Lecture 11

- Statictics review and portfolio theory

- Readings:
  - Reader, lecture 13
  - BM Chapter 7
Where are we?

- We know, in principle, how to make investment decisions:
  - PV/NPV
  - Discount cash flows using an appropriate discount rate
- Where does the discount rate come from?

- We need the **Capital Asset Pricing Model (CAPM)**
  - Gives quantitative tradeoff between risk and expected return.
What we need to know

- How to calculate the expected return on a stock.
- How to calculate the variance of stock returns.
- How to apply the above to a portfolio of stocks:
  - How to calculate the expected return on a portfolio.
  - How to calculate the covariance between the return on two stocks.
  - How to calculate the variance of a portfolio.
A Single Stock: Notation and Definitions

Notation:

- $\tilde{r}_i$ is the (random) return on stock $i$.
- $r_{ix}$ is one possible return for stock $i$, which occurs with probability $p_x$.
- $r_i$ is the expected return on stock $i$.

Expected Return: $E(\tilde{r}_i) = r_i = p_x r_{ix}$

Variance: $\text{Var}(\tilde{r}_i) = \sigma_i^2 = E(\tilde{r}_i - r_i)^2 = p_x (r_{ix} - r_i)^2$

Standard Deviation: $\sigma_i = \sqrt{\text{Variance}}$
A Single Stock: Expected Return

Expected Return $= E(\tilde{r}_i) = r_i = \sum_x p_x r_{ix}$

<table>
<thead>
<tr>
<th>$r_{ix}$</th>
<th>$p_x$</th>
<th>$p_x r_{ix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>0.25</td>
<td>-1.25</td>
</tr>
<tr>
<td>0</td>
<td>0.25</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>1.25</td>
</tr>
<tr>
<td>12</td>
<td>0.25</td>
<td>3</td>
</tr>
</tbody>
</table>

$\sum r_i = 3$
A Single Stock: Variance

Stock Variance = \( \text{Var}(\tilde{r}_i) = E(\tilde{r}_i - r_i)^2 = \sum p_x (r_{ix} - r_i)^2 \)

<table>
<thead>
<tr>
<th>( r_{ix} )</th>
<th>( P_x )</th>
<th>( (r_{ix} - r_i)^2 )</th>
<th>( p_x (r_{ix} - r_i)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>0.25</td>
<td>64</td>
<td>16.00</td>
</tr>
<tr>
<td>0</td>
<td>0.25</td>
<td>9</td>
<td>2.25</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>4</td>
<td>1.00</td>
</tr>
<tr>
<td>12</td>
<td>0.25</td>
<td>81</td>
<td>20.25</td>
</tr>
</tbody>
</table>

\[ \text{Var} = 39.50 \]
Take a portfolio worth $200 that contains
- $40 worth of stock A,
- $60 worth of stock B,
- $100 worth of stock C.

Then the portfolio has investment weights of:
- \( w_A = \frac{40}{200} = .20 \) in stock A
- \( w_B = \frac{60}{200} = .30 \) in stock B
- \( w_C = \frac{100}{200} = .50 \) in stock C

The weights on any portfolio must sum to 1
Portfolios: Expected Return

Formula for the (expected) return on a portfolio:

\[ \tilde{r}_p = w_i \tilde{r}_i + w_j \tilde{r}_j \] (actual returns)
\[ r_p = w_i r_i + w_j r_j \] (expected returns)

Notation:
- the expected return on stock i is \( r_i \),
- the expected return on stock j is \( r_j \),
- the weight for stock i is \( w_i \),
- the weight for stock j is \( w_j \).

Example: \( r_i = 3 \), \( r_j = 5 \), \( w_i = .4 \), and \( w_j = .6 \). Then:
\[ r_p = (.4)(3) + (.6)(5) = 4.2. \]
CD and NBF Sample Problem: Finding Portfolio Weights

- Invest in Corporate Disasters (CD) at 4%.
- Or, invest in Nevada beach front property (NBF): Investment will have an expected return of 10%.

- You want a portfolio with expected return 6%.
- What portfolio fraction should you invest in the Nevada property?
Finding Portfolio Weights: Solution

- Use formula for expected return on a portfolio
  
  \[ 6 = w_{CD}(4) + w_{NBF}(10) \]

- Since \( w_{CD} + w_{NBF} = 1 \), \( w_{CD} = 1 - w_{NBF} \).

- So we can write \( 6 = (1 - w_{NBF})(4) + w_{NBF}(10) \).

- We find \( w_{NBF} = 1/3 \).
Covariance: The relationship between two returns

A zero covariance implies no relationship.

A positive covariance implies that when stock i has an exceptionally high return, so (usually) does stock j.

A negative covariance implies that when the return on stock i is unusually high, the return on stock j tends to be unusually low.

\[
\text{Cov}(\tilde{r}_i, \tilde{r}_j) = E(\tilde{r}_i - r_i)(\tilde{r}_j - r_j) = \sigma_{ij} = p_{xy}(r_{ix} - r_i)(r_{jy} - r_j).
\]
Covariance vs. Correlation

People often talk about the correlation between stock $i$ and $j$ ($\rho_{ij}$) instead of their covariance ($\sigma_{ij}$).

The two measures are very closely related.

We can easily convert from one to the other:

\[
\rho_{ij} = \frac{\text{Cov} (\tilde{r}_i, \tilde{r}_j)}{\text{SD}(\tilde{r}_i) \text{SD}(\tilde{r}_j)} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}.
\]
Example: Computing covariance between two stocks

- Probabilities for returns on each stock

| Joint Probabilities for each pair of returns on stocks i and j | $r_{jy}$ Returns on j |
|---|---|---|
| | 2 | 7 | 8 |
| 0 | 0.1 | 0 | 0 |
| 6 | 0 | 0.2 | 0.1 |
| 12 | 0 | 0.2 | 0.4 |
Example 1: Computation Matrix for Variance and Covariance

<table>
<thead>
<tr>
<th>r_{ix}</th>
<th>r_{jy}</th>
<th>p_{xy}</th>
<th>p_{xy} r_{ix}</th>
<th>r_{ix} - r_{i}</th>
<th>r_{jy} - r_{j}</th>
<th>p_{xy} (r_{ix} - r_{i})(r_{jy} - r_{j})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0.1</td>
<td>0.0</td>
<td>0.2</td>
<td>-9</td>
<td>-5</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>0.2</td>
<td>1.2</td>
<td>1.4</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>7</td>
<td>0.2</td>
<td>2.4</td>
<td>1.4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>0.1</td>
<td>0.6</td>
<td>0.8</td>
<td>-3</td>
<td>1</td>
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<tr>
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<td>8</td>
<td>0.4</td>
<td>4.8</td>
<td>3.2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

r_{i} = 9.0
r_{j} = 7.0

Cov (\tilde{r}_{i}, \tilde{r}_{j}) = 5.4
Rules for covariance:

\[ \text{cov}(w_i \tilde{r}_i, w_j \tilde{r}_j) = w_i w_j \text{cov}(\tilde{r}_i, \tilde{r}_j) \]
\[ \text{cov}(\tilde{r}_i, \tilde{r}_j + \tilde{r}_k) = \text{cov}(\tilde{r}_i, \tilde{r}_j) + \text{cov}(\tilde{r}_i, \tilde{r}_k) \]
\[ \text{var}(\tilde{r}_i) = \text{cov}(\tilde{r}_i, \tilde{r}_i) \]

The variance of a portfolio with two stocks:

\[ \text{var}(w_i \tilde{r}_i + w_j \tilde{r}_j) = w_i^2 \text{var}(\tilde{r}_i) + w_j^2 \text{var}(\tilde{r}_j) + 2w_i w_j \text{cov}(\tilde{r}_i, \tilde{r}_j) \]
Example 1: Computing Portfolio Expected Return and Variance

In previous example, we found \( r_i = 9.0, r_j = 7.0 \).

If we form a portfolio with \( w_i = .4, w_j = .6 \), then
\[
 r_p = E\left(0.4\tilde{r}_i + 0.6\tilde{r}_j\right) = 0.4(9) + 0.6(7) = 7.8.
\]

For homework, show that \( \text{var}(\tilde{r}_i) = 16.2 \), and \( \text{var}(\tilde{r}_j) = 3 \).

We also know that \( \text{cov}(\tilde{r}_i, \tilde{r}_j) = 5.4 \), so
\[
\text{var}(\tilde{r}_p) = \text{var}(0.4\tilde{r}_i + 0.6\tilde{r}_j)
= 0.4^2(16.2) + 0.6^2(3) + 2(0.4)(0.6)(5.4)
= 6.264.
\]
Multiple Stock Portfolios: 
Expected Return and Variance

- Expected return on portfolio, \( r_p \):
  \[
  r_p = w_1 r_1 + w_2 r_2 + \ldots + w_n r_n
  \]
  Just plug in values and add terms together.

- Variance on portfolio, \( \text{Var}_p \):
  \[
  \text{Var}_p = \text{Var}(w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + \ldots + w_n \tilde{r}_n)
  \]
  Use “box method” to compute portfolio variance.

Notation: \( \sigma_i^2 = \text{variance}_i ; \ \sigma_i = \text{SD}_i ; \ \sigma_{ij} = \text{covariance}_{ij} \)
Multiple Stock Portfolios: Computing Variance

Add up all boxes in the weighted variance-covariance matrix:

<table>
<thead>
<tr>
<th>Stock</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$w_1^2 \sigma_1^2$</td>
<td>$w_1 w_2 \sigma_{12}$</td>
<td>...</td>
<td>$w_1 w_n \sigma_{1n}$</td>
</tr>
<tr>
<td>2</td>
<td>$w_1 w_2 \sigma_{12}$</td>
<td>$w_2^2 \sigma_2^2$</td>
<td>...</td>
<td>$w_2 w_n \sigma_{2n}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>$w_1 w_n \sigma_{1n}$</td>
<td>$w_2 w_n \sigma_{2n}$</td>
<td>...</td>
<td>$w_n^2 \sigma_n^2$</td>
</tr>
</tbody>
</table>
Multiple Stock Portfolios: Example with 2 Stocks

\[
\text{Var}(w_1 \tilde{r}_1 + w_2 \tilde{r}_2) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \\
= w_1^2 \text{Var}(\tilde{r}_1) + w_2^2 \text{Var}(\tilde{r}_2) + 2w_1 w_2 \text{Cov}(\tilde{r}_1, \tilde{r}_2)
\]

<table>
<thead>
<tr>
<th>Stock</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(w_1^2 \sigma_1^2)</td>
<td>(w_1 w_2 \sigma_{12})</td>
</tr>
<tr>
<td>2</td>
<td>(w_1 w_2 \sigma_{12})</td>
<td>(w_2^2 \sigma_2^2)</td>
</tr>
</tbody>
</table>
Numerical Example 1

What is the variance of a portfolio with:

\[ w_1 = .2, \ w_2 = .8, \ \sigma_1^2 = 10, \ \sigma_2^2 = 20, \ \text{and} \ \sigma_{12} = 5? \]

\[
\sigma_p^2 = (0.2^2)10 + (0.8^2)20 + 2(0.2)(0.8)5 = 14.8
\]
Numerical Example 2

- You have a portfolio of 15 stocks:
  - The first 5 stocks have portfolio weights 0.1 (10%);
  - The remaining stocks have portfolio weight .05 (5%);
  - Each stock has a variance of 10,
  - All stocks have a covariance of 2 with each other.

- What is the variance of your portfolio?
  - To solve this problem, set up the matrix, and use the patterns in the matrix to simplify the box addition.
### Numerical Example 2, Continued

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<th>6</th>
<th>...</th>
<th>15</th>
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<td>(.1²)(10)</td>
<td></td>
<td>(.1²)(2)</td>
<td>(.1)(.05)(2)</td>
<td></td>
<td>(.1)(.05)(2)</td>
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<td>...</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(.1²)(2)</td>
<td></td>
<td>(.1²)(10)</td>
<td>(.1)(.05)(2)</td>
<td></td>
<td>(.1)(.05)(2)</td>
</tr>
<tr>
<td>6</td>
<td>(.1)(.05)(2)</td>
<td></td>
<td>(.1)(.05)(2)</td>
<td>(.05²)(10)</td>
<td></td>
<td>(.05²)(2)</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>(.1)(.05)(2)</td>
<td></td>
<td>(.1)(.05)(2)</td>
<td>(.05²)(2)</td>
<td></td>
<td>(.05²)(10)</td>
</tr>
</tbody>
</table>

- **Top left hand group of 25 boxes (stocks 1-5):** The 5 diagonals are (.1²)(10). The remaining 20 entries are (.1²)(2). Total = 5x.1²x10 + 20x.1²x2 = .9.

- **Bottom right hand 100 boxes (stocks 6-15):** The 10 diagonals = (.05²)(10). The other 90 entries are .05²x2. Total = 10x.05²x10 + 90x.05²x2 = .7.

- **Top right and bottom left group for total of 100 boxes.** Every box has the entry (.1)(.05)(2). Total = 100x.1x.05x2 = 1.

- **Total portfolio variance = .9 + .7 +1 = 2.6.**
Example: Eliminating “firm specific risk” via Diversification

- Consider a special portfolio, in which \( w_i = 1/n \).

<table>
<thead>
<tr>
<th>Stock</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_1^2/n^2 )</td>
<td>( \sigma_{12}/n^2 )</td>
<td>…</td>
<td>( \sigma_{1n}/n^2 )</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>…</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \sigma_{12}/n^2 )</td>
<td>( \sigma_2^2/n^2 )</td>
<td>…</td>
<td>( \sigma_{2n}/n^2 )</td>
</tr>
<tr>
<td>…</td>
<td></td>
<td></td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>n</td>
<td>( \sigma_{1n}/n^2 )</td>
<td>( \sigma_{2n}/n^2 )</td>
<td>…</td>
<td>( \sigma_n^2/n^2 )</td>
</tr>
</tbody>
</table>

- For portfolio variance, add up the boxes.
Computing Variance of the Fully Diversified Portfolio

- Write the sum of the diagonal elements as

\[ \sigma_i^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \frac{1}{n} (\text{avg. variance}) \]

- Now add up off-diagonal elements. There are \( n \times n \) boxes in total of which \( n \) are diagonals leaving \( n \times n - n \) off-diagonal boxes. Add up the off-diagonal boxes to get

\[ \sigma_{ij} = \frac{n^2 - n}{n^2} \quad \sigma_{ij} = \left(1 - \frac{1}{n}\right) (\text{avg. cov.}) \]
Therefore the variance of the portfolio equals
\[
\text{Var}\left(\frac{1}{n} \tilde{r}_1 + \frac{1}{n} \tilde{r}_2 + \ldots + \frac{1}{n} \tilde{r}_n\right) = \frac{1}{n} \times \text{Avg.Var} + \left(1 - \frac{1}{n}\right) \times \text{Avg.Cov.}
\]

So what is the variance of a “perfectly” diversified portfolio? This occurs as \( n \) goes to infinity. In that case:
\[
\text{Var}\left(\frac{1}{n} \tilde{r}_1 + \frac{1}{n} \tilde{r}_2 + \ldots + \frac{1}{n} \tilde{r}_n\right) = \text{Avg.Cov.}
\]
Variance of a Fully Diversified Portfolio: Comment

- Note that **diversification** eliminates all risk except the average covariance of the stocks.
- Since people can do this for themselves, we’ll see that
  - The market only rewards people for holding “market risk” (covariance between all stocks)
  - No reward for “firm specific risk” (variance of a single stock).