Lecture 21
Options Pricing

Readings
- BM, chapter 20
- Reader, Lecture 21
Outline

Last lecture:
- Examples of options
- Derivatives and risk (mis)management
- Replication and Put-call parity

This lecture
- Binomial option valuation
- Black Scholes formula
Put-Call parity and early exercise

- Put-call parity:
  \[ C = S + P - K / (1 + r)^T \]

- Put-call parity gives us an important result about exercising American call options.
  \[ C = S + P - K / (1 + r)^T \geq S - K / (1 + r)^T \]
  \[ > S - K. \]

- In words, the value of a European (and hence American) call is strictly larger than the payoff of exercising it today.
Early Exercise

- In the absence of dividends, you should thus never exercise an American call prior to expiration.
  - What should you do instead of exercising if you’re worried that your currently “in-the-money” (S > K) option will expire “out-of-the-money” (S < K)?
  - What is the difference between the value of a European and an American call option?

- **Note:** If the stock pays dividends, you might want to exercise the option just before a dividend payment.

- **Note:** This only applies to an American call option.
  - You might want to exercise an American put option before expiration, so you receive the strike price earlier.
Key Questions About Derivatives

- How should a derivative be valued relative to its underlying asset?
- Can the payoffs of a derivative asset be replicated by trading only in the underlying asset (and possibly cash)?
  - If we can find such a replicating strategy, the current value of the option must equal the initial cost of the replicating portfolio.
  - This also allows us to create a non-existent derivative by following its replicating strategy.
- This is the central idea behind all of modern option pricing theory.
We shall be using the following notation a lot:

- \( S \) = Value of underlying asset (stock)
- \( C \) = Value of call option
- \( P \) = Value of put option
- \( K \) = Exercise price of option
- \( r \) = One period riskless interest rate
- \( R = 1 + r \)
Factors Affecting Option Value

The main factors affecting an option’s value are:

<table>
<thead>
<tr>
<th>Factor</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S ), stock price</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( K ), exercise price</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( \sigma ), Volatility</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( T ), expiration date (Am.)</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( T ), expiration date (Eu.)</td>
<td>+</td>
<td>?</td>
</tr>
<tr>
<td>( r )</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Dividends</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

What is not on this list?
Consider a European call option on a stock, price $S$, exercise price $K$, and 1 year to expiration.

Suppose over the next year the stock price will either move up to $uS$, or down to $dS$. For example:

- $u \times S = 104$  
  ($u = 1.3$)
- $d \times S = 64$  
  ($d = 0.8$)
Write $C$ for the price of a call option today, and $C_u$ and $C_d$ for the price in one year in the two possible states, e.g. (suppose $K = 90$):

- $C_u = \text{Max}[104 - 90, 0] = 14$
- $C_d = \text{Max}[64 - 90, 0] = 0$
Example

Consider forming a portfolio today by
- Buying 0.35 shares
- Borrowing $20.3637 (assume interest rate = 10%)

Cost of portfolio today = 0.35 \times 80 - 20.3637
= $7.6363

What’s it worth next year?

0.35 \times 104 - (20.3637 \times 1.1) = $14.00

0.35 \times 64 - (20.3637 \times 1.1) = $0.00

Do these payoffs look familiar?

How much would you pay for the call option?
This is an example of a replicating portfolio. Its payoff is the same as that of the call option, regardless of whether the stock goes up or down.

The current value of the option must therefore be the same as the value of the portfolio, $7.6363

- What if the option were trading for $5 instead?
- Note that this result does not depend on the probability of an up vs. a down movement in the stock price.

The call option is thus equivalent to a portfolio of the underlying stock plus borrowing.

How do we construct the replicating portfolio in general?
Forming a Replicating Portfolio in General

- Form a portfolio today by
  - Buying $\Delta$ shares
  - Lending $B$

- Cost today = $\Delta S + B$ (= option price)

- Its value in one year depends on the stock price:
  - $\Delta u S + B(1+r)$
  - $\Delta d S + B(1+r)$
  - $\Delta S + B$
We can make the two possible portfolio values equal to the option payoffs by solving:

\[ \Delta uS + B(1+r) = C_u, \]
\[ \Delta dS + B(1+r) = C_d. \]

Solving these equations, we obtain

\[ \Delta = \frac{C_u - C_d}{(u-d)S}, \quad B = \frac{uC_d - dC_u}{(u-d)(1+r)}. \]
Example

- $S = 80, u = 1.3, d = 0.8$
- $K = 90, r = 10\%.$

\[ \begin{align*}
\text{Cu} &= 14 \\
\text{Cd} &= 0 \\
80 &\quad 104 \\
\quad 64 &\quad C
\end{align*} \]
From the formulae,

\[ \Delta = \frac{14 - 0}{(1.3 - 0.8)80} = 0.35, \]

\[ B = \frac{1.3(0) - 0.8(14)}{(1.3 - 0.8)1.1} = -20.3637. \]

Hence

\[ C = \Delta S + B \]

\[ = 0.35 \times 80 - 20.3637 \]

\[ = $7.6363 \]
"Delta" (Δ) is the standard terminology used in options markets for the number of units of the underlying asset in the replicating portfolio.

- For a call option, Δ is between 0 and 1
- For a put option, Δ is between 0 and –1 (see HW 11)

\[ \text{option value} = (\text{asset price} \times \text{"delta"}) + \text{lending} \]

For small changes, Δ measures the change in the option’s value per $1 change in the value of the underlying asset.

A position in the option can be hedged using a short position in Δ of the underlying asset.
Define \( R = (1+r) \), and let \( p = (R - d) / (u - d) \).

A little algebra shows that we can write:

\[
C = \frac{pC_u + (1-p)C_d}{1+r}
\]

I.e. to value the call (or any derivative)

- Calculate its “expected” value next period pretending \( p \) is the probability of prices going up.
- Discount the expected value back at the riskless rate to obtain the price today.

**Note**: We don’t need the **true** probability of an up movement, just the “**pseudo-probability**”, \( p \).
- \( p \) would be the true probability if everyone were risk-neutral.
Example

- Using the previous example,

\[ p = \frac{R - d}{u - d} = \frac{1.1 - 0.8}{1.3 - 0.8} = 0.6 \]

- Hence the call price equals

\[
C = \frac{pC_u + (1 - p)C_d}{1 + r} = \frac{(0.6 \times 14) + (0.4 \times 0)}{1.1} = 7.6363.
\]
Shortcomings of Binomial Model

- The binomial model provides many insights:
  - Risk neutral pricing
  - Replicating portfolio containing only stock + borrowing
  - Allows valuation/hedging using underlying stock
- But it allows only two possible stock returns.
- To get around this:
  - Split year into a number of smaller subintervals
  - Allow one up/down movement per subperiod
  - n subperiods give us n+1 values at end of year.
Example, two subperiods

Start at the end, and work backwards through tree.
See reader pp. 164 – 167 for details.
How big should up/down movements be?

For a given expiration date, we keep overall volatility right as we split into \( n \) subperiods [each of length \( t (= T/n) \)], by picking

\[
u = e^{\sigma \sqrt{t}}, \quad d = 1/u = e^{-\sigma \sqrt{t}}.
\]

As \( n \) gets larger, the distribution of the asset price at maturity approaches a **lognormal** distribution, with expected return \( r \), and annualized volatility \( \sigma \).

What happens to option prices as we increase the number of time steps?
Binomial prices vs. # steps
(S=K=60, T=0.5, \(\sigma=30\%\), r=8\%)

What’s this limit?
In the limit, the price of a European call option converges to the **Black-Scholes formula**, 

\[
C = S \, N\left( x \right) - Ke^{-rt} \, N\left( x - \sigma \sqrt{t} \right),
\]

where  

\[
x \equiv \frac{\log\left( \frac{S}{K} \right) + (r + \frac{\sigma^2}{2})t}{\sigma \sqrt{t}}
\]

\[
r\text{ here is a \textit{continuously-compounded} interest rate.}
\]
Interpretation of Black-Scholes

This is just a special case of our old formula
\[ C = \Delta S + B \]
The formula tells us the values of \( \Delta \) and \( B \) in the replicating portfolio.
Note that, for a European call,
- \( \Delta \) is always between 0 and 1.
- \( B \) is negative, and between 0 and \(-\text{PV}(K)\) (i.e. borrow).
Black-Scholes Call Prices for different Maturities

$K = 50$
$r = 0.06$
$\sigma = 0.3$
Combining the Black-Scholes call result with put-call parity, we obtain the Black-Scholes put value,

\[
P = Ke^{-rt} \left[ 1 - N \left( x - \sigma \sqrt{t} \right) \right] - S \left[ 1 - N \left( x \right) \right],
\]

where \( x \equiv \frac{\log \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right)t}{\sigma \sqrt{t}} \).

Note that, for a European put,

- \( \Delta \) is always between 0 and -1 (i.e. short).
- \( B \) is positive, and between 0 and PV(K) (i.e. lend).