we make an important observation: note that even if coordination is achieved along one dimension, that does not necessarily imply that the supply chain makes the same decision as the centralized system along that dimension. For instance, the supply chain achieves quality coordination if the manufacturer selects a quality level that maximizes supply chain profit given the retailer’s inventory choice; however, the retailer may choose inventory that differs substantially from the centralized optimal level, in which case the quality that maximizes supply chain profit will differ from the centralized optimal solution. Nevertheless, given the uncoordinated inventory decision, the supply chain could not achieve higher profits by adjusting product quality, so we say the supply chain achieves quality coordination (but not inventory coordination).
With a wholesale price contract, the sequence of events is as follows. First, the manufacturer chooses a quality level \( \theta \), then the manufacturer chooses a wholesale price \( w \). Given these values, the retailer then chooses the inventory level \( q \). The retailer assumes all inventory risk. The manufacturer’s profit is \( \Pi^d_M(w, \theta) = (w - c(\theta))q \), where \( q \) is the retailer’s order quantity, the superscript \( d \) stands for a decentralized supply chain, and the subscript \( M \) stands for manufacturer. The retailer’s expected profit is the usual newsvendor function at procurement cost \( w \), \( \Pi^d_R(q) = p^R S(q) - wq \). The total supply chain profit is the sum of these two expressions, i.e., \( \Pi^d_M + \Pi^d_R = p^R S(q) - c(\theta)q \), which is identical to the profit function analyzed in the centralized case in the preceding section.

In the next proposition, we identify the unique equilibrium of this sequential game.

**Proposition 2.** A unique equilibrium \( (q^d, \theta^d) \) exists that solves the system of equations

\[
\begin{align*}
\bar{F}(q)(1 - g(q)) &= \frac{c(\theta)}{p^0 \theta}, \quad (7) \\
\bar{F}(q) &= \frac{c'(\theta)}{p^0}.
\end{align*}
\]

Moreover, \( q^d \) and \( \theta^d \) are substitutes.

**Proof.** We proceed with backward induction by considering the retailer’s problem first. Given \( w \) and \( \theta \), the retailer chooses \( q \). Differentiating \( \Pi^d_R = p^0 S(q) - wq \) with respect to \( q \) yields

\[
\frac{\partial \Pi^d_R}{\partial q} = p^0 \bar{F}(q) - w \quad \text{and} \quad \frac{\partial^2 \Pi^d_R}{\partial q^2} = -p^0 f(q) < 0.
\]

Observe \( \lim_{q \to 0} \frac{\partial \Pi^d_R}{\partial q} = p^0 \theta - w \geq 0 \) and \( \lim_{q \to q^d} \frac{\partial \Pi^d_R}{\partial q} = -w < 0 \). Therefore, the retailer chooses \( q = F^{-1} \left( 1 - \frac{w}{p^0 \theta} \right) \). The resulting manufacturer profit is \( \Pi^d_M = \left( w - c(\theta) \right) F^{-1} \left( 1 - \frac{w}{p^0 \theta} \right) \), which can be rewritten as \( \Pi^d_M = p^0 \theta q \bar{F}(q) - c(\theta)q \) by changing the manufacturer’s decision variable from \( w \) to \( q \) using the one-to-one mapping \( w = p^0 \bar{F}(q) \). Differentiating \( \Pi^d_M \) with respect to \( q \) and \( \theta \) yields

\[
\frac{\partial \Pi^d_M}{\partial q} = p^0 \bar{F}(q) (1 - g(q)) - c(\theta) \quad \text{and} \quad \frac{\partial \Pi^d_M}{\partial \theta} = p^0 q \bar{F}(q) - c'(\theta)q.
\]

In Assumption 1 and Lemma A1 we established \( \lim_{q \to 0} c'(\theta) = 0 \), \( c(\theta)/\theta < p_0 \) for all \( \theta \in (0, \Theta) \), and \( \lim_{q \to q^d} c'(\theta) > p_0 \). Using these results together with \( \lim_{q \to 0} \bar{F}(q) = 1 \), \( \lim_{q \to q^d} g(q) = 0 \), and \( \lim_{q \to q^d} \bar{F}(q) = 0 \), we then have the following limiting values: (i) \( \lim_{q \to 0} \frac{\partial \Pi^d_M}{\partial q} = p^0 \theta - c(\theta) > 0 \); (ii) \( \lim_{q \to q^d} \frac{\partial \Pi^d_M}{\partial \theta} = -c(\theta) < 0 \); (iii) \( \lim_{q \to q^d} \frac{\partial \Pi^d_M}{\partial \theta} = p^0 q \bar{F}(q) > 0 \); (iv) \( \lim_{q \to q^d} \frac{\partial \Pi^d_M}{\partial q} = p^0 q \bar{F}(q) - q \lim_{q \to q^d} c'(\theta) < p_0 q \left( \bar{F}(q) - 1 \right) = -p_0 q \bar{F}(q) < 0 \). The inequalities imply that a maximizer exists in the interior. Let \( (\hat{q}, \hat{\theta}) \) be the maximizer. These values should satisfy the first-order conditions

\[
\frac{\partial \Pi^d_M}{\partial q} = \frac{\partial \Pi^d_M}{\partial \theta} = 0
\]

or

\[
p^0 \hat{q} \bar{F}(\hat{q}) (1 - g(\hat{q})) = c(\hat{\theta}) \quad \text{and} \quad p^0 \bar{F}(\hat{q}) = c'(\hat{\theta}).
\]

Dividing the second equation by the second equation yields \( 1 - g(\hat{q}) \epsilon(\hat{\theta}) = 1 \). The first equation requires \( g(\hat{q}) < 1 \), and therefore \( \epsilon(\hat{\theta}) > 1 \). The Hessian \( H \) of \( \Pi^d_M \) has the following components (sub-
scripts 1 and 2 correspond to \( q \) and \( \theta \), respectively: 
\[ H_{11} = -p_0\theta \left[ f(q)(1-g(q)) + F(q)g'(q) \right], \]
\[ H_{12} = H_{21} = p_0F(q)(1-g(q)) - c'(\theta), \]
\[ H_{22} = -c''(\theta)q. \]
Evaluating these at the maximizer \((\hat{q}, \hat{\theta})\) yields
\[ \hat{H} = \begin{bmatrix} -c(\hat{\theta})f(\hat{q})/F(\hat{q}) - \theta c'(\hat{\theta})g'(\hat{q}) & -c'(\hat{\theta})g(\hat{q}) \\ -c'(\hat{\theta})g(\hat{q}) & -c''(\hat{\theta})\hat{q} \end{bmatrix}. \]
Its determinant is then
\[ \det \hat{H} = \left[ c(\hat{\theta})g(\hat{q}) + \theta c'(\hat{\theta})\hat{q}g'(\hat{q}) \right] c''(\hat{\theta}) - c'(\hat{\theta})^2 g(\hat{q})^2. \]
In the proof of Proposition 1 we showed \( c''(\theta) \geq -\frac{c'(\theta)c(\theta)}{\theta^2}[\epsilon(\theta) - 1]. \) Rewriting the expression above using the relation 
\[ c'(\theta) = c(\theta)\epsilon(\theta)/\theta \]
and applying this inequality along with the IGFR condition \( g'(q) > 0, \)
\[ \det \hat{H} = \left[ c(\hat{\theta})g(\hat{q}) + \theta c'(\hat{\theta})\hat{q}g'(\hat{q}) \right] c''(\hat{\theta}) - c'(\hat{\theta})^2 g(\hat{q})^2 > \]
\[ c(\hat{\theta})g(\hat{q})\frac{c(\hat{\theta})\epsilon(\hat{\theta})}{\theta^2}[\epsilon(\hat{\theta}) - 1] - \frac{c'(\hat{\theta})^2\epsilon(\hat{\theta})}{\theta^2} g(\hat{q})^2 = c(\hat{\theta})^2\epsilon(\hat{\theta})^2 \left( 1 - \frac{1}{\epsilon(\hat{\theta})} - g(\hat{q}) \right) g(\hat{q}) = 0, \]
where we used the equilibrium condition \([1 - g(\hat{q})]\epsilon(\hat{\theta}) = 1\) to prove the last equality. Since \( \hat{H}_{11} < 0, \hat{H}_{22} < 0, \) and \( \det \hat{H} > 0, \) the maximizer is unique. In addition, \( \hat{H}_{12} < 0, \) implying that
\[ \frac{\partial \hat{q}}{\partial \theta} = -\frac{\hat{H}_{12}}{\hat{H}_{11}} < 0, \text{ i.e., } \hat{q} \text{ and } \hat{\theta} \text{ are substitutes.} \]

The equilibrium conditions (7) and (8) are counterparts to (4) and (5) for the centralized supply chain case. Combining these equations yields
\[ [1 - g(q^d)]\epsilon(q^d) = 1, \quad (9) \]
which is analogous to (6); the only difference is that the sales function elasticity \( \eta(q) \) is replaced by \( 1 - g(q) \). As in the centralized setting, equation (9) establishes a relationship between the unit cost function elasticity \( \epsilon(\theta) \) and an elasticity originating from the demand distribution. In the centralized setting it was the elasticity of the total expected sales \( S(q) \), namely \( \eta(q) \). The fact that \( 1 - g(q) \) appears in (9) in place of \( \eta(q) \) suggests that the generalized failure rate \( g(q) \) is related to an elasticity, and indeed such an interpretation is offered by Lariviere & Porteus (2001): \( g(q) \) represents “the percentage decrease in the probability of a stockout from increasing the stocking quantity by 1%.” To be more precise, the quantity \( 1 - g(q) \) is equal to the elasticity of the expected sales conditional on the event that a stockout occurs, namely, \( qF(q) \). It is then intuitive that this elasticity is smaller than the elasticity of the total expected sales, i.e., \( 1 - g(q) < \eta(q) \) (this is proved in Lemma A2). Comparing (9) with (6), we see that any deviation in quality and inventory choices under decentralization should be done under the constraint that the same level of joint elasticity is maintained, i.e., the product of two elasticities should be equal to one in both centralized and
decentralized supply chains.

From this comparison, we infer that any change in quality has a smaller impact on the retailer’s inventory decision than it has on the centralized firm’s decision. Proposition 2 also states that quality and inventory levels in equilibrium are substitutes, just as in the centralized setting. In order to clearly delineate how these effects interact under decentralization, we next consider the scenarios in which one of the two decision variables, quality and inventory, is set to exactly the same level as that in the centralized setting: \( q^c \) and \( \theta^c \). The outcomes of these scenarios, summarized below, allow us to isolate the impact of wholesale price contracting on one decision variable at a time, and determine whether wholesale price contracts are capable of either inventory or quality coordination.

**Corollary 3.** (i) With quality fixed at the system-optimal level \( \theta^c \), the retailer is induced to choose the inventory level smaller than \( q^c \).

(ii) In order to induce the retailer to choose the system-optimal inventory level \( q^c \), the manufacturer must set the quality level smaller than \( \theta^c \).

(iii) A wholesale price contract achieves neither inventory nor quality coordination.

**Proof.** (i) and (ii). The equilibrium inventory level with fixed quality \( \theta^c \), denoted by \( \tilde{q} \), is characterized by the equation \( \bar{F}(\tilde{q}) (1 - g(\tilde{q})) = \frac{c(\theta^c)}{p_0 \theta^c} \) (see (7)). Comparing this with (4), we see that \( \frac{\bar{F}(q^c)}{\bar{F}(\tilde{q})} = 1 - g(\tilde{q}) < 1 \), which implies \( \tilde{q} < q^c \). The equilibrium quality with fixed inventory \( q^c \), denoted by \( \tilde{\theta} \), is characterized by the equation \( p_0 \bar{F}(q^c) = c'(\tilde{\theta}) \) (see (8)). Comparing this with (5), we see that \( c'(\tilde{\theta}) = \frac{c'(\theta^c)}{c'(\theta^c) \eta(q^c)} = c'(\theta^c) \eta(q^c) < c'(\theta^c) \), where we used (6) and the fact \( \eta(x) < 1 \) from Lemma A2. Since \( c''(\theta) > 0 \), the inequality \( c'(\tilde{\theta}) < c'(\theta^c) \) implies \( \tilde{\theta} < \theta^c \).

(iii) Inventory coordination is achieved if \( \bar{F}(q) = c(\theta)/p_0 \theta \). However, per equation (7), in the decentralized supply chain \( \bar{F}(q) = c(\theta)/p_0 (1 - g(q)) > c(\theta)/p_0 \theta \), hence for any \( \theta \), the induced inventory level in the decentralized supply chain is too small. Similarly, quality coordination is achieved if \( S(q)/q = c'(\theta)/p_0 \). Since \( S(q) > q \bar{F}(q) \), this implies \( \bar{F}(q) < c'(\theta)/p_0 \); however, per equation (8), in the decentralized supply chain \( \bar{F}(q) = c'(\theta)/p_0 \). Since \( c'(\theta) > 0 \), this implies that quality is too low, for any inventory level, in the decentralized supply chain.

Part (i) of the corollary states the well-known double marginalization effect on inventory choice; because each unit sold brings less profit to the retailer than to the centralized firm, the retailer orders a smaller quantity. More interestingly, part (ii) reveals that a similar double marginalization effect applies to quality choice as well. It shows that, ceteris paribus, decentralization with wholesale pricing leads the manufacturer to choose lower quality than the centralized firm would—the supply chain would be better off with higher quality. The reason behind this result is more subtle because,
unlike the retailer who simply reacts to a given quality and wholesale price in its inventory decision, the manufacturer has to anticipate the retailer’s best response when setting the quality level that influences the wholesale pricing decision. As expected, higher quality allows the manufacturer to charge a higher wholesale price because consumer willingness-to-pay increases. However, a higher wholesale price also leads the retailer to order less; as a result of this retailer response, the manufacturer’s incentive to improve quality is muted. The net effect of these two opposing forces—higher wholesale margin against lower order quantity—is that the manufacturer in a decentralized supply chain chooses quality at a lower level than the centralized firm would. The combination of (i) and (ii) implies that a wholesale price contract cannot achieve full coordination—simultaneous coordination of product quality and inventory, as shown in Table 1. Part (iii) of the corollary further shows that a wholesale price contract achieves no coordination at all—given the equilibrium inventory level, supply chain profit would be greater with higher quality, and given the equilibrium quality level, supply chain profit would be greater with higher inventory. Hence, wholesale price contracts lie in the upper left cell of Table 1.

Corollary 3 isolates the impact of decentralization on one decision variable at a time, but does not compare the equilibrium inventory and quality in the decentralized system to the optimal values in the centralized supply chain. In general, any of the following combinations may arise: lower quality with lower inventory, higher quality with lower inventory, and lower quality with higher inventory. The only combination that never arises under wholesale pricing is higher quality with higher inventory. This wide range of outcomes arises as a result of combining the two effects we have identified. That is, while decentralization leads to underinvestment in both quality and inventory (Corollary 3), substitutability between them (Proposition 2) leaves one variable moving in the opposite direction. Hence, it is possible, e.g., for double marginalization to induce a lower inventory level that, due to the substitution effect between inventory and quality, causes the equilibrium quality in the decentralized system to rise above the centralized optimal quality.

Which combination emerges in equilibrium depends on how demand is distributed and how the unit cost function is shaped. With the isoelastic unit cost function $c(\theta) = c_0 \theta^n$ for $n > 1$, however, we can make sharper predictions. The equilibrium inventory and quality levels with this function are:

\[
q^d = g \left( 1 - \frac{1}{n} \right), \\
\theta^d = \left( \frac{p_0}{nc_0} \mathcal{F}(q^d) \right)^{\frac{1}{n-1}}.
\]

Comparing these values with their counterparts $q^c$ and $\theta^c$ for the centralized supply chain case, we
make the following observation:

**Proposition 3.** Under the isoelastic unit cost function \( c(\theta) = c_0 \theta^n, \) \( q^c > q^d. \) Moreover, \( \theta^c > \theta^d \) if and only if

\[
\frac{\bar{F}(q^c)}{\bar{F}(q^d)} > \frac{1}{n}.
\]

(10)

**Proof.** From (6) and (9) we have \( \eta(q^c) = 1 - g(q^d) = 1/n. \) In Lemma A2 we showed that \( \eta(x) > 1 - g(x). \) Since both sides of this inequality are decreasing, the condition \( \eta(q^c) = 1 - g(q^d) \) implies \( q^c > q^d. \) In addition, \( \theta^c > \theta^d \) can be written as \( \left( \frac{\bar{F}(q^c)}{\bar{F}(q^d)} \right)^{1/(n-1)} > \left( \frac{\bar{F}(q^d)}{\bar{F}(q^d)} \right)^{1/(n-1)} \), which is equivalent to (10).

According to the proposition, the equilibrium inventory level under wholesale pricing never exceeds the inventory level in the centralized setting \( (q^c > q^d) \) when the unit cost function has constant elasticity. Thus, the usual under-ordering due to double marginalization applies. The equilibrium quality level, by contrast, may or may not be greater than the centralized supply chain optimum, i.e., both \( \theta^c > \theta^d \) and \( \theta^c < \theta^d \) are possible. Thus, Proposition 3 shows that double marginalization’s impact on inventory underinvestment dominates when the unit cost function is isoelastic; the quality level absorbs the impact of the substitution effect between inventory and quality, and as a result may be greater or smaller than the centralized optimal quality. Thus, in contrast to the known results from the quality decision literature (Jeuland & Shugan 1983; Economides 1999), decentralization may actually lead to increased product quality even with an extremely simple consumer valuation distribution (unlike, e.g., Shi et al. 2013). We emphasize that what drives this difference is demand uncertainty, which creates a link between quality and inventory choices through the newsvendor dynamics.

An examination of (10) reveals that lower quality is obtained if decentralization does not distort the inventory level too much. If the degree of distortion is sufficiently high, the manufacturer finds it optimal to over-invest in quality, in an attempt to compensate for the loss in sales with higher per-unit wholesale margin. From numerical examples we observe that quality over-investment occurs quite often. The choice of distribution has a significant impact on this result. For example, if demand follows an exponential distribution, it is numerically observed that decentralization always results in quality over-investment: \( \theta^d > \theta^c \) for any \( n > 1. \) However, if demand follows a power distribution \( F(x) = 1 - x^k \) defined on \( x \in [0, 1] \) for \( k > 0, \) which includes the uniform distribution as a special case with \( k = 1 \) (see Lariviere & Porteus 2001 and Cachon 2003 for extensive discussions of this example), we find that there is no quality distortion; quality is set exactly at the centralized
supply chain optimum \((\theta^d = \theta^c)\) as long as the unit cost function has constant elasticity \(n > 1\).\(^7\) However, it is important to observe that despite the fact that \(\theta^d = \theta^c\), the supply chain is not coordinated along the quality dimension; the manufacturer happens to select a quality level equal to the centralized system optimal because that maximizes its profit given the retailer’s inventory level, but double marginalization (Corollary 3) is still at work, meaning the supply chain’s profit would be greater if quality were higher. This leads to a subtle observation: even if a decentralized supply chain chooses the same quality level as a centralized supply chain, that does not mean it is coordinated in terms of product design; because of the two counteracting forces that impact product quality (double marginalization and the substitution effect with inventory) it is possible for quality to be close, or even equal, to the centralized quality level while the supply chain remains completely uncoordinated.

As the preceding discussions have made clear, supply chain coordination cannot be achieved under wholesale price contracting. While this is true even without quality choice, we find evidence that adding quality choice exacerbates the coordination challenge. This is supported by our findings in Corollary 3 which show that the efficiency loss due to double marginalization manifests itself in both quality and inventory decisions. As a consequence, it is impossible to attain coordination for either quality or inventory using a wholesale price contract.

### 6 Coordinating the Decentralized Supply Chain

Because of the effects described above, there is clearly value in coordinating inventory and product design in a decentralized supply chain. Given that wholesale price contracts cannot achieve such coordination, it is important to consider precisely what kinds of contracts are capable of achieving the centralized system outcomes—we explore this question in this section. Specifically, we analyze the capability of revenue sharing contracts to coordinate the supply chain. In a revenue sharing contract, (i) the retailer pays a wholesale price of \(w_r\) per unit to the manufacturer, where the subscript \(r\) stands for “revenue sharing,” and (ii) the retailer keeps a fraction \(r\) of the sales revenue and gives the manufacturer a fraction \(1 - r\) of the sales revenue. This type of contract is well known to coordinate inventory decisions in supply chains in a number of settings (Cachon & Lariviere 2005), and we seek to know whether they can similarly coordinate the supply chain in our setting; with endogenous, manufacturer-determined product quality. For ease of exposition, in this section

\[ q^c = \left( \frac{n+1}{kn+n-1} \right)^{1/k} \]  
\[ \theta^c = \left( \frac{c_0}{k} \right)^{1/(n-1)} \]

for the centralized setting and

\[ q^d = \left( \frac{n-1}{kn+n-1} \right)^{1/k} \]  
\[ \theta^d = \left( \frac{c_0 - k}{k} \right)^{1/(n-1)} \]

for the decentralized setting with wholesale price contracting.

---

\(^7\)The following closed-form solutions are obtained: \( q^c = \left( \frac{n+1}{kn+n-1} \right)^{1/k} \) and \( \theta^c = \left( \frac{c_0}{k} \right)^{1/(n-1)} \) for the centralized setting and \( q^d = \left( \frac{n-1}{kn+n-1} \right)^{1/k} \) and \( \theta^d = \left( \frac{c_0 - k}{k} \right)^{1/(n-1)} \) for the decentralized setting with wholesale price contracting.
we restrict attention to the isoelastic unit cost function $c(\theta) = c_0 \theta^n$, as this will suffice to illustrate the key points associated with supply chain coordination.

We consider two possible event sequences for revenue sharing contracts: either the contract parameters are finalized before the manufacturer determines quality, or the parameters are finalized after quality has been determined. Both are plausible scenarios in practice. The former may happen if, e.g., some elements of product design and quality are not finalized until production begins, and production begins after the manufacturer secures a contract from the retailer; this frequently occurs with apparel as contracts are awarded based on the design of “samples” (prototypes), while mass produced units are often made with different processes or materials and hence may deviate in quality from the initial samples. The latter sequence—in which the contract parameters are determined after quality has been decided by the manufacturer—may occur if the product has a very long design leadtime (e.g., a high technology product), or if mass produced units and prototypes have minimal differences. In reality, most products likely have a mixture of both scenarios: some elements of product quality are fixed before contracts are signed (e.g., overall design and material choice) while some elements are adjustable by the manufacturer after contracts have been determined (e.g., the quality of processing during the manufacturing stage); for simplicity, we examine the two extreme cases depicted in Table 2.

We begin with the “contract first” sequence. In this scenario, given a fixed $r$ and $w_r$, the manufacturer determines quality $\theta$ to maximize its profit. After quality has been determined, the retailer decides the inventory level, $q$. As the following proposition shows, revenue sharing cannot coordinate the supply chain under this sequence of events:

**Proposition 4.** When contracting occurs first, no revenue sharing contract $(r, w_r)$ can achieve full coordination.

*Proof.* We analyze the game beginning in step 2, i.e., after some contract $(r, w_r)$ has been determined in step 1. Under this revenue sharing contract, manufacturer profit is $\Pi_M = (w_r - c_0 \theta^n)q + (1 - r)p_0 \theta S(q)$. Because the wholesale price is fixed, there is a one-to-one relationship between the manufacturer’s quality choice and the retailer’s inventory choice. The retailer’s optimal inventory level for a given $\theta$ is $q = F^{-1}\left(\frac{r_0 \theta - w_r}{r_0 \theta}\right)$. Rearranging this expression, $\theta = \frac{w_r}{r_0 F(q)}$. Thus, $\theta$ is the quality
that achieves an inventory level $q$. Given this transformation, manufacturer profit as a function of the induced inventory $q$ is

$$\Pi_M = w_r \left[ q - \frac{c_0 w_r^n q}{(rp_0 F(q))^n} + \frac{(1-r)S(q)}{r F(q)} \right].$$

Differentiating with respect to $q$,

$$\frac{d\Pi_M}{dq} = w_r \left[ 1 - \frac{c_0 w_r^n q}{(rp_0 F(q))^n} \left( 1 + n q F(q) \right) + \frac{1-r}{r} \left( 1 + \frac{S(q)f(q)}{F(q)^2} \right) \right].$$

Observe that if full coordination occurs, $n q F(q) = S(q)$ and $F(q) = c_0 (\theta^n - 1)/p_0$. This implies $w_r = rc_0 (\theta^n)^n$. Combining these conditions and evaluating the first order condition at a point that satisfies the centralized optimal conditions, $\frac{d\Pi_M}{dq} \Big|_{q=q^c} = -w_r n \frac{S(q) f(q)}{F(q)^2} < 0$. In other words, the manufacturer’s profit function is decreasing at the centralized optimal inventory level—this means that the manufacturer’s optimal induced inventory level is lower than the centralized optimal (corresponding to a smaller quality level than the centralized optimal, per $\theta = w_r/rp_0 F(q)$). Hence, full coordination is not possible.

Coordination fails to occur with revenue sharing because, given a fixed $r$ and $w_r$, the manufacturer only receives part of the benefit of quality (a fraction $(1-r)$ of the increase in the sales revenue) but bears all of the cost of quality (since $w_r$ does not vary with the manufacturer’s quality choice). Consequently, the manufacturer finds it optimal to select a lower quality level than the centralized optimal value. This, in turn, causes the manufacturer to select an inventory level smaller than the coordinating value (because under a revenue sharing contract with fixed parameters, price—and optimal inventory—are both increasing in product quality).

The failure to coordinate is rooted in the fact that the wholesale price does not vary with the quality of the product, hence the manufacturer has reduced incentives to select a high quality level. One might thus suspect that coordination can be achieved if the contract parameters are determined after product quality has been set by the manufacturer, i.e., if the sequence of events follows the “quality first” ordering in Table 2. This is in fact the case:

**Proposition 5.** When quality is chosen first, a revenue sharing contract with $r \in (0, 1)$ and $w_r = rc_0 \theta^n$ fully coordinates the supply chain and arbitrarily allocates profit between the manufacturer and retailer.

**Proof.** Under a revenue sharing contract, retailer profit is $\Pi_R = rp_0 \theta S(q) - w_r q$ while manufacturer profit is $\Pi_M = (w_r - c_0 \theta^n) q + (1 - r)p_0 \theta S(q)$. When $w_r = rc_0 \theta^n$, $\Pi_R = rp_0 \theta S(q) - rc_0 \theta^n q$, and clearly the retailer makes the same inventory decision for any $\theta$ that the centralized supply chain
would, \( q = F^{-1}\left(\frac{p_0 - C_0^{\theta n^{-1}}}{p_0}\right) \). Rearranging this expression, \( \theta = \left(\frac{p_0}{C_0}\tilde{F}(q)\right)^{1/(n-1)} \) is the quality level necessary to induce an inventory level \( \theta \) from the retailer. Given this transformation, manufacturer profit is

\[
\Pi_M^* = (1 - r)\frac{p_0^{n/(n-1)}}{C_0^{1/(n-1)}} \left[ \left(\tilde{F}(q)\right)^{1/(n-1)} (S(q) - \tilde{F}(q)q) \right].
\]

Differentiating, the optimal inventory level induced by the manufacturer satisfies the first order condition

\[
\left(\tilde{F}(q)\right)^{1/(n-1)} (\tilde{F}(q) - \tilde{F}(q) + f(q)q) = \frac{1}{n - 1} f(q) \left(\tilde{F}(q)\right)^{-1+1/(n-1)} (S(q) - \tilde{F}(q)q) = 0.
\]

This reduces to \( nq\tilde{F}(q) = S(q) \), which is the same inventory optimality condition as the centralized supply chain. Moreover, the manufacturer earns a fraction \( 1 - r \) of the total supply chain profit while the retailer earns a fraction \( r \), meaning profit can be arbitrarily allocated.

Indeed, this coordinating contract can also be extended to the sequence in which contracting occurs before quality is chosen—provided the manufacturer and retailer agree during the contracting phase to share the production costs, whatever they may be for the quality level chosen by the manufacturer. This has the effect of transforming the contract from a standard revenue sharing contract into a “revenue + cost sharing” contract, in which it is \textit{ex ante} specified that the manufacturer will charge the retailer a fixed fraction \( r \) of the production cost, and the retailer will return a fraction \( 1 - r \) of the sales revenue to the manufacturer.

Based on the result of Proposition 5, it would seem that coordination is surprisingly straightforward in a supply chain with quality choice. Unlike many supply chain contracting models (e.g., with endogenous pricing, such as Bernstein & Federgruen 2005) it is not necessary to introduce additional parameters to achieve supply chain coordination, despite the presence of an additional decision (manufacturer quality) within the supply chain. If quality is determined first, before contracting occurs, supply chain coordination can be achieved with a standard revenue sharing contract. If quality is determined after contracting, it is necessary to also specify \textit{ex ante} some percentage of cost sharing, but this does not introduce significant additional complexity to the contract between the firms.

Unfortunately, though, there is one key challenge regarding the implementation of the coordinating contract described above, and it is rooted in the fact that to achieve coordination, the wholesale price must be a fraction of the manufacturing cost. As the following proposition shows, this is \textit{not} the wholesale price that the manufacturer prefers:
Proposition 6. For a fixed $r$, the manufacturer prefers a higher wholesale price than the coordinating value, $w_r = rc_0\theta^n$.

Proof. Suppose the manufacturer determines product quality, then the manufacturer determines the wholesale price $w_r$ that maximizes its profit given a fixed $r$; then the retailer determines the inventory level that maximizes its profit. For a given $r$ and $w_r$, retailer profit is $\Pi_R = r p_0 \theta S(q) - w_r q$ while manufacturer profit is $\Pi_M = (w_r - c_0\theta^2)q + (1 - r)p_0\theta S(q)$. The retailer’s optimal inventory level for a given $\theta$ and $w_r$ is $q = F^{-1}\left(\frac{r p_0 \theta - w_r}{r p_0 \theta}\right)$. Rearranging this expression, $w_r = r p_0 \theta F(q)$ is the wholesale price that achieves an inventory level $q$ given a quality level $\theta$. Given this transformation, manufacturer profit is

$$\Pi_M = (r p_0 \theta F(q) - c_0\theta^n)q + (1 - r)p_0\theta S(q).$$

Differentiating with respect to $q$,

$$\frac{d \Pi_M}{dq} = p_0\theta F(q) - c_0\theta^n - q r p_0 \theta f(q).$$

For any $q$ and $\theta$, this is less than the first order condition for inventory in the centralized system, $p_0\theta F(q) - c_0\theta^n$, if $r > 0$. Hence, for any $\theta$, any optimal $q$ is less than the unique optimal $q$ in the centralized system. This means that if $r > 0$, the manufacturer chooses a wholesale price such that the inventory is strictly smaller than the coordinating value (and hence charges a higher wholesale price).

In other words, Proposition 6 shows that, if the supply chain members agree on a fraction of sales revenue to share (say 50%), the manufacturer would prefer to charge a higher wholesale price, selling fewer units to the retailer than the coordinating level but earning a higher margin on each sale. Figure 1 provides a graphical example. Indeed, this effect occurs at any given quality level, and thus happens whether quality is endogenously determined or not; however, it is particularly treacherous when quality is endogenously chosen, for the following reasons.

In models of revenue sharing contracts with exogenous quality (Dana & Spier 2001; Gerchak & Wang 2004; Cachon & Lariviere 2005), it is assumed that the supply chain members bargain over the allocation of supply chain profits by selecting from a menu of coordinating contracts which all maximize total supply chain profit. In those models, a crucial assumption is that quality is fixed and manufacturing costs are known to the retailer. Thus, the retailer and manufacturer can restrict their bargaining to the set of coordinating contracts.

When quality is endogenous and manufacturing cost is variable, there are two significant hurdles to revenue sharing contracts. First, to achieve coordination the retailer must have detailed
knowledge of the manufacturer’s cost/quality trade-off curve. While it is plausible to assume that retailers can infer the cost of a single existing product when formulating a revenue sharing contract (as in Cachon & Lariviere 2005), understanding the entire cost/quality trade-off curve requires detailed knowledge about the design process, raw materials costs, and manufacturing process that a retailer simply may not have. If the retailer does not know the cost/quality trade-off curve, Proposition 6 implies that the manufacturer has incentive to lie and inflate the costs reported to the retailer to earn a higher profit for any given $r$.

Second, for a newly designed product (as opposed to an existing product with an established design and cost history), it is unlikely that the firms in the supply chain—particularly the retailer—have a firm grasp of the true manufacturing cost. Indeed, in many cases manufacturing costs are not fully understood until an item has been in production for some time and managers have developed a solid understanding of labor, material, and transportation costs for the product. Thus, achieving coordination with revenue sharing—which requires both parties know the cost/quality trade-off curve—is especially challenging if product design is endogenous. Consequently, while a supply chain with endogenous quality can be coordinated by a revenue sharing (or revenue + cost sharing) contract, it may prove to be quite challenging to do so in practice.

In addition, Propositions 4-6 extend to the case of buyback contracts because, as in the simple newsvendor supply chain described in Cachon & Lariviere (2005) with exogenous quality, revenue

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8While we do not explicitly model information asymmetry, it is straightforward to see that if cost is private information, the manufacturer will have an incentive to tell the retailer cost is higher than the true value to induce a higher wholesale price, yielding greater manufacturer profit in accordance with Proposition 6.
sharing and buyback contracts are mathematically equivalent in our setting, even with endogenous quality. To see this, observe that if \( b = p_0 \theta (1 - r) \) and \( w_b = w_r + p_0 \theta (1 - r) \), the two contracts are identical. Hence, in a buyback contract, the wholesale price must also exhibit some element of cost sharing in order to coordinate the supply chain, and both the buyback and wholesale prices must be a function of product quality to achieve coordination—consequently, buyback contracts suffer from the same difficulties in implementation as revenue sharing contracts.

This result sheds light on one reason why centralized supply chains have value for seasonal products when quality is a decision variable: because, depending on when contracts are signed relative to when the product design is finalized, standard supply chain contracts (wholesale price, revenue sharing, and buyback) either cannot coordinate the supply chain or are challenging to implement in practice. Centralization—having the same firm both design and sell the product—avoids this potential trap. This may, in part, help to explain the rise of direct-to-consumer apparel retailers (such as Bonobos, Lululemon, Warby Parker, and Zara) and the relative decline of decentralized solutions (separate brands and department stores, such as JC Penney’s) despite widespread awareness of conventional coordinating contracts, particularly as the fixed costs of operating a direct supply chain have declined due to the proliferation of the internet and e-commerce capabilities. While the literature and popular press have identified many advantages and disadvantages of vertical integration, our results illustrate that one potential benefit is the capability to coordinate product design and operational decisions that may otherwise be difficult or impossible to achieve.

### 7 Quality-Dependent Market Size

The results derived in §§3-6 depend on the assumption that product quality impacts only the selling price and not the size of the market \( (D) \), which is exogenous. This assumption is, without question, a simplification of a more complex reality—in practice we expect product quality would impact both price and demand. In essence, our model allows the manufacturer to select the profit maximizing product from a menu of many products with different price/cost combinations, but with identical demand distributions. To that end, we analyze the optimal price/cost tradeoff given a fixed demand distribution. Clearly, if the demand distribution also depends on product quality our results may no longer hold.

There are, however, several advantages to making this assumption. First, this is representative of the case where the size of the market is determined by strategic choice of fundamental product type, features, functionality, or market positioning, while within that market consumer willingness-to-pay is determined by the vertical quality level of the product features. For example, market size...
for a sweater may be determined by the overall style and design, while consumer willingness-to-pay is influenced by the quality of the production process or raw materials. Second, this assumption allows us to link our model and results to both the marketing literature on product design and vertical quality (e.g., Moorthy & Png 1992 and Netessine & Taylor 2007, which use similar assumptions) and the newsvendor literature on inventory choice. Lastly, this assumption facilitates straightforward and clean analysis of optimal firm decisions. Despite these advantages, it is clearly useful to illustrate whether our results extend to the case where quality impacts market size. As the following proposition shows, this is the case, at least under the isoelastic quality cost function:

**Proposition 7.** Suppose that:

(i) Market size is equal to \( \theta D \);

(ii) Consumer willingness to pay for the product is equal to \( p_0 \) regardless of quality;

(iii) The unit cost of quality is \( c_0 \theta^{n-1} \);

(iv) Inventory per unit quality is defined to be \( \hat{q} \equiv q/\theta \).

Then both the centralized and decentralized supply chains face mathematically equivalent optimization problems to those analyzed in §§3-6, replacing \( q \) with \( \hat{q} \) and \( c(\theta) \) with \( c_0 \theta^n \).

**Proof.** Under the revised assumptions, firm profit is \( \hat{\Pi}^c (\theta, q) = p_0 \mathbb{E} [\min \{ \theta D, q \}] - c_0 \theta^{n-1} q \). Letting \( \hat{q} \equiv q/\theta \), \( \hat{\Pi}^c (\theta, \hat{q}) = p_0 \theta \mathbb{E} [\min \{ D, \hat{q} \}] - c_0 \theta^n \hat{q} \), which is an equivalent optimization problem to (3), when \( c(\theta) = c_0 \theta^n \), replacing \( q \) with \( \hat{q} \). Similarly, with a wholesale price contract \( \hat{\Pi}^d_R (q) = p_0 \mathbb{E} [\min \{ \theta D, q \}] - wq \) and \( \hat{\Pi}^d_M (w, \theta) = (w - c_0 \theta^{n-1}) q \). Let \( \hat{w} \equiv w \theta \). Then, \( \hat{\Pi}^d_R (\hat{q}) = p_0 \theta \mathbb{E} [\min \{ D, \hat{q} \}] - \hat{w} \hat{q} \) and \( \hat{\Pi}^d_M (\hat{w}, \theta) = (\hat{w} - c_0 \theta^n) \hat{q} \), which are identical to the profit functions analyzed in §5 for an isoelastic quality cost, replacing \( q \) with \( \hat{q} \) and \( w \) with \( \hat{w} \). The same change of variables applies under revenue sharing contracts.

Proposition 7 shows that if quality impacts market size in a multiplicative manner, and if price is independent of product quality, then a simple change of variables leads to identical optimization problems to the ones analyzed in both the centralized and decentralized supply chain models in §§3-6. This means that all of our previous results continue to hold under this new set of assumptions, with some reinterpretation around the new variable \( \hat{q} \), the inventory per unit quality.\(^9\) For instance, in the centralized system, the optimal \( \hat{q} \) is independent of \( p_0 \) and \( c_0 \) (Corollary 1), but the optimal \( q = \theta \hat{q} \) is clearly dependent on quality and hence price and cost coefficients. Most notably, the derivation of the optimal inventory and quality, as well as the challenges associated with supply chain coordination, persist under this alternative interpretation.

\(^9\)Observe that the in-stock probability under this modified demand function, \( Pr(\theta D < q) \), is equal to \( F(\hat{q}) \). Hence, \( \hat{q} \) can be interpreted in terms of the newsvendor model for this new demand formulation.
We conclude from this proposition that, at least under some circumstances, our model and most of our results extend to the case of quality-dependent market size. Hence, while the assumption that market size is independent of quality clearly has some limitations, it allows for relatively straightforward analysis while still managing to capture the dual scenarios in which quality impacts price or the size of the market.

8 Numerical Study

While demonstrating that a coordination problem exists when both inventory and design are endogenously determined, the results of §§3-6 do not illustrate the magnitude of this coordination problem. In this section, we explore precisely this issue by performing an extensive numerical study. Our goals are threefold: first, we will explore properties and behavior of the centralized optimal inventory and quality levels; second, we will determine the efficiency loss associated with wholesale price and revenue sharing contracts; and third, we will compare the magnitude of supply chain inefficiency when quality is endogenous to the inefficiency when quality is exogenous, as is typically assumed in the literature.

We begin by examining the optimal centralized supply chain decisions. We assumed an isoelastic unit cost function \( c(\theta) = c_0 \theta^n \), and we considered three different cost curves \( n = 1.5, 2, \) and 3. We constructed a set of economic parameters that represent a wide range of scenarios from very costly products (high \( c_0 \)/low \( p_0 \)) to very profitable products (low \( c_0 \)/high \( p_0 \)). These parameters are presented in Table 3. We selected seven different families of IGFR demand distributions that represent a wide range of possible shapes: normal, lognormal, beta, gamma, Weibull, exponential, and uniform. For each distribution we further selected a large number of distributional parameters to represent varying shapes of the density and levels of demand uncertainty, presented in Table 4. Using every combination of the parameter values in Tables 3 and 4, we analyzed a total of 18,150 examples, calculating the optimal inventory and quality for a centralized firm in all cases.

With quadratic costs, we find that optimal inventory is greater than the expected demand in 92% of problem instances; this is to be expected, since most of our demand distributions are symmetric or right skewed (except for the beta distribution under some parameter combinations) and Corollary 1 showed the optimal critical ratio is greater than one half when \( n \geq 2 \). On average,
centralized inventory was 111% of the expected demand—the maximum was 134% of mean demand and the minimum was 99% of mean demand. Hence, we conclude that when quality is endogenous and costs are quadratic, in most cases the firm finds it optimal to select a product quality that leads to an inventory level slightly higher than expected demand.

With \( n = 3 \) (meaning quality is relatively more expensive than the quadratic case), the results are even stronger than the quadratic case: inventory is greater than expected demand in 100% of problem instances; the average inventory is 139% of mean demand (the range is 103% to 193% of mean demand). This confirms the result derived in Corollary 1 part (iii), which showed that the optimal in-stock probability is at least \((n - 1)/n\); with \( n = 3 \), this implies an in-stock greater than \( 2/3 \), corresponding to a higher inventory level than was optimal with quadratic costs. However, when \( n = 1.5 \) (quality is less expensive than the quadratic case), the results are strikingly different: we find inventory is greater than expected demand in only 2% of problem instances; the average inventory is 87% of mean demand (the range is 69% to 101% of mean demand). In this case, because quality is “cheaper,” relatively speaking, than the quadratic case, the firm finds it optimal to invest in higher quality which leads to a lower optimal inventory level. Optimal inventory remains relatively close to (but less than) the expected demand, however.

We next move to the impact of decentralization on inventory, quality, and supply chain profit with a wholesale price contract. Using a similar methodology to that employed for the centralized system, we analyzed the equilibrium quality, inventory, and supply chain profit under all parameter combinations. The results are summarized in Table 5. With all cost functions, quality was almost always slightly higher in the decentralized system than in the centralized system; however, as discussed in Corollary 3, supply chain profit would always be greater with higher quality (e.g., quality that is twice as high if \( n = 2 \)). Thus, while the quality distortion reported in Table 5 seems small, this is in fact quite misleading due to the counteracting force generated by the inventory substitution effect—quality may be close to what the centralized supply chain would choose, but that does not mean that the decentralized system achieves quality coordination; because inventory varies

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal ((\mu, \sigma))</td>
<td>(\mu \in {10, 13, 15}, \sigma \in {1, 3, 5})</td>
</tr>
<tr>
<td>Lognormal ((\mu, \sigma))</td>
<td>(\mu \in {1, 2, 3}, \sigma \in {0.25, 0.5, 0.75})</td>
</tr>
<tr>
<td>Beta ((\alpha, \beta))</td>
<td>(\alpha \in {1, 2, 3}, \beta \in {1, 2, 3})</td>
</tr>
<tr>
<td>Gamma ((k, \theta))</td>
<td>(k \in {1, 5, 10}, \theta \in {5, 10, 15})</td>
</tr>
<tr>
<td>Weibull ((\lambda, k))</td>
<td>(\lambda \in {1, 5, 10}, k \in {5, 10, 15})</td>
</tr>
<tr>
<td>Exponential ((1/\lambda))</td>
<td>(1/\lambda \in {0.1, 1, 5, 10, 20, 40, 50, 75, 100})</td>
</tr>
<tr>
<td>Uniform ((0, 1))</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4. Demand distributions used in numerical experiments.
substantially from the centralized optimal level, the supply chain is significantly uncoordinated along the quality dimension, leading to an average profit loss for the supply chain of 22-24%.

Lastly, we compare the inefficiency of the supply chain with endogenous quality to a supply chain with exogenous quality. Naturally, the comparison depends on the point of reference—what is the exogenous quality level used for the comparison? Thus, rather than performing another large scale numerical study, we analyze one representative example in detail, varying the exogenous quality level. Figure 2 provides that example, showing the loss in supply chain profit resulting from decentralization, i.e., centralized supply chain profit minus decentralized supply chain profit, as a function of the exogenous quality level. In this example, with endogenous quality the profit loss is 10.6, and the optimal quality level is 0.73 in the centralized system while the equilibrium quality is 0.75 in the decentralized system.\(^\text{10}\)

As the figure illustrates, profit loss is at its peak with endogenous quality. When the exogenously

\(^\text{10}\)In the example, demand is normally distributed with mean 10 and standard deviation 3, and \(p_0 = 10, c_0 = 5,\) and \(n = 3.\)
specified quality level is very close to the centralized optimal quality, the profit loss is nearly identical to the endogenous quality case, as one might expect; but when quality deviates significantly from the centralized optimal level, the supply chain profit loss is reduced. This example demonstrates that endogenous quality exacerbates the coordination problem by worsening the loss in supply chain profit due to decentralization—a fact which supports our conclusion that coordination is particularly valuable when product quality is a firm decision.

9 Conclusion

In this paper we have analyzed the coordination of product design (quality) and operational (inventory) decisions in a supply chain for a seasonal product. Our contributions are threefold. First, we develop a tractable modeling framework for product quality decisions in both centralized and decentralized supply chains that allows for the study of coordinated product design and operational activities for seasonal goods. We provide simple characterizations for optimal quality and inventory in terms of the sales and cost elasticities that illustrate the interactions between quality and inventory—namely, that quality and inventory are substitutes. We demonstrate that if quality costs are isoelastic and reasonably “costly” (elastic), then it is optimal for a centralized firm to select a quality level leading to a critical ratio greater than one half.

Second, we analyze the performance of wholesale price contracts when quality is endogenous. We show that in contrast to results shown in the previous literature with deterministic demand, with uncertain demand decentralization can lead to an increase in product quality when using a simple wholesale price contract, even with extremely simple consumer valuations. Nevertheless, the supply chain suffers from both inventory and quality that are too low, exacerbating the well-known inventory coordination problem. Our numerical results show that this can lead to particularly large efficiency gaps, on average 22-24% of the centralized system profit—substantial amounts in the retail sector, where profit margins are usually razor-thin.

Third, we show that, surprisingly, simple supply chain contracts such as revenue sharing and buyback agreements can coordinate the supply chain with endogenous product quality if quality is determined before contract parameters are finalized; however, such contracts require truthful revelation of the cost/quality trade-off curve by the manufacturer (which is not in the manufacturer’s interests to reveal). Hence, coordination in this setting is challenging in practical settings, perhaps helping to in part explain the success of centralized firms that both design and sell their own seasonal goods such as Bonobos, Warby Parker, and Zara.

In sum, our results highlight several consequence of the centralization of design and sales for
seasonal products. Given the increasing popularity of this business model, it is important to understand the impact of vertical integration on tactical firm decisions (quality and inventory), strategic firm decisions (to centralize or not), and overall supply chain performance. There are many reasons why firms may—and may not—want to vertically integrate that we do not model, such as transaction costs (Coase 1937), information (Arrow 1975), and contractual rights of control (Grossman & Hart 1986). Our model illustrates that the coordination of product design and operational decisions—coordination that may prove challenging or even impossible to achieve in a decentralized supply chain—may provide a new explanation for the value of centralization.

A Appendix: Supporting Results

Lemma A1. Under Assumption 1, (i) \( c(\theta) / \theta < p_0 \) for all \( \theta \in (0, \theta) \) and (ii) \( \lim_{\theta \to \theta} c'(\theta) > p_0 \).

Proof. To prove (i), we show that the supremum of the ratio \( c(\theta) / \theta \) in the interval \( \theta \in (0, \theta) \) occurs at the boundary \( \theta = \theta \), where \( c(\theta) / \theta = p_0 \) by definition. To see this, observe \( \lim_{\theta \to 0} \frac{c(\theta)}{\theta} = \lim_{\theta \to \infty} \frac{c(\theta)}{\theta} = \infty \) where we used l’Hospital’s rule along with Assumption 1, and \( \frac{d}{d\theta} \left( \frac{c(\theta)}{\theta} \right) = \frac{c(\theta)}{\theta^2} [\epsilon(\theta) - 1] \). Hence, the ratio \( c(\theta) / \theta \) starts at zero, is increasing at sufficiently large \( \theta \) where \( \epsilon(\theta) > 1 \) (see Assumption 1), and approaches infinity as \( \theta \to \infty \).

The ratio may be increasing or decreasing in the vicinity of \( \theta = 0 \). Together, they imply that the supremum of \( c(\theta) / \theta \) in the interval \( \theta \in (0, \theta) \) occurs at the boundary \( \theta = \theta \). It then follows that \( c(\theta) / \theta < p_0 \) for all \( \theta < \theta \). To prove (ii), recall that the ratio \( c(\theta) / \theta \) increases if and only if \( \epsilon(\theta) > 1 \) and that it is increasing at \( \theta = \theta \). This implies \( \lim_{\theta \to \theta} \epsilon(\theta) = \lim_{\theta \to \theta} \frac{\theta c'(\theta)}{c(\theta)} = \frac{1}{p_0} \lim_{\theta \to \theta} c'(\theta) > 1 \), or equivalently, \( \lim_{\theta \to \theta} c'(\theta) > p_0 \).

Lemma A2. For \( x \in (0, D] \), the sales function elasticity \( \eta(x) \) defined in (2) has the following properties: (i) \( \lim_{x \to 0} \eta(x) = 1 \); (ii) \( \lim_{x \to D} \eta(x) = 0 \); (iii) \( \eta'(x) < 0 \); (iv) \( \frac{F(x)}{x} \) is \( \eta(x) < 1 \); (v) \( \eta(x) + g(x) > 1 \); (vi) \( x \eta'(x) / \eta(x) = \eta(x) + g(x) - 1 \).

Proof. Note that \( \eta(x) \) can be written as \( \eta(x) = \frac{x F(x)}{S(x)} = 1 - \frac{\int_0^x y f(y) dy}{\int_0^x F(y) dy} \) since \( S(x) = E \min \{D, x\} = \int_0^x F(y) dy = \int_0^x y f(y) dy + x F(x) \). By l’Hospital’s rule,

\[
\lim_{x \to 0} \eta(x) = 1 - \lim_{x \to 0} \frac{\int_0^x y f(y) dy}{\int_0^x F(y) dy} = 1 - \lim_{x \to 0} \frac{x f(x)}{F(x)} = 1 - \lim_{x \to 0} g(x) = 1,
\]

\[
\lim_{x \to D} \eta(x) = 1 - \frac{\int_0^D y f(y) dy}{\int_0^D F(y) dy} = 1 - \frac{\mu}{\mu} = 0.
\]
Differentiating $\eta(x)$ yields $\eta'(x) = -\frac{F(x)}{S(x)^2} \left( g(x) S(x) - \int_0^x yf(y) \, dy \right)$. Observe that

$$\lim_{x \to 0} \left( g(x) S(x) - \int_0^x yf(y) \, dy \right) = 0,$$

$$\frac{d}{dx} \left( g(x) S(x) - \int_0^x yf(y) \, dy \right) = g'(x) S(x) + g(x) F(x) - xf(x) = g'(x) S(x) > 0,$$

which together imply $g(x) S(x) - \int_0^x yf(y) \, dy > 0$. Therefore, $\eta'(x) < 0$. Combined with $\lim_{x \to 0} \eta(x) = 1$ and $\lim_{x \to \mathcal{D}} \eta(x) = 0$, this implies $\eta(x) < 1$. Moreover, $\eta(x) = \frac{xF(x)}{\int_0^y F(y) \, dy} > \frac{xF(x)}{\int_0^y S(y) \, dy}$ since $F(y) < 1$ for $y < \mathcal{D}$. Finally, rewriting $\eta'(x)$ yields $\eta'(x) = -\frac{F(x)}{S(x)} \left( g(x) - \frac{\int_0^x yf(y) \, dy}{S(x)} \right) = -\frac{\eta(x)}{x} [\eta(x) + g(x) - 1]$. Since $\eta'(x) < 0$, this implies $\eta(x) + g(x) > 1$. □

References


