Overview of Lecture 4

- Examples of Quoted vs. True Interest Rates
  - Banks
  - Auto Loan

- Forward rates, spot rates and bond prices
  - How do things change when interest rates vary over different periods?
  - Present Value in terms of spot/forward rates
  - Converting between forward and spot rates
  - Calculating forward/spot rates from bond prices (what information can we get out of bond prices?)
Bank Interest Calculations

The sign in the window reads:
- Interest rate: 8% compounded daily.
- Effective annual yield: 8.33%.

What do these numbers mean?

Which (if either) is a true annual interest rate?
- What amount will you actually earn in 1 year per dollar invested?
Bank Interest Calculations

- “Compounded daily”: Each day the bank multiplies the prior day’s balance by \((1 + \frac{.08}{365})\)
  - True daily interest rate = \(\frac{.08}{365} = 0.022\%\)
- Over 1 year your earnings per dollar are
  \[
  \left[1 + \frac{.08}{365}\right]^{365} = 1.0833.
  \]
- Quoted rate of 8\% (APR) is not a true interest rate
- Effective annual yield, 8.33\%, is a true interest rate.
An Actual Auto Loan

The advertisement says: “12 month car loans. Only 9%!”

The small print shows the payments are calculated on a $10,000 car as follows:

– On a 12 month $10,000 loan with 9% interest rate, you owe a total of $10,900.
– Twelve equal payments come out to $10,900/12 = 908.33 per month.

Is 9% the actual interest rate on the loan?
The true interest rate must set the present value of the payments equal to the initial loan.

\[
10,000 = \frac{908.33}{1+r_{\text{monthly}}} + \frac{908.33}{(1+r_{\text{monthly}})^2} + \ldots + \frac{908.33}{(1+r_{\text{monthly}})^{12}}
\]

Therefore \( r_{\text{monthly}} = 1.35\% \).

The annual interest rate \( r_{\text{annual}} = (1+r_{\text{monthly}})^{12} - 1 = 17.5\% \) percent per year!
Auto Loan: Rate Calculations

- How to find these numbers on your calculator.
  
  \{ 10000 \} [CHS][PV]  
  \{ 908.33 \} [PMT]  
  \{ 12 \} [n]  
  [i]  

- (Note that the 1.35 is a percentage)
Auto Loan - Why is the Rate so High?

- Each month you pay off part of the principal, thereby borrowing less later in the year.
- The calculation by the dealer assumes that you borrow all $10,000 for the whole year.
- A loan with a true 9% annual interest rate comes out to 12 monthly payments of $872.89.
If the annual interest rate equals 9%, then:

1. \((1.09)^{1/12} = 1 + r_{\text{monthly}}\), so
2. \(r_{\text{monthly}} = 0.0072073\).

The monthly payment therefore solves:

\[
10,000 = \frac{PMT}{1.0072} + \frac{PMT}{(1.0072)^2} + \cdots + \frac{PMT}{(1.0072)^{12}}.
\]
Auto Loan -
Monthly Payments at \( r = 9\% \)

Using your calculator:

\[
\begin{align*}
\{10000\}[\text{CHS}][\text{PV}] & \quad \text{Loan.} \\
\{.72073\}[\text{i}] & \quad \text{Monthly rate.} \\
\{12\}[\text{n}] & \quad \text{Number of payments.} \\
[\text{PMT}] & \quad \text{Calc. Payment.}
\end{align*}
\]
Interest rates for different periods usually differ.

There are two standard ways to summarize this:

- **Forward rate, $f_t$:** Tells us how much interest we earn if we agree today to invest $ from end of year $t-1$ to end of year $t$.
  
  - It’s the rate we earn just **in period** $t$ (for one future period).

- **Spot rate, $r_t$:** Tells us the rate we earn on an **annual** basis if we leave our money in the bank for $t$ years.
  
  - It’s a **sort of** average of the forward rates in periods $1, 2, 3, \ldots, t$. 

Rate subscripts all defined by the period of final cash flow. Spot rates all start at date 0. Forward rates all last 1 period.
How much do you earn leaving C in the bank for t years?
- In terms of forward rates: $C(1+f_1)(1+f_2)\ldots(1+f_t)$
- In terms of spot rates: $C(1+r_t)^t$.

These must give the same result, so

$1+f_1 = 1+r_1$

$(1+f_1)(1+f_2) = (1+r_2)^2$

$(1+f_1)(1+f_2)(1+f_3) = (1+r_3)^3$

etc.

These allow us to convert forward to/from spot rates.
Example - Calculating Spot Rates from Forward Rates

Suppose \( f_1 = 0.14, f_2 = 0.15, f_3 = 0.10 \). Then

- Solving for \( r_1 \):
  
  \[
  r_1 = f_1 = 0.14
  \]

- Solving for \( r_2 \):
  
  \[
  (1 + r_2)^2 = (1 + f_1)(1 + f_2) = (1.14)(1.15)
  \]
  
  \[
  r_2 = [(1.14)(1.15)]^{1/2} - 1 = 0.1450
  \]

- Solving for \( r_3 \):
  
  \[
  (1 + r_3)^3 = (1 + f_1)(1 + f_2)(1 + f_3) = (1.14)(1.15)(1.10)
  \]
  
  \[
  r_3 = [(1.14)(1.15)(1.10)]^{1/3} - 1 = 0.1298
  \]
Example - Calculating Forward Rates from Spot Rates

Suppose $r_1 = .05$, $r_2 = .08$, $r_3 = .11$. Then

- Solving for $f_1$:
  \[ f_1 = .05 \]

- Solving for $f_2$:
  \[ 1.05(1 + f_2) = 1.08^2 = 1.1664 \]
  \[ f_2 = .1109 \]

- Solving for $f_3$:
  \[ (1.05)(1.1109)(1 + f_3) = 1.11^3 = 1.3676 \]
  \[ f_3 = .1725 \]
Calculating PV with time-varying interest rates

We can write the PV formula in terms of either forward rates or spot rates:

\[
PV = C_0 + \frac{C_1}{(1+f_1)} + \frac{C_2}{(1+f_1)(1+f_2)} + \frac{C_3}{(1+f_1)(1+f_2)(1+f_3)} + \ldots
\]

\[
= C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)^2} + \frac{C_3}{(1+r_3)^3} + \ldots
\]

Example: \( C_0=10, C_1=11, C_2=12, f_1=.14, f_2=.15. \)

\[
PV = 10 + \frac{11}{1.14} + \frac{12}{(1.14)(1.15)} = 28.8
\]

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A bond is defined by its **maturity date**, **coupon rate**, and **face (par) value**.
- The bond makes coupon payments every 6 months until maturity.
- Each coupon payment = \( \frac{1}{2} \) (coupon rate)(face value)
- The face value is paid out at the maturity date (together with the last coupon payment).

For example, consider a 10 year bond with a face value of $1000 and 10% annual coupons (paid semi-annually).
- This bond has twenty coupons, each of which pays $50.
- After 10 years the company redeems the bond for $1000.
Calculating interest rates from bond prices

- In the real world no one gives you forward/spot rates. You must estimate them.
- We do observe bond prices, which equal the present value of the bond’s cash flows.
- We can use observed bond prices to work out what forward/spot rates must be.
You are given the following 6 bond prices:

<table>
<thead>
<tr>
<th>Name</th>
<th>Period</th>
<th>FV</th>
<th>Coup. ($)</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1,000</td>
<td>100</td>
<td>1000.00</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1,200</td>
<td>60</td>
<td>1086.49</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>800</td>
<td>160</td>
<td>1004.50</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>1,500</td>
<td>50</td>
<td>1233.21</td>
</tr>
<tr>
<td>E</td>
<td>11</td>
<td>300</td>
<td>0</td>
<td>20.00</td>
</tr>
<tr>
<td>F</td>
<td>12</td>
<td>10,000</td>
<td>0</td>
<td>512.82</td>
</tr>
</tbody>
</table>
Calculating interest rates from bond prices

- Goal: find \( f_1 \), then \( f_2 \), then \( f_3 \), then \( f_4 \) and finally \( f_{12} \).
- Also find \( r_1 \), \( r_2 \), \( r_3 \), \( r_4 \), \( r_{11} \) and \( r_{12} \).
  - Note: We are finding 6 month forward/spot rates.
- Answers:
  
  \[
  \begin{align*}
  f_1 &= .1 \\
  r_1 &= .1 \\
  f_2 &= .11 \\
  r_2 &= .105 \\
  f_3 &= .08 \\
  r_3 &= .0966 \\
  f_4 &= .06 \\
  r_4 &= .087 \\
  f_{12} &= .3 \\
  r_{11} &= .27914 \text{ and } r_{12} = .28086
  \end{align*}
  \]
Calculating interest rates from bond prices

- **f₁**: PV = 1000, C₁ = 1100
  
  \[ 1000 = \frac{1100}{1+f₁} \]

  so \( 1 + f₁ = \frac{1100}{1000} = 1.1 \)

  \( f₁ = .1 \), so \( r₁ = f₁ = .1 \).

- **f₂**: PV = 1086.49, C₁ = 60, C₂ = 1260

  \[ 1086.49 = \frac{60}{1+f₁} + \frac{1260}{(1+f₁)(1+f₂)} \]

  \[ 1086.49 = \frac{60}{1.1} + \frac{1260}{1.1(1+f₂)} \],

  so \( f₂ = .11 \)

  Finally, \( 1+r₂ = [(1+f₁)(1+f₂)]^{1/2} \), so \( r₂ = .105 \)

- To find \( f₃, f₄, r₃, r₄ \) repeat this procedure.
Calculating interest rates from bond prices

- For $f_{12}$ use fact that E and F are zero coupon bonds.
- Bond E:
  
  \[
  PV = 20, \ C_1=C_2=C_3=\ldots=C_{10}=0, \text{ and } C_{11}=300
  \]
  
  \[
  20 = \frac{300}{(1+f_1)(1+f_2)\cdots(1+f_{11})}
  \]
  
  - So \((1+f_1)(1+f_2)\cdots(1+f_{11}) = 15\)
  - But \((1+r_{11})^{11} = (1+f_1)(1+f_2)\cdots(1+f_{11}) = 15, \text{ so } r_{11} = .27914\)

- Lesson: You can read spot rates right off zero coupon bonds
Calculating interest rates from bond prices

**Bond F:**

\[ PV = 512.82, C_1 = C_2 = \ldots = C_{11} = 0, C_{12} = 10,000 \]

\[ 512.82 = \frac{10,000}{(1+f_1)(1+f_2)\ldots(1+f_{11})(1+f_{12})} \]

- **Now use** \((1+f_1)(1+f_2)\ldots(1+f_{11}) = 15\) **to get**

\[ 512.82 = \frac{10,000}{15(1+f_{12})}, \text{ so } f_{12} = .3 \]

- **Read** \(r_{12}\) **off the bond price and promised payment:**

\[ (1+r_{12})^{12} = (1+f_1)(1+f_2)\ldots(1+f_{12}) \]

\[ = \frac{10,000}{512.82} = 19.5, \text{ so } r_{12} = .28086 \]