The Intuition Behind Black-Litterman Model Portfolios

In this article and as our title suggests, we demonstrate a method for understanding the intuition behind the Black-Litterman asset allocation model.

To do this, we use examples to show the difference between the traditional mean-variance optimization process and the Black-Litterman process. We show that the mean-variance optimization process, while academically sound, can produce results that are extreme and not particularly intuitive. In contrast, we show that the optimal portfolios generated by the Black-Litterman process have a simple, intuitive property:

- The unconstrained optimal portfolio is the market equilibrium portfolio plus a weighted sum of portfolios representing an investor’s views.
- The weight on a portfolio representing a view is positive when the view is more bullish than the one implied by the equilibrium and other views.
- The weight increases as the investor becomes more bullish on the view as well as when the investor becomes more confident about the view.

December 1999
Appendix C

1. Given the expected returns $\mu$ and the covariance matrix $\Sigma$, the unconstrained maximization problem $\max_w w^\top \mu - \frac{1}{2} w^\top \Sigma w$ has a solution of $w^* = \delta \mu / \Sigma$.

2. Given the covariance matrix $\Sigma$, the minimum variance portfolio is $w_{\text{min}} = -\delta \Sigma / \mu^\top \Sigma$, where $\delta$ is a vector with all elements being one.

3. The solution to the risk constrained optimization problem, $\max_w w^\top \mu$, subject to $w^\top \Sigma w = \sigma^2$, can be expressed as $w^* = -\delta \mu / \Sigma$, which is the solution of the unconstrained problem.

4. The risk, budget, and beta-constrained optimization problem can be formulated as $\max_w w^\top \mu$, subject to $w^\top \Sigma w = \sigma^2$ and $w^\top \Sigma w_b = \beta^2$, where $w$, $\beta$, and $\sigma$ are chosen in the way both risk and budget constraints are satisfied.

5. The risk-, budget-, and beta-constrained optimization problem can be formulated as $\max_w w^\top \mu$, subject to $w^\top \Sigma w = \sigma^2$ and $w^\top \Sigma w_b = \beta^2$, where $w$, $\beta$, and $\sigma$ are chosen in the way all three constraints are satisfied.

References


Appendix B

1. There are N assets in the market. The market portfolio (equilibrium portfolio) is \( \mu_p \). The covariance of the returns is \( \Sigma \). The expected returns \( \Pi \) is a vector of normally distributed random variables with mean \( \mu_p \)

2. The error term in the Black-Litterman model is represented by the risk aversion parameter \( \lambda \). The equilibrium expected returns are \( \Pi^E \). The CAPM prior distribution for the expected returns is

\[
\Pi = \mu_p + \lambda \Sigma^{-1} (\Pi^E - \mu_p) \]

where \( \lambda \) is a scalar measuring the investor's risk aversion.

3. The vector \( \Lambda \) contains the user views about the market. \( \Lambda \) is a vector of normally distributed random variables with mean \( \Lambda^E \) and covariance \( \Gamma \).

4. The expected returns are the neutral reference point of the Black-Litterman model. The unconstrained optimal portfolio is

\[
\omega^{*} = \left( \Sigma + \Lambda \Sigma \Lambda^T \right)^{-1} \left( \Pi^E - \mu_p \right) \]

5. The investor's views can be summarized as

\[
\omega = \omega^{*} + \lambda \Sigma^{-1} (\Pi^U - \Pi) \]

where \( \Pi^U \) is the vector of additional views. The investor’s weights \( \omega \) are the result of combining the equilibrium expected returns with the investor’s views.

6. Let \( \Pi \) be the expected returns from the investor's point of view, \( \tilde{\Pi} \) be the equilibrium expected returns, and \( \Pi^U \) be the user views. The investor’s weights \( \omega \) are given by

\[
\omega = \left( \Sigma + \Lambda \Sigma \Lambda^T \right)^{-1} \left( \tilde{\Pi} - \mu_p \right) + \lambda \Sigma^{-1} (\Pi^U - \Pi) \]

The Black-Litterman model provides both a reference point for expected return assumptions as well as a mechanism for combining equilibrium expected returns with the user's views.

7. For a particular user, the new expected returns \( \tilde{\Pi} \), is an increasing function of its expected return. The absolute value of the weight \( \omega \), is an increasing function of its confidence level \( \lambda \).

Goldman Sachs Investment Management

Since publication in 1990, the Black-Litterman asset allocation model has gained widespread application in major financial institutions. As developed in the original paper, the Black-Litterman model provides the flexibility to analyze, model, and implement views construction into the mean-variance framework. Since publication, views have been incorporated into the standard portfolio optimizers, providing both the set of expected returns in addition to the optimal portfolio weights.

In contrast to the Black-Litterman model, in the traditional mean-variance approach the user inputs a complete set of expected returns and the portfolio optimizer generates the optimal portfolio weights. However, users often find that their specification of expected returns is unrealistic. The Black-Litterman model provides the investor with a unique ability to incorporate views about the expected returns of arbitrary portfolios, and the model combines the investor's views with equilibrium, producing both the set of expected returns and realized optimal portfolio weights that are consistent with the investor's views and the market equilibrium.

In this article, we use examples to illustrate the difference between the traditional mean-variance optimization process and the Black-Litterman process. In so doing, we demonstrate how the Black-Litterman approach provides both a reference point for the expected return assumptions, as well as a mechanism for incorporating user views into the portfolio optimization process.

The Black-Litterman framework provides both a reference point for expected return assumptions as well as a mechanism for incorporating user views into the portfolio optimization process.

First, investment managers tend to focus on small segments of their potential investment universe—sector rotation, industry rotation, the Black-Litterman model provides both a reference point for expected return assumptions and a mechanism for incorporating user views into the portfolio optimization process. The model combines the investor’s views with equilibrium, producing both the set of expected returns and optimal portfolio weights that are consistent with the investor’s views and the market equilibrium.
The following example demonstrates the unstable behavior of the optimal weights. Starting from equal expected returns as the starting point results in optimal weights of -33.5% in Germany and 71.4% in Australia. A small shift in the expected returns (e.g., increase the expected return in Germany by 5%) results in weights of 25.0% in Germany and 75.0% in Australia. Indeed this was the original motivation for Black and Litterman to develop their approach.

Second, investment managers tend to think in terms of weights in a constrained (e.g., long-only, long-short) investment universe where the portfolio weights returned by the optimizer (when not overly constrained) tend to appear to be extreme and not particularly intuitive. In the examples, the optimizer appears to have no consideration of the different levels of risk in assets of different countries and tends to generate very extreme portfolios. For example, the starting point using equal means does not compensate for the different asset class volatility in the different countries. The resulting portfolio weights for the different countries are inconsistent with the views given by the manager. For example, the portfolio would have 68% in the United Kingdom, 10% in the United States, and 22% in Germany - a result that seems to be inconsistent with the views of the manager. This difference in the weights is due to the optimizer using the different levels of risk in assets of different countries.

There are many different ways to translate the views to expected returns. Throughout our examples, we use the equilibrium expected returns as the neutral starting point, being optimal for an investor using the mean-variance approach. Now, the manager modifies this neutral starting point to reflect her views on the expected returns. As the manager’s views are not expected return independent, we must also modify the market capitalization weights so as to reflect the manager’s views on the weights of these assets.

Black and Litterman also demonstrated the shortcomings of several other optimizers. Indeed, this was the original motivation for Black and Litterman to develop their approach.
Investment Management Research

The Intuition Behind Black-Litterman

Model Portfolios

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<td>0.491</td>
<td>0.00</td>
</tr>
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</table>

Table 1: Annualized volatilities, market-capitalization weights, and equilibrium expected returns for the equity markets in the seven countries.

Table 2: Correlations among the equity index returns.

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Goldman Sachs Investment Management

Appendix A

Equal Expected Returns

1. The investor has only one view about the market: German equities will outperform European equities by 5% per year. Since our investor does not have any views about the rest of the world, the investor shifts the expected returns for the rest of the world to those of Germany (i.e., the investor sets the expected return for Germany 5% higher than the market capitalization-weighted average of the expected returns for the rest of the world). This approach may suggest that Germany will be over-weighted and the other countries will be under-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1A.

2. There are many different ways to translate the views to expected returns. For example, the investor could simply shift the expected return for German equities by 5% higher than the market capitalization-weighted average expected returns expected for the rest of the world. This approach may suggest that Germany will be over-weighted and the other countries will be under-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1B.

3. What this investor finds is that the investor needs extremely high expected returns to achieve a diversified portfolio. This is because the investor is unable to diversify away the risk of the German equity market.

4. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1C.

5. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1D.

6. What this investor finds is that the investor needs extremely low expected returns to achieve a diversified portfolio. This is because the investor is unable to diversify away the risk of the German equity market.

7. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1E.

8. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1F.

9. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1G.

10. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1H.

11. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1I.

12. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1J.

13. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1K.

14. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1L.

15. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1M.

16. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1N.

17. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1O.

18. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1P.

19. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1Q.

20. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1R.

21. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1S.

22. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1T.

23. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1U.

24. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1V.

25. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1W.

26. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1X.

27. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1Y.

28. The investor can also shift the expected returns in order to achieve a diversified portfolio. For example, the investor can shift the expected return for Germany by 5% lower than the market capitalization-weighted average expected return for the rest of the world. This approach may suggest that Germany will be under-weighted and the other countries will be over-weighted. The optimal portfolio weights generated by this approach are presented in Chart 1Z.
The Black-Litterman asset allocation model addresses these practical issues in solving the Markowitz optimization by allowing the portfolio manager to express views about the expected returns of assets as well as the optimal portfolio weights. Since publication in 1990, the Black-Litterman asset allocation model has gained wide application in many financial institutions. As developed in the variance approach, the Black-Litterman model gives the optimal weights for these portfolios. 

The model is relatively easy to understand and implement. A user of the standard portfolio optimizers often finds that their specification of a set of deviations from market capitalization weights in the potential investment universe implies unrealistic requirements on stated expected returns on the portfolios. The Black-Litterman model places the optimal weights for these portfolios on a set of potential investment universe, where the investor has the world average risk tolerance. The objective of the investor is to maximize the utility function of the investor.

The investor has the world average risk tolerance. The expected return is the neutral reference point of the Black-Litterman model. The expected return is the mean-variance problem. Using the equilibrium expected returns, the optimal portfolio weights are the market portfolio weights. However, when the given expected returns are not the market capitalization weights, the optimal portfolio weights are quite different from what one would have expected. The unconstrained optimal portfolio is the market portfolio plus a weighted sum of the portfolios in the user views. The weights for these portfolios are given by the elements of the vector $\omega^*$. Since the columns of matrix $\Sigma$, $\Sigma = \Omega \Lambda$, the additional steps are the following formula:

$\omega^* + \Lambda \Sigma^{-1} \Delta = \Omega \Lambda \Lambda^{-1} \Lambda \Sigma^{-1} \Delta = \Omega \Sigma^{-1} \Delta$.

The variance process is the following formula:

$\Pi \Sigma^{-1} \Delta$.

The additional steps are the following formula:

$\Pi \Sigma^{-1} \Delta$.

The mean of the expected returns is the neutral reference point of the Black-Litterman model. The expected returns: $\Pi \Sigma^{-1} \Delta$. The new view on the portfolio is an increasing function of its confidence level. The investor has the world average risk tolerance. The objective of the investor is to maximize the utility function of the investor.

The Black-Litterman model gives the optimal weights for these portfolios. The Markowitz formulation of the portfolio optimization problem is a systematic approach to deviating from this point to express one's views about portfolios, rather than a complete vector of expected returns. As developed in the variance approach, the user inputs a complete set of expected returns, and the output is a vector of optimal portfolio weights. This set of deviations from market capitalization weights in the potential investment universe implies unrealistic requirements on stated expected returns on the portfolios. The Black-Litterman model places the optimal weights for these portfolios on a broad benchmark. As developed in the variance approach, the user inputs any number of views about the expected returns of assets as well as the optimal portfolio weights. The Markowitz formulation of the portfolio optimization problem is a systematic approach to deviating from this point to express one's views about portfolios, rather than a complete vector of expected returns. As developed in the variance approach, the user inputs a complete set of expected returns, and the output is a vector of optimal portfolio weights.
The Black-Litterman model is the central framework for our modeling process. Our process starts with finding a set of views that are profitable. In the Quantitative Strategies group at Goldman Sachs Asset Management, we develop quantitative models and use these models to manage portfolios. The practical application of the Black-Litterman model is to adjust a set of expected returns on each asset. Although we manage many portfolios, using different benchmarks, different targeted risk levels, and different constraints on the portfolios, the same set of expected returns from the Black-Litterman model is used to construct a portfolio of diversified assets.

The view that German equity will outperform the rest of Europe is now expressed in the view. This is quite natural because the view includes a position in German equity and short positions of market capitalization-weighted portfolio of France and the United States. If the investor is only half as confident as in the unconstrained case, the weight on a portfolio of Canada versus USA is unchanged at 4% expected return. The remaining the same. These effects are illustrated in chart 4.

When will the weight on a portfolio be positive, negative, or zero? It turns out that the sign of the weight on a portfolio also has a very intuitive interpretation. The weight on a portfolio is an increasing function of the strength of the view. Without the view, the view has no impact at all. Since we already know that the weight on a portfolio is an increasing function of the strength of the view, the weight can be interpreted as the degree of confidence associated with the view. This is quite intuitive. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view. The weight can be interpreted as the degree of confidence associated with the view. The weight is an increasing function of the strength of the view.
The solutions to the unconstrained optimization problem as well as to several special constrained optimization problems are given in Goldman Sachs Investment Management.

The Constrained Optimal Portfolio

Arriving at the optimal portfolio is somewhat more complex in the presence of constraints. In general, when there are constraints, the exact way to find the optimal portfolio is to use the Black-Litterman model to generate the market equilibrium portfolio and then solve the constrained optimization problem. In these situations, the intuition of the Black-Litterman model is more difficult to see. However, one can see the intuition in slightly modified form in a new special constrained case. Suppose the portfolio about which the investor has a view is the optimal portfolio in the unconstrained case.

Risk Constraint

In the case of having a risk constraint, the investor's objective of selecting the optimal portfolio to maximize the expected return while keeping the risk at a maximum can be solved by solving the unconstrained optimization problem to obtain the optimal weights, then applying the risk constraint. In Chart 4, the investor has two views. If the investor is targeting a risk level of 20% per annum, the model is exactly the portfolio of a long position in Germany and a short position of market capitalization-weighted Germany. Since both France and the United Kingdom are under-performing the United States, their portfolio weights increase as well. The same intuition applies to the other countries in Chart 4.

Optimal Deviation: One View on Germany versus Rest of Europe

From the graph of the expected returns in Chart 3A, it seems counter-intuitive to see that the expected returns for both France and the United Kingdom increase as well. The same intuition applies to the other countries in Chart 3A. The optimal portfolio weights in Chart 3B are computed by solving the constrained optimization problem. Compared to the equilibrium weights, the optimal portfolio increases the weight in Germany and decreases the weights in the United States, Canada, and Japan. This is because the investor has views about the market portfolio for Germany.

Portfolio Weights, Black-Litterman Model

In Chart 5, one can see the deviation of the optimal portfolio from the equilibrium weights are exactly proportional to the proportion of a long position in Germany and a short position of the United States portfolio.

Portfolio Weights, Black-Litterman Model

The optimal portfolio weights in Chart 5 are computed by solving the constrained optimization problem. Compared to the equilibrium weights, the optimal portfolio increases the weight in Germany and decreases the weights in the United States, Canada, and Japan. This is because the investor has views about the market portfolio for Germany.
The solutions to the unconstrained optimization problem as well as to several special constrained optimization problems are given in Chart 3.

In Chart 3A, the investor can calculate the constrained optimal portfolio weights by taking the solution of the unconstrained optimization problem to the desired risk level. In Chart 3B, the investor can calculate the constrained optimal portfolio weights by taking the solution of the unconstrained optimization problem to the desired risk level. In Chart 3C, the investor can calculate the constrained optimal portfolio weights by taking the solution of the unconstrained optimization problem to the desired risk level. In Chart 3D, the investor can calculate the constrained optimal portfolio weights by taking the solution of the unconstrained optimization problem to the desired risk level. In Chart 3E, the investor can calculate the constrained optimal portfolio weights by taking the solution of the unconstrained optimization problem to the desired risk level. 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In general, factors. have exposures to the same set of historically profitable return-generating factors. The Black-Litterman model is the central framework for our modeling process. Our process starts with finding a set of views that are profitable. We develop quantitative models and use these models to manage portfolios. The Black-Litterman model is the central framework for our modeling process. Our process starts with finding a set of views that are profitable.

The view that German equity will outperform the rest of Europe is now natural for the investor to invest more in the portfolio when she believes the view on the portfolio? Suppose now the investor still believes Germany versus Rest of Europe?

The next question is: how does the weight on a portfolio change when the view changes? What will happen if the investor is more bullish? For example, the expected return on the United States is more bullish? For example, the expected return on the United States is more bullish. Inputting all these parameters into the Black-Litterman model, the weight of the view increases.

When will the weight on a portfolio be positive, negative, or zero? It turns out that the weight on a portfolio is an increasing function of the strength of the view, we can deduce that the weight on the portfolio is positive, if and only if the expected return of the view increases.

In Chart 4, we show the weights of portfolios in the views and optimal deviations. In Chart 3A, in addition to the original view that German equity will outperform the rest of the European markets, the investor has another view that Canada versus the United States is more bullish. This set of expected returns is the neutral market equilibrium expected returns. The deviations of optimal portfolio weights from the neutral market equilibrium (capitalization weights) portfolio. In the unconstrained case, the manager can be confident that the same tradeoff of risk and return equal the outstanding supply. This set of expected returns is the neutral market equilibrium expected returns. The deviations of optimal portfolio weights from the neutral market equilibrium (capitalization weights) portfolio.

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In the case when the investor lacks views, the Litterman Model uses the market equilibrium as the starting point for the portfolio. These views are combined with the equilibrium weights to adjust the portfolio. The weights for the rest of the European markets are determined in the following way: 

Deviations from the market equilibrium expected returns are calculated. The Black-Litterman model uses these inputs to generate a set of expected returns. A tilted optimal portfolio is then determined. The practical application of this process is illustrated in Chart 1A. In Chart 1A, in addition to the original view that German equity will outperform the rest of Europe, the market also contains two views: 

1. Germany versus the United States: Germany should outperform the United States. The Black-Litterman model captures the degree of uncertainty associated with this view, and thus the Black-Litterman portfolio is tilted away from the market portfolio in the direction of the view. The optimal tilt is represented in Chart 2A. 

2. Canada versus the United States: Canada should underperform the United States. The deviations of the optimal portfolio weights from the market weights show an overweight in Germany and an underweight in Canada. This is the optimal portfolio when the investor is very confident in her views. If the investor is only half as confident in her view, then the magnitude of the weight on Germany versus the rest of Europe decreases, which is also very intuitive. If the investor has less confidence in a view, she would take less risk in the view, everything else remaining the same. These effects are illustrated in Chart 4.

The next question is: how does the weight on a portfolio change when the view changes? What will happen if the investor expresses a stronger or a weaker view on a country versus the market? Suppose now the investor still believes Germany versus the United States is a 5% per annum. The deviations of optimal portfolio weights from the market weights show an overweight in Germany and a strong underweight in the United States, which is the tilt that the investor intended views in the model.

In contrast to the traditional approach, the Black-Litterman model adjusts weights to make the market equilibrium expected returns consistent with the view being expressed. Because the view is expressed in terms of deviations from the value implied by the equilibrium weights, it is not necessary to separately specify the value of the portfolio. The weights of the portfolio, representing the view, are given by: 

\[ w = \frac{1}{\tau} \left( \Sigma - \tau^{-1} \mathbf{q} \right)^{-1} \mathbf{q} \]
Optimal Portfolios: Variance Optimizer versus Black-Litterman Model

The solutions to the unconstrained optimization problem as well as to several special constrained optimization problems are given in Appendix C.

The intuition behind the Black-Litterman model is that the views represent a long portfolio, and the equilibrium weights are exactly proportional to the portfolio of a long investor. 

Arriving at the optimal portfolio is somewhat more complex in the presence of constraints. In general, when there are constraints, the easiest way to find the optimal portfolio is to use the Black-Litterman model and then solve the constrained optimization problem. In these situations, the solution of the Black-Litterman model is more difficult to use. However, there are two special cases where the solution of the Black-Litterman model can be used as part of the solution of the constrained portfolio construction, even in the constrained case.

One View on Germany versus the Rest of Europe

In Chart 2A, one can see that the deviations of the optimal portfolio from the equilibrium weights are exactly proportional to the portfolio of a long investor. The views on Germany versus the rest of Europe are negative, and the optimal portfolio increases the weight in Germany and decreases the weight in France or the United Kingdom.

In Chart 2B, one can see that the deviations of the optimal portfolio from the equilibrium weights are exactly proportional to the portfolio of a long investor. The views on Germany versus the rest of Europe are negative, and the optimal portfolio increases the weight in Germany and decreases the weight in France or the United Kingdom.

In Chart 2C, one can see that the deviations of the optimal portfolio from the equilibrium weights are exactly proportional to the portfolio of a long investor. The views on Germany versus the rest of Europe are negative, and the optimal portfolio increases the weight in Germany and decreases the weight in France or the United Kingdom.

All these examples display a very important property of the Black-Litterman model. In these examples, in addition to the risk constraint, the investor faces a budget constraint which forces the sum of the total portfolio weights to be one. In general, when there are constraints, the easiest way to find the optimal portfolio is to use the Black-Litterman model and then solve the constrained optimization problem. In these situations, the solution of the Black-Litterman model is more difficult to use. However, there are two special cases where the solution of the Black-Litterman model can be used as part of the solution of the constrained portfolio construction, even in the constrained case.

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Model Portfolios' scaling of the market equilibrium portfolio (Chart 5).

In many cases, in addition to the risk constraint, the investor faces a budget constraint which forces the sum of the total portfolio weights to be one. It can be shown that the optimal portfolio is the market portfolio to be one. It can be shown that the optimal portfolio is the market capitalization-weighted portfolio on the one hand. Since both France and the United Kingdom are positively correlated with the view portfolio and the view raises the expected return of this portfolio, it is natural to see the expected returns on both France and the United Kingdom increase as well. The same intuition applies to the other countries in Chart 2A.

In Chart 2B, the investor has a view about this portfolio, she simply invests in this portfolio, on top of her normal weights, the equilibrium weights.

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From the graph of the expected returns in Chart 2A, it seems counterintuitive to see that the expected returns for both France and the United Kingdom are negative. On the contrary, the view expressed does not say France or the United Kingdom will do badly, but that they will underperform with respect to the equilibrium portfolio. This result is not coincidental, as it is intuitive to see that the expected returns for both France and the United Kingdom increase as well. The same intuition applies to the other countries in Chart 2A.

In Chart 2A, we compare the expected returns for both France and the United Kingdom, and observe that they are positively correlated with the view portfolio and the view raises the expected return of this portfolio. It is natural to see the expected returns on both France and the United Kingdom increase as well. The same intuition applies to the other countries in Chart 2A.

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The optimal portfolio weights in Chart 2B are computed by solving the constrained optimization problem. Compared to the equilibrium weights, the optimal portfolio assigns weights to France and Germany, and short France and the United Kingdom results in the portfolio representing the investor's view. This result is also intuitive. Since the investor has a view about this portfolio, on top of her normal weights, the equilibrium weights.

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In the last example, a beta constraint is added to the budget and risk constraints. A beta constraint forces the beta of the portfolio to be unity. With this constraint, there is a special global minimum variance portfolio with the highest Sharpe ratio in the space of all possible portfolios satisfying the constraints. It can be shown that the optimal portfolio is a linear combination of the constrained optimal portfolio and the market equilibrium portfolio. This portfolio is chosen such that the combination satisfies both the risk constraint and the budget constraint (Chart 5).

Golden Sachs Investment Management

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In the Quantitative Strategies group\(^8\) at Goldman Sachs Asset Management, we develop quantitative models and use these models to manage portfolios. The Black-Litterman model is the central framework for our modeling process. Our process starts with finding a set of views that are profitable. For example, it is well known that portfolios based on certain value factors and portfolios based on momentum factors are consistently profitable. We forecast the expected returns on portfolios which incorporate these factors and portfolios based on momentum factors are consistently profitable. We forecast the expected returns on portfolios which incorporate these factors and construct a set of views. The Black-Litterman model takes these views and constructs a set of expected returns on each asset. Although we manage many portfolios for many clients, using different benchmarks, different targeted risk levels, and different constraints on the portfolios, the same set of expected returns from the Black-Litterman model is used throughout. Even though the final portfolios may look different due to the differences in benchmarks, targeted risk levels and constraints, all portfolios are constructed to be consistent with the same set of views, and all will have exposures to the same set of historically profitable return-generating factors.

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\(^8\) This group is part of the Quantitative Resources Group and was formerly known as Quantitative Research.
The Black-Litterman asset allocation model differs from traditional approaches in allowing the portfolio manager to express views about the expected returns of specific assets or groups of assets. These views are used to adjust the equilibrium expected returns, which are determined by market data and historical relationships. The adjusted expected returns are then used as inputs to the mean-variance optimization process. This allows for the incorporation of subjective judgments while maintaining the benefits of diversification and statistical inference.

The model involves a few key steps:

1. **Expression of Views**: The portfolio manager expresses views about the expected returns of specific assets or groups of assets. These views can be in the form of mean, variance, or correlations.

2. **Elicitation of Prior**: A prior distribution is elicited for the expected returns. This is often done by using the market portfolio as a reference point and adjusting the views accordingly.

3. **Adjustment of Expected Returns**: The equilibrium expected returns are adjusted by the views to create the posterior expected returns.

4. **Portfolio Optimization**: The adjusted expected returns are then used to optimize the portfolio weights using standard mean-variance techniques.

The Black-Litterman model is designed to address some of the limitations of traditional mean-variance optimization, such as the influence of extreme views, the difficulty of specifying expected returns for all assets, and the inability to incorporate subjective judgments.

An example of the output of the Black-Litterman model is shown in the chart. The chart illustrates how the model adjusts the weights of various assets in the portfolio based on the views expressed by the portfolio manager. In this example, the model has adjusted the weights of the United States, United Kingdom, and France markets, while the weights for other regions have remained relatively unchanged.

In conclusion, the Black-Litterman model provides a framework for incorporating subjective views into the asset allocation process while maintaining the benefits of a data-driven approach. It allows for the expression of views about asset returns without unrealistic requirements for expected returns. The model has gained wide application in many financial institutions and has been implemented in various forms and with different adaptations to suit specific needs.
### Goldman Sachs Investment Management

#### General

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### Goldman Sachs Investment Management

#### Model Portfolios

<table>
<thead>
<tr>
<th>Country</th>
<th>Equity Index</th>
<th>Volatility (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada</td>
<td>20.3</td>
<td>2.2</td>
</tr>
<tr>
<td>Germany</td>
<td>27.1</td>
<td>5.5</td>
</tr>
<tr>
<td>USA</td>
<td>18.7</td>
<td>61.5</td>
</tr>
</tbody>
</table>

Table 2: Correlations among the equity index returns.

<table>
<thead>
<tr>
<th>Country</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada</td>
<td>0.488</td>
</tr>
<tr>
<td>France</td>
<td>0.478</td>
</tr>
<tr>
<td>Germany</td>
<td>0.515</td>
</tr>
<tr>
<td>USA</td>
<td>0.491</td>
</tr>
</tbody>
</table>

### BlackBerry Investment Management

#### Starting from Equilibrium Expected Returns

Black and Litterman also demonstrated the shortcomings of several other methods for specifying a starting point for expected returns. They proposed the following equation to derive expected returns:

\[
\hat{E} = \left( I + \rho \Sigma \right)^{-1} \mu
\]

where \( \hat{E} \) represents the equilibrium expected returns, \( \rho \) is the risk aversion parameter representing the world average risk tolerance, \( \Sigma \) is the variance-covariance matrix, and \( \mu \) is the vector of required returns.

Starting from the equilibrium expected returns with a risk aversion parameter of 5, the expected return for Canada is 0.488%, for France is 0.478%, and for Germany is 0.515%. These expected returns are not particularly intuitive as they are derived from the equilibrium portfolio weights.

### Unstable Behavior in Optimizers

For example, starting with a portfolio that is already leaning towards Europe, the optimizer may result in large changes in the portfolio weights, even when the expected returns are not significantly different from the equilibrium expected returns. This can lead to suboptimal portfolio weights and may not align with the investor's risk tolerance.

### Starting from Equilibrium Expected Returns

The starting point for expected returns derived from the equilibrium portfolio weights can be used as a more stable starting point for the optimization process. This approach aims to reduce the sensitivity of the portfolio weights to small changes in the expected returns.

### Expected Returns Shifted for European Countries

In this example, the expected return for Germany is set to 5% higher than the equilibrium expected return, while keeping the other expected returns unchanged. The weight for France now is -94.8 percent! This demonstrates how small changes in expected returns can lead to large changes in portfolio weights.

### Conclusion

Starting from the equilibrium expected returns provides a more stable and intuitive starting point for the optimization process, reducing the sensitivity of the portfolio weights to small changes in expected returns.
Investment Management Research
The Intuition Behind Black-Litterman

Appendix A

Table 1: Assumed volatilities, market capitalization weights, and equilibrium expected returns for the equity markets in the seven countries.

<table>
<thead>
<tr>
<th>Country</th>
<th>Volatility (%)</th>
<th>Market Cap Weight (%)</th>
<th>Equilibrium Expected Return (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada</td>
<td>6.5</td>
<td>9.2</td>
<td>6.9</td>
</tr>
<tr>
<td>France</td>
<td>8.2</td>
<td>9.3</td>
<td>6.6</td>
</tr>
<tr>
<td>Germany</td>
<td>7.7</td>
<td>11.4</td>
<td>7.8</td>
</tr>
<tr>
<td>Japan</td>
<td>0.310</td>
<td>11.4</td>
<td>0.310</td>
</tr>
<tr>
<td>UK</td>
<td>7.7</td>
<td>11.4</td>
<td>7.8</td>
</tr>
<tr>
<td>USA</td>
<td>6.5</td>
<td>9.3</td>
<td>6.7</td>
</tr>
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<td>6.3</td>
<td></td>
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</tbody>
</table>

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<table>
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<tr>
<th>Country</th>
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<th>France</th>
<th>Germany</th>
<th>UK</th>
<th>USA</th>
<th>Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada</td>
<td>0.605</td>
<td>0.64</td>
<td>0.605</td>
<td></td>
<td></td>
<td>0.605</td>
</tr>
<tr>
<td>France</td>
<td>0.605</td>
<td>0.64</td>
<td>0.605</td>
<td></td>
<td></td>
<td>0.605</td>
</tr>
<tr>
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<td>0.605</td>
<td>0.64</td>
<td>0.605</td>
<td></td>
<td></td>
<td>0.605</td>
</tr>
<tr>
<td>UK</td>
<td>0.605</td>
<td>0.64</td>
<td>0.605</td>
<td></td>
<td></td>
<td>0.605</td>
</tr>
<tr>
<td>USA</td>
<td>0.605</td>
<td>0.64</td>
<td>0.605</td>
<td></td>
<td></td>
<td>0.605</td>
</tr>
</tbody>
</table>

Through the lens of Markowitz’s Modern Portfolio Theory, we can see that the expected return for Germany is 5% higher than the weighted average equilibrium expected return for the rest of Europe. To be precise in expressing her view, the investor translates her view into expected returns.

There are many different ways to translate the view to expected returns. This chart is to be used for illustrative purposes only. Chart 1A demonstrates that when the investor translates her views into expected returns, the optimal weights are sensitive to the investor’s views.

The investor’s views are reflected in the optimal weights and can be used to construct diversified portfolios. The following example demonstrates the sensitivity of the optimal weights to the investor’s views.

When an investor translates her views into expected returns, the optimal weights are sensitive to the investor’s views. Chart 1A demonstrates that when the investor translates her views into expected returns, the optimal weights are sensitive to the investor’s views. The investor’s views are reflected in the optimal weights and can be used to construct diversified portfolios.
Appendix B

1 There are $N$ assets in the market. The market portfolio (equilibrium portfolio) is $w$, the covariance of the return is $\Sigma$. The expected returns are $\mu$, the mean of normally distributed random variables with mean $\tau$. $\tau$ is a vector of normally distributed random variables with $\tau \sim N(0, \tau^2)$. The covariance of $\tau$ is $\Sigma$. $\Sigma$ is a symmetric, positive definite matrix. $\tau$ is a vector of normally distributed random variables with mean $0$, covariance $\Sigma$, and $\mu$ is a vector of $N$ expected returns. The covariance $\Sigma$ is normally distributed with mean zero and covariance $\Sigma$. The uncertainty of the CAPM prior is independent of the CAPM prior and independent of each other. $\Sigma$ is the covariance matrix and $\mu$ is the vector of expected returns.

2 The average risk tolerance of the world is represented by the risk-aversion parameter $\lambda$. The mean-variance problem is to maximize the utility of the portfolio weight $w$. The Black-Litterman model provides the appropriate weights on the portfolios, based on these views or statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights.

3 The investor has the Markowitz formulation of the portfolio optimization problem. In so doing, we demonstrate how the Black-Litterman approach presents a complete set of expected returns, and the portfolio solution presents the optimal portfolio weights. However, the Black-Litterman model along with the covariance matrix) in a portfolio framework has had surprisingly little impact. Why is that the case? The Markowitz formulation of the portfolio-optimization problem is a brilliant quantification of the two basic objectives of investing: maximizing returns and minimizing risk. The Black-Litterman model provides the appropriate weights on the portfolios, based on views or statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights.

4 The investor should first invest in the market portfolio, then deviate from the Black-Litterman model along with the covariance matrix. When the investor has constraints, or a different risk tolerance level, the Black-Litterman model will always generate an optimal portfolio whose weights are relatively easy to implement. The investor should invest in the market portfolio first, then deviate from the Black-Litterman model along with the covariance matrix. When the investor has constraints, or a different risk tolerance level, the Black-Litterman model will always generate an optimal portfolio whose weights are relatively easy to implement.

5 Throughout this paper for simplicity we use the phrase "biased return" to refer to "expected return," and the portfolio solution presents the optimal portfolio weights. However, the Black-Litterman model along with the covariance matrix) in a portfolio framework has had surprisingly little impact. Why is that the case? The Markowitz formulation of the portfolio-optimization problem is a brilliant quantification of the two basic objectives of investing: maximizing returns and minimizing risk. The Black-Litterman model provides the appropriate weights on the portfolios, based on views or statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights.

6 In the case of the Black-Litterman model, the traditional mean-variance approach the user inputs a complete set of expected returns, and the portfolio solution presents the optimal portfolio weights. However, in the Black-Litterman model, the user inputs any number of views or statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights.

7 Let $w_*$ be the vector of the returns of the new portfolios. The new returns are the elements of the vector $w_*$, which can be written as $w_*$ with probability $f_*$ and $w_i = w_i$ with probability $1 - f_*$ for $i = 1, 2, ..., N$. The investor should invest in the market portfolio first, then deviate from the Black-Litterman model along with the covariance matrix. When the investor has constraints, or a different risk tolerance level, the Black-Litterman model will always generate an optimal portfolio whose weights are relatively easy to implement.
Appendix C

1. Given the expected returns $\mu$ and the covariance matrix $\Sigma$, the unconstrained maximization problem

$$\max \ w' \mu - \frac{1}{2} \delta \Sigma w$$

has a solution of $w^* = \frac{-\delta}{\mu' \Sigma \delta} \Sigma \mu$.

2. Given the covariance matrix $\Sigma$, the minimum variance portfolio is

$$\min \ w' \Sigma w$$

subject to $w' \mu = \mu_0$, where $\mu_0$ is a vector with all elements being one.

3. The solution to the risk constrained optimization problem, $w^*$, subject to $w' \Sigma w = \sigma^2$, can be expressed as

$$w^* = \frac{-\delta}{\mu' \Sigma \delta} \Sigma \mu$$

where $\mu$ and $\delta$ are chosen in the way both risk and budget constraints are satisfied.

4. The risk and budget constrained optimization problem can be formulated as

$$\max \ w' \mu$$

subject to $w' \Sigma w = \sigma^2$ and $w' \mu = \mu_0$, where $\mu_0$ is the market portfolio. The solution to the problem has the form $w^* = \frac{-\delta}{\mu' \Sigma \delta} \Sigma \mu$, where $\delta$, $\mu$, and $\Sigma$ are chosen in the way all these constraints are satisfied.

5. The risk-, budget-, and beta-constrained optimization problem can be formulated as

$$\max \ w' \mu$$

subject to $w' \Sigma w = \sigma^2$, $w' \mu = \mu_0$, and $w' \Sigma w = \sqrt{\beta^2}$, where $\beta^2$ is the market portfolio. The solution to the problem has the form $w^* = \frac{-\delta}{\mu' \Sigma \delta} \Sigma \mu$, where $\delta$, $\mu$, $\Sigma$, and $\beta$ are chosen in the way all these constraints are satisfied.

References


In this article and as our title suggests, we demonstrate a method for understanding the intuition behind the Black-Litterman asset allocation model.

To do this, we use examples to show the difference between the traditional mean-variance optimization process and the Black-Litterman process. We show that the mean-variance optimization process, while academically sound, can produce results that are extreme and not particularly intuitive. In contrast, we show that the optimal portfolios generated by the Black-Litterman process have a simple, intuitive property:

- The unconstrained optimal portfolio is the market equilibrium portfolio plus a weighted sum of portfolios representing an investor's views.
- The weight on a portfolio representing a view is positive when the view is more bullish than the one implied by the equilibrium and other views.
- The weight increases as the investor becomes more bullish on the view as well as when the investor becomes more confident about the view.