Limits to Arbitrage and Lockup Contracts

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Abstract

This paper studies the interaction between a fund manager who has information regarding a long-term opportunity and investors who are uncertain about their manager. Investor behavior determines the fund liquidation risk. Manager portfolio decisions interact with investor behavior through the learning channel. A loop between limits to arbitrage and liquidation arises: higher liquidation risk pushes the manager to invest less in the long-term opportunity, which leads investors to liquidate the manager earlier, which feeds back into higher liquidation risk. The introduction of a lockup reduces limits to arbitrage, but also leads to managerial entrenchment. The optimal lockup maturity balances these two forces. For a calibration that matches moments of a hedge fund database, the model produces quantitatively large limits to arbitrage distortions and lockup maturities consistent with the data.

In contrast to mutual funds where 95% of the funds are open-ended, hedge fund managers often impose restrictions on the ability of investors to redeem their cash. These restrictions typically take the form of an initial black out window, the so called lockup. Recent research [Aragon, 2007b, Agarwal et al., 2009] has shown that investors in funds with lockup restrictions are well compensated, earning higher net of fees excess returns of more than 4% per year. Since managers choose to lock investors in, they ought to benefit from a restriction on outflows. In this paper, I study an economy where lockups are beneficial because they ease constraints on manager arbitrage activity. In my model, investors use returns to assess the skill of their money manager, and then vote with their feet. Concerned about their reputations, managers focus on short-term strategies, foregoing long-term opportunities. Lockups mitigate this distortion, but they are costly as investors fear being stuck with a bad manager.
Investor rational learning links the cost and the benefit of lockup restrictions. This insight gives us a theory that has quantitative predictions regarding how much lockups can reduce limits to arbitrage, how large the lockup premium should be, and which lockup maturities maximize value.

Recently, researchers have documented compelling evidence of the link between the behavior of fund investors and asset pricing anomalies. For example, Coval and Stafford [2007] show that assets of funds that experience large outflows tend to experience high long-term returns going forward. In the context of hedge funds, Brunnermeier and Nagel [2004] analyze the ability of hedge funds to time the end of the technology boom. The authors give a vivid description of the different outcomes for two hedge funds during this period. Both George Soros’s Quantum Fund and Julian Robertson’s Jaguar Fund took bets against tech stocks during most of 1999, and had outflows during the whole year. Whereas Julian Robertson’s fund had a limited exposure to the tech sector and was liquidated just before March 2000, Soros’s fund increased its exposure to the tech sector toward the end of the year and attracted new capital. Although the Quantum Fund performed poorly during the March crash, it did survive. The survival of the Quantum Fund illustrates why deviating from the highest expected return strategy is extremely tempting when fund liquidation is a short-term threat.

In my model, rational investors liquidate managers like Julian Robertson because in the short-term they struggle to differentiate a good manager pursuing a profitable long-term strategy from a bad manager. So even though investors understand that Mr. Robertson might be betting against something really profitable, as time goes by investors become increasingly worried that Mr. Robertson does not know what he is doing. My model captures this tension in the investor learning problem by assuming a skilled manager faces a time-varying opportunity set. Sometimes the highest expected return strategy is to follow a positive carry strategy, but other times the highest expected return is in a long-term reversal strategy. The finance literature documented several examples of positive carry strategies, such as the carry trade [Burnside et al., 2010], merger arbitrage [Mitchell and Pulvino, 2001], momentum [Jegadeesh, 1993], among several other strategies that look like liquidity provision by sophisticated financial intermediaries [Mitchell and Pulvino, 2011]. In addition to paying positive returns most of the time, the common feature of these strategies is that sometimes they crash, yielding very negative returns. In the model, short-term momentum and long-term reversals are strategies that are exactly the opposite of each other. In the
former, the manager collects a small carry but bears a large loss when a crash arrives; in the latter the manager pays the carry, but wins big in a crash event. A skilled manager has insight as to how large a crash will be, enabling them to optimally alternate between momentum and reversal strategies. In the model, a skilled manager invests in strategies that share the key features of the hedge fund strategies documented in the literature, but uses private information to only take on crash risk when it is not too high.

This paper contributes to the literature on portfolio delegation by studying the investor and manager problem jointly. Basak et al. [2007] and Chapman et al. [2009] are good examples of papers that study manager portfolio choices in economies where fund flows are assumed to chase performance. Berk and Green [2004], Berk and Stanton [2007], Dangl et al. [2008] have modeled the investor learning problem thoroughly, but assume that the manager’s portfolio choice is always efficient. This paper complements this literature by filling this gap, studying how delegation is impacted by reputation concerns in a setting where investors are rational. In addition to studying the manager-investor problem jointly, the paper goes beyond risk-shifting incentives and puts the trade-off between short-term and long-term performance at the center of the manager considerations.

Most importantly, the introduction of rational investors naturally leads to a trade-off theory of optimal lockup maturity. Endogenous investor behavior provides us with a link between the costs and benefits of longer lockups. An increase in contract maturity leads to both an increase in entrenchment costs and a reduction in limits to arbitrage. Entrenchment costs increase because the longer a contract is, the more likely investors will be in a situation in which they would like to withdraw but cannot. At the same time, limits to arbitrage decrease with contract maturity because a longer contract increases the probability that a long-term strategy pays off before investors force the fund liquidation. The longer time horizon improves managers ex-ante incentives to invest in long-term bets with high expected returns, because it makes these bets less risky from the manager’s perspectives. From the trade-off between entrenchment and limits to arbitrage I solve for the optimal contract maturity.

My paper is not the first to study the choice of liability maturity of corporations or financial intermediaries. Diamond [1991], and more recently Stein [2005] study an environment in which good firms/managers attempt to signal their type by choosing short-term contracts. While this type of channel is certainly plausible and is likely a driver of some of the contract
choices we see in the data, it provides no insight as to what the efficient contract maturity should be. The trade-off between entrenchment and limits to arbitrage explored in this paper is a natural complement to these type of models, as it can teach us what the optimal contract maturity should depend on. In their model investors out of equilibrium beliefs about deviating managers drive contract maturity choices, something that is obviously very hard to measure. In my theory, contract maturities are driven by measurable quantities, among them the degree of competition for money management skills, the signal-to-noise ratio of fund performance, and the horizon of the manager reversal strategy.

The cost of long-term contracting arises from the interaction between learning and competition. When competition for managerial skill is weak, fund investors face a symmetric bet on their manager skill during the life of the contract. When the manager is better than expected, investors capture some of the rents going forward but also stand to lose when the news is bad. As competition for skill increases, better-than-expected managers are able to capture more of the increase in the perceived value of their skill. This means that assuming everything else is constant, investors in funds with long lockups stand to earn lower expected returns on average over the life of the contract. This competition channel predicts that as manager’s track records become more visible, investors will demand higher compensation to invest in a lockup fund, leading to shorter lockups. The strength of the learning magnifies the effects of competition. As investors expect to learn more quickly about the quality of their manager, they demand higher premiums to write long-term contracts.

While the speed of learning increases the cost of a lockup, it also increases its benefits. The interaction of learning and time-varying trading horizon leads managers that currently pursue long-term strategies to be liquidated too often. The increase in liquidation risk leads these managers to distort their portfolio towards the less profitable short-term strategy. The result is a powerful amplification mechanism. The shorter the horizon of a manager, the more investors expect her to focus on short-term strategies. This leads investors to rationally learn more from short-term performance, what feeds-back in even stronger incentives for the manager to focus on short-term strategies. In such an environment an increase in the lockup maturity produces large reductions in limits to arbitrage. Longer lockups are effective not only because a longer trading horizon reduces limits to arbitrage, but because more long-term choices lead investors to focus less on short-term performance, which further amplifies the increase in the manager horizon.
This amplification mechanism is a direct consequence of rational investor behavior in an environment where portfolio choice is not observable. This channel predicts that as the implementation of strategies becomes more complex, investors will increasingly find themselves unable to evaluate whether their manager is really pursuing a long-term reversal strategy, resulting in large limits to arbitrage distortions. Managers understand that a higher speed of learning is related in the short-term to a higher liquidation risk, but in the medium term it results in a faster reputational build up. This dynamic mechanism is an important qualification to Shleifer and Vishny [1997] seminal paper. Shleifer and Vishny were the first to emphasize that fund manager strategies are constrained by investors will, and managers are likely to perceive long-term strategies as too risky when they need to convince investors of their worth in the short-term. My paper builds on Shleifer and Vishny’s insight and pushes their idea one step further by showing that their mechanism works even in an environment where everyone is rational. While it is certainly plausible that the manager trading environment is so rich that investors end up using rules of thumb when learning about their manager’s skill, a fully articulated theory of investor behavior allows us to study how long we can expect a skilled money manager to be in a situation where the doubt about her skill is strong enough to distort her portfolio choices. This restriction between the magnitude of the distortion and its persistence is something that only a fully dynamic model can speak to.

Using a calibration that matches key moments of hedge fund returns, I show that the learning mechanism produces quantitatively large distortions with the typical new manager having 170 b.p. lower expected return in the first year of operation due to limits to arbitrage. This number implies that the average manager has 14% lower expected returns than they would if they were always maximizing expected returns. Limits to arbitrage shrink as managers build reputational capital. The average half-life of the limits to arbitrage distortion is 1.4 years, but in paths without crashes distortions increases to up to 230 b.p. after 5 years with no crashes. This same calibration produces lockup maturity choices that are consistent with those observed in the hedge fund data. Aragon [2007a] documents that conditional on having a lockup contract, the median fund has a lockup of 12 months. In the model the median maturity choice is 13 months. The success of the model in reproducing this feature of the lockup maturity data is a consequence of the large lockup premium existent in the data. To match a premium of 400 b.p. for a fund with a one-year lockup maturity, the
model requires that the cost of being unable to fire a bad manager be high. This large cost of entrenchment drives both the premium and the relatively short maturity choices in the data.

Optimal lockup maturities lead to substantial decreases in limits to arbitrage: the average fund manager has an expected return 110 b.p. higher than if they were forced to have a one month contract, the typical performance evaluation periods among hedge funds. While the reduction is substantial, optimal contracts are too short to fully drive limits to arbitrage to zero and the average manager still has 60 b.p. lower expected returns over the first year of operation. The calibration indicates that the maturity of financial intermediary liability has quantitatively large implications for portfolio choice.

I use the model to compare infrequent redemption periods with lockup contracts, and I show the importance of the option to reduce contract maturity when necessary. Lockup contracts are only renewed when the manager has a high enough reputation to attract new money, while investors in a fund with an infrequent redemption window are forced to decide between cashing-out and reinvesting for another redemption period. I show that this option to shorten the contract maturity embedded in lockup contracts is an important aspect of its ability to reduce limits to arbitrage. Maturity shortening allows survival of managers who would otherwise be liquidated. In contrast with Brunnermeier and Oehmke [2010] who emphasize the role of externalities between investors in making maturities too short, in my model, maturity shortening is an efficient response to a decrease in manager reputation. The ability to liquidate increases in value the more likely it becomes that a manager is bad.

Another new aspect of maturity choice that the model highlights is that from the perspective of a skilled manager the advantages of writing long-term contracts are inversely related to the ability to writing them. Managers who have a relatively poor reputation benefit the most from writing longer contracts, but only attract capital if they choose very short contracts. As the manager’s reputation increases, longer contracts become feasible, but they became less valuable as the liquidation risk that the manager faces is lower.

The model produces a tight link between entrenchment costs and lockup maturities as entrenchment costs go from -200 b.p. to -800 b.p. per year, the optimal maturity shrinks from 50 to eight months. This decrease in maturities perversely happens exactly as limits to arbitrage distortions become larger. Entrenchment costs simultaneously increases the downside of being locked in a long-term contract and the liquidation risk that a skilled manager faces,
since short-term performance becomes more informative when.

Competition for skill increases entrenchment costs, it drives up lockup premiums and leads to shortening of lockup maturities. As the expected time to a outside offer decreases, current investors expect faster dilution of positive news about their manager. Dilution asymmetrically shifts the burden of negative news to current investors. Quantitatively, a reduction in the expected time to an outside offer from one year to one month, reduce lockup maturities from 23 to 11 months.

While the calibration matches key properties of hedge fund data, the possibility of heterogeneity across strategies can easily let us think that other parameter combinations are also plausible. I explore the model implication for limits to arbitrage and lockup maturities across a wide set of parameters. The general message of these comparative statics is that the quantitative importance of limits to arbitrage is a robust feature of the environment. As long as the manager has a meaningful long-term investment opportunity, and faces the amount of liquidation risk implied by the hedge fund data\(^1\), the model implies large and persistent limits to arbitrage distortions.

\(^1\)Brown et al. [2001] documents a 18% probability of liquidation for a new manager in the first year of operation.
1 Model

The model examines the problem of a new fund manager and the fund investors. The economy has two stages: first the manager chooses the fund lockup maturity, then she manages the fund until liquidation. A skilled risk-neutral manager discounts cash-flows at the rate $\rho$ and generates excess returns by picking securities and timing the market. For an investment of one dollar the selection strategy evolve as follows:

$$dP^{\alpha} = (\alpha^s + r)dt + \sigma dB^i + \tilde{\omega}^i dN^\zeta,$$

where $dB^i$ and $\tilde{\omega}^i$ are shocks specific to the manager portfolio. $dB^i$ is a standard Brownian motion and $\tilde{\omega}^i$ is a normally distributed random variable with mean zero and variance $\omega^2$. $dN^\zeta$ is a Poisson process with intensity $\delta_{\zeta}$ and $\alpha^s$ is the risk-adjusted excess return a skilled manager is able to generate by selection. Security selection exposes the manager to idiosyncratic risk during normal times and during crashes. The timing strategy consists of optimally alternating between a momentum and a long-term reversal strategy. The returns on the momentum asset evolve as follows,

$$dP_{t}^{M} = (\lambda + r)dt - \zeta dN^\zeta.\quad (2)$$

The momentum strategy consists of receiving a premium $\lambda$ as long there is no crash ($dN^\zeta = 0$), but when there is a crash ($dN^\zeta = 1$), the strategy loses $P_{t-}^{M} - P_{t}^{M} = -\zeta$. Since crashes are infrequent, the momentum strategy generates positive returns most of the time, even though it is assumed to have no alpha on average ($\lambda = E[\zeta dN^\zeta]$). So a manager without timing skill that invests in this momentum strategy has performance that looks good in the short-run, but in the long-run her performance delivers no alpha. A manager can invest at most one dollar per dollar managed, long (momentum) or short (reversal), in this asset, before fees fund returns evolve as:

$$dR = \quad dP^{\alpha} + \pi dP^{M} - (\pi - 1)rdt$$
$$dR = \quad (r + \alpha^s + \pi \lambda)dt + \sigma dB^i - (\pi \zeta + \tilde{\omega}^i)dN^\zeta \quad (3)$$

The manager can choose any portfolio weight between fully long and fully short ($\pi \in [-1, 1]$), but pure return maximization will always drive the manager to the corners. Going long the
momentum asset ($\pi = 1$) maximize expected returns if $E_t[dP^m - rdt] > 0$, which happens if the expected crash is small enough. While in states that the expected crash is large, a reversal strategy will maximize expected returns. I model this time-variation in crash-risk by assuming the crash severity alternates between two values ($\zeta^R > \zeta^M$) with equal unconditional probability $p^R = p^M = \frac{1}{2}$. When $\zeta = \zeta^M$, we have that $\lambda + \zeta^M E_t[dN^c] > 0$ and a momentum strategy maximizes expected returns ($\pi^{ef}(\zeta^M) = 1$). When $\zeta = \zeta^R$, the opposite is true and is more profitable to invest in a reversal strategy ($\pi^{ef}(\zeta^R) = -1$). A skilled manager that maximizes expected returns generates a total abnormal performance of

$$\alpha = \alpha^s + \delta_\zeta \frac{\zeta^R - \zeta^M}{2}. \tag{4}$$

Since leverage is fixed, the larger the time-variation is crashes detected by the manager, the higher the profits of timing.

1.1 An illustrative example

Consider a manager who knows that crash risk is high ($\zeta^R$) and has to decide her position for a period of length $dt$. The manager gets a continuation value $G$ if she survives until the next period. Investors stick with their manager as long as the manager track-record is not too bad. Let the manager reputation $X$ summarize the manager past track-record, investors will want to cash out if $X + \Delta R_{dt} < X_L$, where $X_L$ is the liquidation boundary. However, the manager is only liquidated if investors want to cash out and the lockup contract has expired ($L = L^e$). Assuming the expiration event is driven by a Poisson process, longer lockups are less likely to expire in any given period, $\frac{\partial \text{Prob}(L = L^e)}{\partial T} < 0$. In addition to the lockup, the fund contract compensates the manager with a performance fee $k$ for the excess performance the manager is able to deliver. The manager problem can be written as

$$\max_k \left( \alpha^s dt + \pi (\lambda dt - \zeta^R (1 - e^{-\delta_\zeta dt})) + (1 - \text{Prob}(L = L^e) \text{Prob}(X + \Delta R_{dt} < X_L)) G \right). \tag{5}$$

Managers are less likely to be liquidated if they have good reputations or longer contracts. The lockup acts as a substitute for reputation in the near term. Liquidation risk will be higher if investors perceive performance to be more informative ($\Delta \uparrow$). If managers perceive the franchise value of managing the fund as high ($G \uparrow$), liquidation risk will impact how aggressively the manager pursues the long-term reversal strategy The manager first order
condition is,

\[
k(\lambda \Delta - \zeta R (1 - e^{-\delta \Delta})) - (1 - e^{-\delta \Delta}) \frac{\partial \text{Prob}(X + \phi R \Delta < X_L)}{\partial \pi} G. \tag{6}
\]

While the performance pay pushes the manager to maximize expected returns and go short the momentum asset, the impact of liquidation risk can push the manager either way. Let \( \phi(\cdot) \) be the standard normal density\(^2\), then liquidation concerns can be written as,

\[
\frac{\partial \text{Prob}(X + \phi R \Delta < X_L)}{\partial \pi} = e^{-\delta \Delta} \phi \left( \frac{\mu_x - \phi \lambda \pi \Delta}{\phi \sigma \Delta} \right) \left( -\frac{\lambda}{\sigma} \right) + (1 - e^{-\delta \Delta}) \phi \left( \frac{\mu_x - \phi \pi (\lambda \Delta - \zeta R)}{\phi \sqrt{\sigma^2 \Delta^2 + \omega^2}} \right) \left( \frac{(\zeta R - \lambda \Delta)}{\sqrt{\sigma^2 \Delta^2 + \omega^2}} \right). \tag{7}
\]

The managers short-term liquidation concerns (the first term), pushes the manager to go long the momentum asset. The manager knows that if the crash does not happen, the more aggressively she invests in the long-term strategy (\( \pi \downarrow \)), the lower her fund returns in periods without crashes will be, and the more likely fund liquidation becomes. Long-term liquidation concerns will push the manager to go short the momentum asset and reinforce the positive effect of pay for performance, since performing well in crash events reduces the probability of liquidation. The relative importance of these two competing forces depends on the horizon of the reversal arbitrage (\( \delta \zeta \)), and the expected marginal impact of the portfolio choice on the probability of liquidation. The longer the horizon of the arbitrage (\( \delta \zeta \downarrow \)), the more weight the manager places on short-term liquidation risk. The expected marginal impact depends on the level of liquidation risk at the manager choice and the impact that changes in the portfolio have on the manager reputation. The manager knows that if she is aggressive, the probability of liquidation during a crash is very small as her returns are likely to be very large. This low level of liquidation risk implies that the marginal long-term liquidation concerns are smaller when the manager maximizes expected returns, which results in very weak incentives to maximize expected returns.

This shows how short-term liquidation concerns distort the manager portfolio from maximizing expected returns. Liquidation risk leads to limits to arbitrage. It is easy to see how a lockup provision can reduce the link between performance and liquidation risk, leading the

\(^2\)And let \( \mu_x \) be all the other terms that do not depend on the manager choice: \( \mu_x = X_L - \phi E[R \Delta - \lambda \pi \Delta] \)
manager to put more weight on her performance incentives than on liquidation concerns. In particular, consider a manager of a closed-end fund \((T \to \infty)\). This manager faces no liquidation risk and any positive performance fee would be enough to incentivize her to maximize expected returns. As the lockup maturity is reduced \((T \downarrow)\), fund liquidation becomes more sensitive to fund performance, resulting in a more short-term oriented manager. In short, the manager becomes more exposed to limits of arbitrage as maturities become shorter. Progressively larger performance fees are needed to induce the manager to do the right thing. In the model, limits to arbitrage arise from bad incentives related to short-term liquidation risk, and not exogenous constraints on arbitrage activity. In the limit, the future of a manager of an open-ended fund \((T \to 0)\) will depend exclusively on her performance. Liquidation risk and limits to arbitrage are maximum in this case.

This analysis is similar in spirit to Shleifer and Vishny’s seminal paper, and illustrates the link between the contingency of funding and the manager desire to pursue an opportunity that is likely to pay poorly in the short-term. However, such a model leaves several questions unanswered: Why would investors liquidate managers with negative short-term performance if the high expected return opportunity takes time to pay off? If the link between performance and reputation is due to learning shouldn’t managers move away from this liquidation boundary relatively quickly? If they do not move away from the liquidation boundary isn’t a sign that learning is weak and liquidation risk low? How does one reconcile these intuitions? Can learning generate quantitatively large limits to arbitrage without generating plausible amount of fund liquidation? If the threat of short-term liquidation is so costly why not lock investors in? Why will long-term reputational concerns not mitigate short-term liquidation concerns?

To answer all of these questions I spell out a theory of how investors learn about their manager (section 1.2) and how they evaluate investment in a fund given their current information and expected future news about about their manager (section 1.4). I derive the manager endogenous valuation of operating the fund (section 1.6). Most importantly, I impose consistency between behavior and beliefs of managers and investors (section 1.8). The equilibrium behavior between managers and investors yields rich feedback effects with self-reinforcing dynamics. These effects are illustrated in Figure 1. As we saw in the example, managers will reduce her position in the long-term reversal strategy after increases in fund liquidation risk. Rational investors understand the manager’s incentives, and anticipating
lower expected returns, investors choose to liquidate the manager earlier than if the manager was investing optimally. Endogenous investor behavior further amplifies fund liquidation risk and the incentives to pursue short-term oriented strategies. The second amplification mechanism works through the informativeness of short-term performance. The position reduction in the long-term strategy makes short-term performance more informative, leading investors to rationally learn more from short-term returns. Stronger learning leads to a more volatile reputation and ultimately to an increase in liquidation risk.

1.2 Endogenous Reputation Dynamics

In this section, I characterize the evolution of the manager’s reputation, taking as given the investors’ beliefs about her portfolio choices. Consider a manager who begins managing the fund in period zero, whose skill is unknown. Let $\phi = 1$ if the manager is skilled and $\phi = 0$ if she is bad. Investors share a prior belief that the manager is skilled with probability $a_0 = E_0[\phi]$ and do not observe the fund portfolio ($\pi$) or the crash state ($\zeta$). I will refer to
investors beliefs about the manager type as the manager reputation. After observing fund returns, investors update their priors. Let $a_t = E^I_t[\phi]$ denote time $t$ manager reputation. Bayesian updating is made taking investors beliefs about the manager portfolio as given.

**Assumption 1.** A bad manager expected short-term performance is given by $r + \nu$, where $\nu < 0$ is the amount the bad manager wastes in transaction costs. She takes the same level of idiosyncratic risk $(\sigma, \omega)$ as the skilled manager, and she is always long the momentum asset $\pi = 1$.

The bad manager is non-strategic and is only introduced in order to formalize the investor learning problem. It is important that the bad manager under-performs relative to the investor’s best outside options. This guarantees that there are fund liquidations in equilibrium. It is also important that idiosyncratic risk is equal across skilled and bad managers, so that the learning problem doesn’t become trivial.

In particular, let $\Omega^a_t$ be all relevant information available to agent $a$ up to time $t$. In particular, $\Omega^{m\phi}_t$ denotes all relevant information available to a manager of type $\phi$. Define $\pi^I(\Omega^{m1}_t)$ as investors beliefs about a skilled manager choices given history $\Omega^{m1}_t$. I divide the learning process into two states: the short-term (periods without a crash), and the long-term (periods with a crash). The statistical learning problem investors face is equivalent to that of an econometrician who tries to differentiate between two processes of identical variance but different means. Consider investors optimal filtering after observing a return $r_{t+\Delta}$ in a period of length $\Delta$. Taking as given investors’ beliefs about the skilled manager portfolio, Bayes rule implies,

$$a_{t+\Delta} = \frac{e^{-\frac{(r_{t+\Delta} - \mu^s)\Delta^2}{2\sigma^2 \Delta}} a_t}{e^{-\frac{(r_{t+\Delta} - \mu^b)\Delta^2}{2\sigma^2 \Delta}} a_t + e^{-\frac{(r_{t+\Delta} - \mu^b)\Delta^2}{2\sigma^2 \Delta}} (1 - a_t)}$$

where $\mu^s$ and $\mu^b$ are the investor’s expectations regarding the average short-term performance of skilled manager and a bad manager.

It is useful to represent manager reputation in terms of its log-likelihood ratio. Let $X = \ln(\frac{a}{1-a})$, then

$$X_{t+\Delta} = \ln \left( \frac{e^{-\frac{(r_{t+\Delta} - \mu^s)\Delta^2}{2\sigma^2}} a_t}{e^{-\frac{(r_{t+\Delta} - \mu^b)\Delta^2}{2\sigma^2}} (1 - a_t)} \right) = \frac{1}{\Delta \sigma} \left( r_{t+\Delta} - \frac{\mu^s + \mu^b}{2} \right)$$

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The filter works as follows: if investors expect that, on average skilled managers deliver better short-term performance than bad managers \((\mu^s - \mu^b > 0)\) then positive return surprises \((r_{t+\Delta} - \frac{\mu^s + \mu^b}{2}\Delta > 0)\) will signal good news about manager skill. The informational content of short-term performance is measured by \(\Delta dB = \frac{\mu^s - \mu^b}{\sigma}\), the short-term performance signal-to-noise ratio. Performance news is measured relative to how an average manager is expected to perform. Recognizing that the manager’s expected short-term performance depends on investors’ beliefs about their portfolio choice, the evolution of investors priors can be characterized as follows:

**Proposition 1. (Short-term reputation evolution)**

Let \(\Delta dB_t = \frac{\alpha + E[\pi^f(\Omega^m_t)]\lambda - (\lambda + \nu)}{2}\) be the signal-to-noise ratio of short-term performance, and let \(\mu dB_t = \frac{\alpha + E[\pi^f(\Omega^m_t)]\lambda + (\lambda + \nu)}{2}\) be the average expected short-term performance across manager types. Let \(X_t \equiv \ln \left( \frac{\alpha_t}{1 - \alpha_t} \right)\) be the log-likelihood ratio that the manager is skilled, then the log-likelihood process evolves as \(dX_t = \frac{\Delta dB_t}{\sigma} \left( dR_t - \mu dB_t dt \right)\).

**Proof.** See Lipster and Shiryaev [2002].

Note that from the perspective of investors, the manager reputation is a martingale. Under the manager’s own information set, their reputation will be a sub or a super-martingale depending on the manager type and asset allocation. A key aspect of the learning problem is that investors don’t know if the manager is currently pursuing a momentum or a reversal strategy, hence investors have to form beliefs about the manager choices, and learn from performance by averaging across the two crash-risk states.

**Assumption 2.** If the manager maximize expected returns \(\pi = \pi^{ef}\), positive short-term performance surprises are interpreted as good news about manager type \((\Delta dB_f = \frac{\alpha + E[\pi^f(\zeta - 1)]\lambda - \nu}{\sigma} > 0)\).

When a crash happens, investor’s update their beliefs differently as they observe a very large return (positive or negative). The extent that this large return leads to a lot of learning will depend on investors beliefs about differences between the choices of skilled and bad managers. If investors believe that both managers have the same exposure to crash risk, they will not update their beliefs regardless of the actual return realization. They interpret returns as a function of pure luck. When investors believe skilled managers have a different crash exposure than bad managers, crashes will be times of sharp learning.
Assumption 3. Once the crash hits, investors immediately learn the (old) crash state $\zeta$. \footnote{This assumption is convenient to keep the learning problem tractable. Nothing substantial changes if one assumes that investors don’t know the crash state. But the learning problem would become more complicated as the return distribution of the skilled manager fund would become a mixture of normals.} The crash state does not change between crashes $(N^\zeta_{t_1} - N^\zeta_{t_2} = 0 \Rightarrow \zeta_t = \zeta_{t_2})$, and it is sampled from the unconditional after a crash $(Pr(\zeta_{t_1} = \zeta^R | \zeta_t = \zeta) = Pr(\zeta_{t_2} = \zeta^M | \zeta_t = \zeta) = \frac{1}{2})$.

I interpret crash arrivals as the “long-term”, since crashes are infrequent. During crashes learning behavior is almost identical to during normal times, but it is lumpy. Because investors know the crash state ex-post, the filtering problem is again equivalent to the problem of distinguishing between two normal distributions of different means.

Proposition 2. (Reputation evolution)

Let $\Delta_t^\zeta = -\zeta(E[\pi^I(\Omega^{m}_t) | \Omega^I_t, \zeta] - 1)$ be the signal-to-noise ratio of crash performance, and let $\mu_t^{dN} = \langle -\zeta \rangle (1 + E[\pi^I(\Omega^{m}_t) | \Omega^I_t, \zeta])$ be the average of the expected crash performance of both skilled and bad managers, and $dR_t$, the fund crash event return. Immediately after a jump event, the manager reputation evolves as

$$dX_t = \frac{\Delta_t^{dB}}{\sigma} (dR_t - \mu_t^{dB} dt) + \frac{\Delta_t^{dN}}{\omega} (dR_t - \mu_t^{dN}) dN^\zeta$$

(10)

Proof. Follows from equation (9). \qed

This process in a martingale under the investors information set both in periods without a crash and periods with a crash. Under the information set of a manager who knows her type and choices, this process could be a sub or super-martingale. This learning dynamics is general but it depends on investors beliefs about the manager’s portfolio choice $E[\pi^I(\Omega^{m}_t) | \Omega^I_t]$. Law of iterated expectations implies $E[\pi^I(\Omega^{m}_t) | \Omega^I_t] = \sum_{\zeta \in \{\zeta^M, \zeta^R\}} E[\pi^I(\Omega^{m}_t) | \Omega^I_t, \zeta] E[1|\Omega^I_t]$. Beyond knowing her own type, the only additional information that a skilled manager has over investors is the knowledge of the crash state $\zeta$. This restricts investors beliefs to be deterministic conditional on information $\{\Omega^I_t, \zeta\}$, that is $E[\pi^I(\Omega^{m}_t) | \Omega^I_t, \zeta] = \pi^I(\Omega^{m}_t)$. Equilibrium restricts beliefs to be equal to manager choices. This is a key aspect of the model and it will be analyzed at length when I discuss the manager portfolio choice. The second piece is investors beliefs about the probability of a particular crash state $E[1|\Omega^I_t]$. Investors could
potentially use fund performance to learn about $\Omega$, in a similar fashion to the way they learn about manager type. This learning would make $E[1_\Omega|\Omega]$ evolve over time, further enriching the dynamic of this economy. I abstract from this second learning channel by assuming investors do not learn about the crash state, so $E[1|\Omega] = E[1] = \frac{1}{2}$.

1.3 Contracts

I model the maturity of the contract as a stochastic constraint on investor’s ability to redeem. Longer contracts are captured by constraints that expire less frequently. The lockup maturity is represented by contract term $T$ that specifies that the lockup expiration is a Poisson event with intensity $\delta_T = \frac{1}{T}$. So an investors will have their monies locked up for an average of $T$ years after the initial investment. As it is practice under real world lockup contracts investor’s monies become fully liquid after the initial lockup period. Only if the fund manager is able to attract new investors that are willing to be locked up, she gets the fund effective lockup extended. For tractability, I assume that all investors in the fund have the same liquidity. Either all the fund shares are liquid, or they all have to wait until the lockup expires. I use the lockup status $L \in \{L^a, L^e\}$ to keep track if the lockup is active ($L^a$) or expired ($L^e$).

Another important and well studied feature of money management contracts is how the manager is compensated. Manager compensation is modeled as a management fee and a symmetric performance fee for the manager abnormal performance. In an extension, I allow for high-water-marks and show that it does not change the results of the paper. Contractual terms are described by three parameters $\Xi = \{f, k, T\}$, management fees, performance fees, and a lockup maturity.

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4 This stochastic assumption is not necessary but buys considerable tractability relative to taking into account the time dependence of the life of the contract. In the studied economy where strategy horizons are also stochastic, this assumption is innocuous. If, for example, the manager were trading on an information that will be released in known event date, then this modeling assumption would lead to very different incentives than the fixed maturity contracts observed in practice. The crash events that I have in mind are unpredictable like a bubble crashing, or an unexpected bad news about a sector of the economy, or a massive currency depreciation, or any other large financial event that the manager might have insight about the probability but not it’s exact timing.

5 I assume there is a hurdle rate for the performance payment but I set it to the risk-free rate, since it doesn’t play any interesting role in the analysis.
1.4 Fund Investors

Investors are risk-neutral, discount cash-flows at the rate $r$, and can borrow and lend freely at this rate. Let $X_t$ denote the manager reputation and $W_t$ the current fund size, and let $Z_t^I = \{X_t, W_t\}$ be the manager track-record. Given contractual terms $\Xi$, investors valuation of one fund share with an active lockup, in a fund with a unit mass of shares and assets under management $W$, manager reputation $X$, is given by $V(Z^I, L^a)$:

$$V(Z^I, L^a) = E \left[ \int_0^\tau e^{-rt}W_t dD_t + e^{-r\tau} \left( 1_{\tau_p} W_\tau + 1_{\tau_e} V(Z^I, L^c) \right) \right] | Z^I$$ (11)

**Definition 1.** Let $a, b, \ldots$ denote events and let $\tau_a, \tau_b, \ldots$ each associate first arrival event time. Let $\tau$ be the first arrival time, then define $1_{\tau_i} \equiv \{1 \text{ if } \tau_i \leq \tau\}$.

If the manager accepts an outside offer for her services ($\tau = \tau_p$), investors have their money returned. When a lockup expires ($\tau = \tau_e$), investors shares become fully liquid. By contract, the investors liquidation policy is constrained when the lockup is active ($l^I(Z^I, L^a) = 0$), but once the lockup expires, investors can optimally decide when to cash out,

$$V(Z^I, L^c) = \max_{l \in \{0, 1\}} (1 - l) E \left[ \int_0^{\tau_p} e^{-rt}W_t dD_t + e^{-r\tau_p} W_\tau | Z^I \right] + lW.$$ (12)

1.5 The Market for Skill

The key assumption about the market for skill is that competition will ultimately drive any rents of the manager human capital back to the manager. The theoretical literature [Berk and Green, 2004] and the empirical literature interprets money flow behavior as doing exactly that. The theoretical literature often used other mechanisms to capture this rent transfer, Berk and Stanton [2007] assume that a better-than-expected manager is eventually poached by outside investors. Sirri and Tufano [1998] assume manager are promoted to manage a larger fund. In the real world all these mechanisms and others work simultaneously, there is often news of successful managers leaving a fund, opening new funds, returning some investors money but letting other investors in, using a feeder-master arrangement to sell fund space, among others. And of course as in the mutual fund business, in the hedge fund industry better performing funds do attract new money, but only when they
are relatively small [Goetzmann et al., 2003]. While all these mechanisms have their own particularities, the common thread is that regardless of the actual contractual arrangement governing the manager-investor relationship, the manager will ultimately receive increases in pay if investors perceive her human capital as valuable[Harris and Holmstrom, 1982]. Any mechanism that has this feature will induce investors to redeem too early.

A competitive market for skill not only reduces the option value of sticking with a loosing manager but leads to an entrenchment when the fund has a lockup. A better-than-expected manager will extract compensation increases, and worse than expected managers will be protected from liquidation or compensation reductions. This entrenchment effect was highlighted by Berk and Stanton [2007] as an explanation for the close-end fund discount puzzle. I follow Berk and Stanton [2007] and I assume that competition for skill works through poaching offers by outside investors. I model the timing of these offers as exogenous, but their value as an endogenous function of the amount of value that current fund investors are capturing.

More specifically, I assume that a fund track record $Z^I$ gets privately noticed by a pool of outside investors with intensity $\delta_a$. This intensity parameter measures the degree of competition in market for skill. Let the arrival times of these awareness shocks be $\{\tau^1_a < \tau^2_a < \ldots\}$. Once a pool of outside investors becomes aware of a fund, they check the fund contract terms $\Xi$. If they value the fund at a premium ($\frac{V(Z^I_t, L^a)}{W_t} > 1$) they bid for fund access offering to pay $B^I$ per dollar invested. The equivalent of a load fee. In equilibrium all outside investors offer $B^I = \beta \left[ \frac{V(Z^I_t, L^I)}{W_t} - 1 \right]$, where $\beta \in [0, 1]$ controls the competitiveness of the bidding process. If $\beta = 1$, the manager receives all the surplus. But independently of how much of the surplus managers capture, current investors always get the short end of the stick, receiving their money back exactly when a dollar in the fund is more valuable than their best outside option. If the manager’s track record is such that $\frac{V(Z^I_t, L^e)}{W_t} \geq 1 > \frac{V(Z^I_t, L^e)}{W_t}$, then outside investors find it unprofitable to invest under the initial lockup. In this case they make offers contingent on the manager waiving the initial lockup for then, in this case they bid $B^I = \beta \left[ \frac{V(Z^I_t, L^e)}{W_t} - 1 \right]$. If the manager track-record is so bad that investors value the fund at a discount even if the manager were to waive the lockup $\frac{V(Z^I_t, L^e)}{W_t} < 1$, they just walk away from the fund. The manager can always choose to reject an offer. If she rejects the

\footnote{In contrast to previous work that used this modeling device, I am explicit about the continuation value of the manager since I am studying her portfolio choices.}
offer, the investors walk away and they never meet again. Let \( L^I(Z^I_t) \in \{ L^c, L^a \} \) describe if the investors offer is contingent on the manager waiving the lockup restriction for them. So outside investor policy is described by three choices \( \{ O^I, L^I, B^I \} \in \{ 0, 1 \} \times \{ L^a, L^c \} \times [0, +\infty] \), where \( O^I \) describes if they make any offer at all. Taking fund investors valuations as given, their equilibrium offers can be written as:

\[
\{ O^I, L^I, B^I \}(Z^I) = \begin{cases} 
1, L^a, \beta \left( \frac{V(Z^I, L^a)}{W} - 1 \right) & V(Z^I, L^a) \geq W \\
1, L^c, \beta \left( \frac{V(Z^I, L^c)}{W} - 1 \right) & V(Z^I, L^c) > W \geq V(Z^I, L^c) \end{cases}
\]

1.6 Endogenous Value Functions

Fix contractual terms \( \Xi \), then the manager valuation is given by \( G(Z^I_t, L_t, Z^{m_\phi}_t) \), where track record \( Z^I_t \) and lockup status are known to managers and current investors, but \( Z^{m_\phi}_t \) is the information only held by a manager of type \( \phi \). \( Z^{m_1} = \{ \phi, \zeta \} |_{\phi=1} \) is the private information of a skilled manager and \( Z^{m_0} = \{ \phi \} |_{\phi=0} \) is the private information of a bad manager. The state space \( \Omega^{m_\phi} = \{ Z^I_t, L_t, Z^{m_\phi}_t \} \) subsumes all relevant information for manager type \( \phi \). Risk-neutral managers know their type \( \phi \), discounting cash-flows at rate \( r \), and receiving outside option \( UW \) when liquidated. This notation allows me to write the manager problem as

\[
G(Z^I_t, L_t, Z^{m_\phi}_t) = \max_{\pi(\cdot)} \mathbb{E} \left[ \int_0^T e^{-rt} W_t \left[ k(dR_t - (f + r)dt) + f dt \right] + 1_{\tau_p} e^{-r\tau_p} \left[ G(Z^I_t, L^I_t, Z^{m_\phi}_t) + W_t B^I_t \right] + 1_{\tau_e} e^{-r\tau_e} G(Z^I_{\tau_e}, L^c_{\tau_e}, Z^{m_\phi}_{\tau_e}) + 1_{\tau_l} e^{-r\tau_l} UW | \Omega^{m_\phi} \right]
\]

where the three different event-times are the acceptance of an outside offer \( (\tau_p) \), of the lockup expiration \( (\tau_e) \), or the fund liquidation \( (\tau_l) \). \( dR_t \) is the fund gross return and \( (f + r) \) is the performance fee hurdle rate. Outside offers arrive if outside investors become aware and choose to make an offer. This happens with intensity \( O^I_t \delta_\phi \). The manager accepts if \( G(Z^I_{\tau_e}, L^I_{\tau_e}, Z^{m_\phi}_{\tau_e}) + B^I_{\tau_e} \geq G(Z^I_t, L_t, Z^{m_\phi}_t) \), so the arrival time of an accepted offer and its arrival rate can be written as a function of the exogenous awareness shocks as the first awareness.
shock where an offer is made and accepted:

\[ \tau_p = \min_{t} \{ \tau_i | \tau_i \geq 0 \text{ and } O_{\tau_i}^I = 1 \text{ and } G(Z_t^{\tau_i}, L_t^{\tau_i}, Z_m^{\phi}) + W_{\tau_i}B_t^{\tau_i} \geq G(\Omega_{\tau_i}^{\phi}) \} \]

\[ \delta_p(\Omega_{\tau_i}^{\phi}) = O_t^{I} \delta a 1 \{ G(Z_t^{\tau_i}, L_t(Z_t^{\tau_i}), Z_m^{\phi}) + W_{\tau_i}B_t^{\tau_i} \geq G(\Omega_{\tau_i}^{\phi}) \} \]

The expiration event \( (\tau_e) \) changes the lockup from active to expired, if the lockup was previously active. The last event is the fund liquidation. Liquidation happens when investors decide to redeem all their monies. The liquidation event can be defined as \( \tau_l = \min_{t \geq 0} \{ t | t^{I}(Z_t^{I}) = 1 \} \). The manager and her investors share the same beliefs about how the fund size and manager reputation evolve as a function of the realized performance:

\[ \frac{dW_t}{W_t} = r dt + (1 - k)(dR_t - (r + f)dt) - dD_t \]  \( (15) \)

\[ dX_t = \frac{\Delta_{dB}}{\sigma} (dR_t - \mu_{dB} dt) + \frac{\Delta_{dN}}{\omega} (dR_t - \mu_{dN} dt) dN^\zeta \]  \( (16) \)

As in Berk and Stanton [2007], I assume that skilled managers are born with the ability to generate value for the first dollar (or any other fixed initial scale), and this scale grows (and decreases) with organic fund growth. I do not believe this assumption is a great approximation of skill scale dynamics in the real world, but it buys a lot of tractability to the extent that the problem becomes homogeneous in fund size. This same assumption is implicit in the large literature that studies high-water-marks in hedge fund contracts [Panageas and Westerfield, 2009, Goetzmann et al., 2003, Drechsler, 2011].

**Assumption 4.** The dividend policy is of the form \( dD_t = \mu_d dt + \beta_d (dR_t - (r + f)dt) \), with \( \beta_d = (1 - k) \) and \( \mu_d = r \).

This dividend policy should be thought of as a modeling assumption used to control scale effects. While the model is fully tractable and can be solved for any choice of \( \beta_d \) and \( \mu_d \), I focus on the case of fixed scale \( (\beta_d = (1 - k), \mu_d = r) \). Note that fixing the size of the fund does not shut down reputational incentives, as the manager still expects to receive raises \((B_t^I \text{ offers})\) if she performs well, and to be liquidated if she performs poorly\(^7\).

\(^7\)Fixing the scale allow us to abstract from inter-temporal incentives that arise when the scale of the manager skill depends on past performance. If one believes leverage constraints are unimportant and the scale of manager skill does not increase with fund size, then by my assumption is a good one. If one believes leverage constraints puts limits on the ability of managers to explore their skill, then these inter-temporal incentives are important. Goetzmann et al. [2003] finding that the flow performance relationship
1.7 Return Dynamics

While all agents are rational and agree on the evolution of the economy as a function of fund performance, they have different information regarding the manager type, the current crash state, and the manager portfolio choice. These different perspectives require introduction of some notation. Let $\pi^a(\Omega_{it}^{m1})$ be the belief system of agent $a$ about the skilled manager portfolio choices. Note that for the skilled manager this will be trivially equal to her choice $\pi^{m1}(\Omega_{it}^{m1}) = \pi(\Omega_{it}^{m1})$, and for the investor it will be given by $\pi^I(\Omega_{it}^{m1})$ as previously defined. For completeness, let the beliefs of bad managers be equal to those of investors. The return process is a jump diffusion that can be written as:

$$dR = (r + \mu(\Omega_{it}^a, \pi^a))dt + \sigma dB^i + (\eta(\Omega_{it}^a, \pi^a) + \tilde{\omega}) dN^\zeta,$$

where both the drift $\mu(\Omega_{it}^a, \pi^a)$ and the crash event expected return varies with the agent information set and beliefs about portfolio choices.\(^8\)

$$\mu(\Omega_{it}^a, \pi^a) = E[\phi(\alpha + \pi^a(\Omega_{it}^{m1})\lambda) + (1 - \phi)(\nu + \lambda)|\Omega_{it}^a].$$

Crash average returns can have four different means depending on the state of the economy, as $\{\phi, \zeta\} \in \{1, 0\} \times \{\zeta^R, \zeta^M\}$, but have the same conditional variance ($\tilde{\omega} \sim N(0, \omega)$). Since before the crash investors put non-zero probability in all of these four possibilities, crash returns are a mixture of four normally distributed random variables from their perspective. Managers on the other hand can condition on some of this uncertainty. Bad manager’s condition on their type and put non-zero weight in only two of the possibilities. A skilled manager puts non-zero weight on only one of the possibilities as she can condition on her type and the current crash state $\zeta$. Just after the crash happens all agents can conditional on the realized crash. Bad managers put non-zero probability weight on only one of the possibilities, and for investors realized crash returns is a mixture of two normals.

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\(^8\)For investors, the expression becomes $a_t(\alpha + E[\pi^I(\Omega_{it}^{m1})\lambda] + (1 - a_t)(\nu + \lambda)$, where $a_t = \frac{v_t}{v_{it}^M}$ is the manager reputation in probability space. For skilled managers, the expression becomes $(\alpha + \pi(\Omega_{it}^{m1})\lambda)$.
\[ \eta(\Omega^a_t, \pi^a_t) = \begin{cases} -\zeta^R \pi^a(\Omega^m_1) & \text{w. prob. } E[1_{\zeta^R} \phi|\Omega^a_t] \\ -\zeta^M \pi^a(\Omega^m_1) & \text{w. prob. } E[1_{\zeta^M} \phi|\Omega^a_t] \\ -\zeta^R & \text{w. prob. } E[1_{\zeta^R}(1-\phi)|\Omega^a_t] \\ -\zeta^M & \text{w. prob. } E[1_{\zeta^M}(1-\phi)|\Omega^a_t] \end{cases} \] (19)

Define \( \alpha(\Omega^a_t, \pi^a_t) \equiv \mu(\Omega^a_t, \pi^a_t) + \delta \zeta E[\eta(\Omega^a_t, \pi^a_t)|\Omega^a_t] \) as the fund gross excess expected returns given agent \( a \) information and agent \( a \) beliefs about the skilled manager manager portfolio policy.

### 1.8 Equilibrium

As I showed in section 1.2, the manager reputation evolution depends on realized returns and investors beliefs about a skilled manager portfolio choice. In general, this property will lead to the existence of multiple equilibria. In this paper I am interested in an equilibrium that maximizes expected returns. So investors and managers are assumed to always play the equilibrium that maximizes expected returns (and hence, everyone valuations). I focus on Markovian equilibria, where \( Z = \{X, W\} \), \( L \), and \( Z^{m_0} \) are the state variables. Investors decisions can only depend on the manager track-record, and the state of the lockup contract, but investors beliefs about manager actions can depend on unobservable states (manager type and crash state). Manager decision can depend on her type and information about the crash state. Since the bad manager does not take any decision, this equilibrium definition should be seem as imposing mutual consistency between actions and beliefs of investors and skilled managers. I take as given the offer behavior of outside investors to keep the focus on the relationship between current investors and the fund manager.

**Definition 2.** Given contract terms \( \Xi \) and outside offer policies \( \{O^I, L^I, B^I\}(Z^I) \), a Markovian equilibrium is given by portfolio policies \( \pi(\Omega^{m_1}) \), investors beliefs about portfolio choices \( \pi^I(\Omega^{m_1}) \), a law of motion for investors beliefs \( X \) about manager type, and liquidation policies \( l^I(Z^I) \), such that

1. The manager’s portfolio choices are optimal given investors beliefs, liquidation policies and poaching offers.
2. Investors beliefs about manager portfolio choices are consistent with manager portfolio choices.

3. Investors beliefs $X$ are consistent with Bayes’ law on the equilibrium path.

4. Current investors liquidation policies are optimal given other investors policies and investors beliefs about manager portfolio choice and manager type.

5. Manager portfolio choice maximizes expected returns given conditions above.

1.9 Maturity Choice

I assume lockup maturity is chosen once when a new manager announces that she is opening a fund. At inception all managers are fully confident to be skilled, even though they have different reputations. Investors beliefs about the manager type are summarized by the manager reputation. Once managers choose the contract, outside investors make their offer, the manager starts trading and immediately learn its type and the crash state $\zeta^9$ (If skilled). These set of assumptions make contracting simple as they are written under symmetric information. At the time of entry managers choose the lockup maturity subject to investors breaking even ($V(X_0, W_0, L^a|T) \geq 1$), and takes into account how much of the rent she will extract due to outside investors bidding behavior.

$$
T^*(Z_0) = \arg \max_T B^T + E^\zeta [G(X_0, W_0, L^a, \phi = 1, \zeta(T))]
$$

s.t

$$
1 \leq V(X_0, W_0, L^a|T)
$$

$$
B^T = \beta(V(X_0, W_0, L^aT) - 1),
$$

In the baseline case outside investors have zero bargaining power ($\beta = 1$), the manager maximizes the the total surplus of the relationship $V(X_0, W_0, L^a|T) + E^\zeta [G(X_0, W_0, L^a, \phi = 1, \zeta(T))]$.

---

$^9$A perhaps more realistic assumption would be that managers slowly learn their type through trading as in Berk and Green [2004], but this would lead to an additional state variable as managers’ beliefs would evolve differently from investors as they know the state of the economy, while investors do not. One could think the model happening after this initial learning phase took place.
I make these particular assumptions regarding the contracting stage because the paper does not have anything new to say about potential screening and signaling consequences of contract choices, and they allow the model to focus on the role that future asymmetric information between managers and investors in generating limits to arbitrage and shaping maturity choices. For static economies this signaling literature is very large, for an application for maturity choice in the context of money management see Stein [2005]. The dynamic literature is still very young, an important paper is Demarzo and Sannikov [2011] but they only consider contracts that screen out bad types, ruling out any learning in the equilibrium path.

2 Model Solution

I start by taking investors policies as given and derive the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the solution of the manager problem. I then demonstrate some key properties of the solution with special focus on the manager’s optimal portfolio policies. I then take these policies as given and derive the HJB satisfied by the investors problems, and characterize investors optimal policies. I will show that both problems are homogeneous in fund size, and the HJB’s simplify to a system of four coupled equations for the manager problem and a system of two coupled equations for the investor problem. In the Appendix, I discuss how this system of equations can be solved using off-the-shelf linear algebra techniques. The solution takes the initial contract terms and the bidding behavior of outside investors as given. Most of the models analysis is numerical, but in this section I will use particular cases to illustrate some key aspects of the model. The Appendix provides proofs for the propositions in the main text.

2.1 Manager

Fix contract terms and let $G_t = G(Z_t^I, L, Z_t^{m o})$ denote the value function of the manager, given by the solution to equation (14). Then, for $t < \tau$ the value process $G_t$ is a martingale and satisfies the following HJB:
when the lockup is expired, the valuation respects the boundary condition imposed by investors’ liquidation policy: $\forall Z_{l}^l(\cdot) = 1$, we have $G(Z_{l}, \cdot) = UW$, the manager outside option. The HJB formulation is convenient for providing intuition about all the dynamic forces that shape the manager valuation. In line 1, the first two terms are the expected payoff of the performance fee and the management fee. The third term is standard time discounting. The fourth term captures the valuation effect of expected reputation changes in periods without crashes and will play a very important whole in the analysis. The last term in the first line captures the valuation effect of unexpected shocks to the manager’s reputation, which is driven by the idiosyncratic risk the manager takes to generate alpha. In the second line we have the valuation effects of changes in the fund scale. The first term captures expected changes, the second term unexpected, and the third the co-variation between scale and reputation shocks\textsuperscript{10}. In the third line we have the valuation consequences of the three different Poisson events. The first term is the valuation gain of outside offers, $B_{l}^l \geq 0$ is the load fee that new investors pay, and the difference in continuation values measures valuation gains if new investors’ shares have different liquidity from those of old investors. The second term measures the valuation loss of a lockup expiration\textsuperscript{11}. The last term measures the expected valuation effects of a crash. The first part is the expected pay for performance in a crash, and the difference in continuation value captures the crash impact on the fund size and manager reputation. These effects will be more transparent in the next expression. I solve this problem by exploring homogeneity with respect to fund size, which is a direct consequence of the one-to-one relationship assumed between the fund size and the scale of the manager’s skill assumed in this paper.

\textsuperscript{10}Throughout the analysis I eliminate these effects by setting $\beta_d = (1 - \kappa)$. These effects can be very important if you think managers can easily scale up their bets as the fund grows.

\textsuperscript{11}This term is obviously zero if the lockup was already expired as $Z^m_{l+} = Z^m_{l}$. 

\[ 0 = \max_{\pi} kW_t(\mu(\Omega_t^{m, o}, \pi) - f) + fW_t - \rho G + G_X \frac{\Delta dB}{\sigma^2} (\mu(\Omega_t^{m, o}, \pi) - \mu dB) + \frac{1}{2} G_{XX} \left( \frac{\Delta dB}{\sigma} \right)^2 + G_W W_t \left( r + (1 - k - \beta_d)(\mu(\Omega_t^{m, o}, \pi) - f) - \mu_d \right) + \frac{1}{2} G_{WW} W_t^2 \sigma^2 (1 - k - \beta_d)^2 + G_{XW} W_t (1 - k - \beta_d) \Delta dB + \delta_{p,t} (W_t B_t^l + G(Z_t^l, L_t^l, Z_t^{m, o}) - G) + \delta_{c} (G(Z_t^l, L^c, Z_t^{m, o}) - G) + \delta_{e} \left( kW_t((\pi \phi + (1 - \phi))(- \zeta) - 1) + E[G(Z_{l+}, L, Z_{l+}^{m, o})\Omega_t^{m, o}] - G \right) \]
Proposition 3. (The manager HJB)

Let investors offer policy \( \{ O^I, L^I, B^I \} \), and investors beliefs \( \pi^I \) be homogeneous of degree zero in fund size, then manager value function is homogeneous of degree one in fund size \( G(Z^I, L, Z^{m,\phi}) = W G(X, L, Z^{m,\phi}) \), and satisfies:

\[
0 = \max_{\pi^I} k(\mu(\Omega_i^{m,\phi}, \pi) - f) + f - \rho G + G_X \frac{\Delta dB}{\sigma} (\mu(\Omega_i^{m,\phi}, \pi) - \mu_i^{BdB}) + \frac{1}{2} G_{XX} \left( \Delta dB \right)^2 + \delta_{p,t} \left( G(X_t, L^I_t, Z^{m,\phi}_t) + B^I_t - G \right) + \delta_e \left( G(X_t, L^e_t, Z^{m,\phi}_t) - G \right) + \delta_\zeta \left( k(\pi\phi + (1 - \phi))(-\zeta_t) + E \left[ \left( \frac{\Delta W_t}{W_t} + 1 \right) G(X_{t+}, L, Z^{m,\phi}_t) | \Omega_i^{m,\phi} \right] - G \right),
\]

where

\[
\left[ \frac{X_{t+} - X_t}{\Delta W_t/W_t} \right] \sim \left[ \frac{\Delta_{dN} \left( (\pi\phi + (1 - \phi))(-\zeta_t) - \mu_i^{dN} \right)}{1 - k - \beta_d (1 - \zeta_t)(\pi\phi + (1 - \phi))} \right] + \left[ \frac{\Delta_{dN}}{1 - k - \beta_d |\omega|} \right] \times N(0, 1)
\]

and the following boundary condition is satisfied \( G(Z^I, \cdot) = U \), for any \( Z^I \) such that \( L^I(Z^I) = 1 \).

Proof. This follows directly from equation 22. I conjecture that the manager valuation is homogenous with respect to fund size given that all policies are homogenous. This conjecture proves correct as fund size drops out of the manager HJB. \( \square \)

A crash event has a potential large impact on the manager valuation, as it leads to a jump in the fund size \( \left( \frac{\Delta W_t}{W_t} \right) \) and in the manager’s reputation. Proposition 3 fully characterizes the system of equations that I solve numerically. Note that this equation is actually a system of four coupled integro-differential equations. It is “integro” because in contrast with ordinary differential equations it features “non-local” movements. These non-local changes can be seen in the last term, where learning in the jump state leads to a jump in manager reputation. It is a system of four equations because in addition to variation in the manager reputation, the system jumps between strategy profitability states and lockup contract active/inactive state \( \{ \zeta^R, \zeta^M \} \times \{ L^e, L^a \} \).

Proposition 4. (Manager valuations are S-shaped in her reputation)

Let \( \delta_a = 0 \), \( U = 0 \), \( f \geq 0 \), \( k \geq 0 \), \( \alpha^s > 0 \), \( \alpha^m \geq 0 \), \( \Delta_{ef}^{dB} > 0 \), \( \omega \rightarrow \infty \) and fix the liquidation threshold \( X_L \) across states \( (\zeta, L) \), the manager portfolio \( \pi \), then
(A) a skilled manager who manages an open-ended fund is risk-averse with respect to reputational risk \( (X \geq X_L \Rightarrow G_{XX} < 0) \).

(B) a skilled manager who manages a lockup fund is risk-averse with respect to reputational risk \( (X \geq X^*(T) \Rightarrow G_{XX} < 0) \), but likes reputational risk when her reputation is very low \( (X < X^*(T) \Rightarrow G_{XX} > 0) \).

Propositions 4 and 11 (in the Appendix) prove two key properties of the equilibrium value functions under particular conditions. Valuations are increasing and S-shaped in reputation. The S-shape induces state dependent variation in aversion to reputation shocks which is the main driver of limits to arbitrage in the model. More generally these two properties hold across all parametrization explored in the paper. Proposition 5 proves that lockups reduce the manager’s short-term reputation concerns. This effect is how lockup contract improve incentives and reduce equilibrium limits to arbitrage.

**Proposition 5. (Longer lockups reduce short-term reputation concerns)**

Under the same conditions as Proposition 10, and if idiosyncratic volatility is not too large, then for any \( X > X_L \) we have \( \frac{\partial G_X(T)}{\partial T} < 0 \).

### 2.2 Optimal Portfolio Choice

We can get insight about what matters for the portfolio choice by isolating the terms where \( x_m \) shows up. Let the skilled manager value function be \( G \), then the manager portfolio solves:

\[
0 = \max_\pi \pi k(\lambda - \delta \zeta) + \pi \frac{\Delta dB}{\sigma} G_X + \pi \lambda \{(1 - k - \beta_d)\} G

\delta \zeta E \left[ \{(1 - k - \beta_d)(-\pi \zeta + \bar{\omega} - 1)\} + 1\right] G \left( X + \frac{\Delta dN}{\omega} \left( -\pi \zeta + \bar{\omega} - \mu I dN \right), L, \zeta' \right] \Omega^{m1}_{m1}.
\]

The first take away is that the objective is almost all linear in the portfolio choice. The only non-linearity shows up in the after-crash reputation. So this expression tells us that absent this long-term reputational concern, the optimal portfolio would be always in one
of the corners. The second important finding is that pay-off incentives always push the manager to maximize expected returns, and everything else constant, as the performance fee increases, the more weight the manager will place in maximizing expected returns. This expected returns term\(^{12}\) will be positive when the momentum strategy is profitable and will be negative when the reversal strategy is profitable. The second term measures the marginal short-term reputational concerns. This term is always positive as long as \(\Delta_t^{AB} > 0\) (see condition (2)). This short-term reputation concern pushes the manager to always invest in the momentum strategy, regardless of its profitability. The magnitude of this term determines how much the manager distorts her choice. The two terms inside the curly brackets are the fund size effects of different choices. The remaining term inside the expectation is the after-crash continuation value. I call this term the long-term reputational concerns, because the portfolio choice impacts the manager’s reputation once the crash happens. It acts as a balance to the short-term concerns, pushing the manager to maximize expected returns. The relative strength of these two reputation concerns depends on the manager’s horizon. The manager’s horizon shows up in Equation (23) in the relative slopes\(^{13}\) of the value function, before and after a crash. A manager with a long horizon will have similar before and after crash slopes, while a manager close to liquidation will have a much steeper value function before the crash.

Proposition 13 has two parts. Under some conditions it shows that when the momentum strategy is profitable the manager will always maximize expected returns. In states \(\zeta^R\), the manager incentive compatibility constraint might bind, so the optimal choice maximizes expected returns subject to also solving the manager problem when investors beliefs are consistent. These requirement are always satisfied in the case of \(\zeta^M\). The proposition then shows that in the reversal state the manager will alternate between three different policies. If the short-term reputation concerns are small enough, the manager will maximize expected returns and short the momentum asset as much as she can (\(\pi = -1\)). If the short-term concerns are very large, then the manager will go to the other corner and buy the momentum asset as much as possible (\(\pi = 1\)). Under suitable conditions on the maximum/minimum amount of learning during a crash, there is also an intermediate region where the optimal policies are interior. In this case, at the optimal choice the marginal long-term reputational

\(^{12}\)For clarity, \(k x^m (\lambda - \delta \zeta)\)

\(^{13}\)With respect to the manager reputation.
concern will exactly balance out the short-term concern and payoff incentives.

**Proposition 6. (Equilibrium Portfolio Choice)**

Let condition (2) hold, \( G, G_X \geq 0 \), and let \( \pi(X, L, \zeta) \) be manager optimal portfolio choices, then in an equilibrium as defined in (2) the following holds:

\[
\pi(X, L, \zeta) = \begin{cases} 
1 & \zeta = \zeta^M \\
\pi^R(X, L) & \zeta = \zeta^R 
\end{cases}
\]

and \( \pi^R \) satisfies,

\[
\pi^R = \arg \max_{\pi^R} \pi^R(\lambda - \delta \zeta^R) \\
\text{s.t.:} \\
\pi^l = \arg \max_{\pi^m} k\pi^m(\lambda - \delta \zeta^R) + G_X \frac{\alpha + p^R(\pi^l - 1)\lambda - \nu}{\sigma^2} \frac{\pi^m \lambda + G(1 - k - \beta_d)\pi^m \lambda}{\sigma^2} \\
\delta \zeta \left\{ (1 - k - \beta_d)(\omega z - \pi^m \zeta^R - 1) + 1 \right\} \xi \left\{ \left( \frac{(\pi^l - 1)\zeta^R}{\omega^2} \left( \omega z - \pi^m \zeta^R + \frac{1}{2}(\pi^l + 1) \zeta^R \right), L, \zeta \right) \right\} \, d\Phi(z)dz,
\]

Assume the dividend policy is equal to \( \beta_d = (1 - \kappa) \), then we have:

(A) \( \pi^R(X, L) = -1 \) if

\[
\left| k(\lambda - \zeta^R \delta \zeta) - \frac{2(\zeta^R)^2}{\omega^2} \delta \zeta \xi \int G_X \left( X + \frac{2(\zeta^R)}{\omega^2} (|\zeta^R| + z\omega), L, \zeta \right) \, d\Phi(z)dz \right| > G_X \frac{\alpha - 2E(\zeta^R | \lambda - \nu)}{\sigma^2} \lambda
\]

(B) \( \pi^R(X, L) = \min\{\pi\} \) if (A) does not hold and \( \exists \pi \in (-1, 1) \) such that

\[
k(\lambda - \zeta^R \delta \zeta) + G_X \frac{\alpha + p^R(\pi - 1)\lambda - \nu}{\sigma^2} \lambda + \frac{(\pi - 1)\zeta^R^2}{\omega^2} \delta \zeta \xi \int G_X (X + z, L, \zeta) \, d\Phi \left( \frac{z - \frac{(\pi - 1)\zeta^R^2(\pi - (\pi + 1))}{\omega^2}}{\zeta^R^2(\pi - (\pi + 1))} \right) dz = 0
\]

(C) \( \pi^R(X, L) = 1 \) if (A) and (B) do not hold

**Proof.** This proposition follows directly from three conditions being respected: the manager first order condition, and investors beliefs about the manager portfolio choice, and return maximization given that the first two conditions are respected. In the state \( \zeta^M \), the manager FOC is satisfied when evaluated at the portfolio that maximizes expected returns \( \pi(\zeta^M) = 1 \), regardless of investors’ beliefs about \( \pi(\zeta^R) \), and the other state variables. So as long investors believe that the manager is choosing \( \pi(\zeta^M) = 1 \), this will be the equilibrium choice at \( \zeta^M \).
Note that the manager is in a corner. If she could she would choose an even more aggressive portfolio. State $ζ^R$ is the interesting case with three possibilities (A,B, and C). First we check if $π^R(X,L) = -1$ satisfies the manager FOC. If yes, it is the equilibrium portfolio choice. If not we check what the lowest value of $π$ (expected returns are decreasing in $π$) such that the manager FOC is satisfied. Sometimes this will be an interior solution (case B), sometimes a corner (case C).

Inspecting the expression in case (A), we can see that at the efficient choice the long-term reputation concern will be close to zero both when the long-term signal-to-noise ratio is very high or very low. If learning during crashes is very weak, than the after-crash reputation will be very close to her reputation just before the crash. So marginal long-term and short-term reputation concerns will be of similar magnitude, but the marginal impact of portfolio changes on her after-crash reputation is small as the signal-to-noise ratio is small in this weak learning case. On the other hand, if learning is very strong, the marginal impact of portfolio changes on her long-term reputation is large, but the marginal value of reputation is low if the manager is maximizing expected returns. Since learning is strong, a manager who is maximizing expected returns will likely have a very high reputation after the crash hits. As long reputational concerns are decreasing in reputation ($G_{XX} < 0$), the marginal value of reputation after the crash will be much lower than before the crash, leading to a strong temptation to deviate towards more short-term oriented strategies.

While long-term reputational concerns will typically not be strong enough to keep the manager from distorting her choices, they will be effective in keeping the manager from fully maximizing short-term performance. Short-term reputational concerns introduce complementarity between investors beliefs and manager actions. The more the manager is expected to deliver high short-term performance $ΔdB ↑$, the higher her incentives to do so. The non-linearity build in long-term learning make reputational incentives more subtle. On one hand the sensitivity of reputation to performance works exactly as for short-term performance. The better the manager is expected to do when the crash hits $π^I(−ζ^R) ↑$, the higher the signal-to-noise ratio $ΔdN = \frac{−(π^I−1)ζ^R}{ω ↑}$. But because revelation of information during a crash is lumpy, the long-term reputational concern $∫G_XdΦ(z)$ is impacted by the manager choice $π^m$ and investors beliefs $π^I$. Inspecting the distribution of crash reputation growth, we can see that as $π^m → (π^I + 1)/2$, the higher the probability that the manager will experience decreases in her reputation, leading the expectation $∫G_XdΦ(z)$ to put more weight is high
reputation concern states $G_X \uparrow$. Reputation risk introduce strategic substitutability between investors beliefs and manager actions. The investors think the manager is relative to the bad manager, the less risky is for the manager to take action that reduce her performance at the margin. In the Appendix, I show conditions under which the manager always has positive reputation concerns ($G_X > 0$) and reputation concerns are decreasing in reputation ($G_{XX} < 0$).

Definition 3. Define limits to arbitrage as the difference between the expected returns of the maximum expected return portfolio and the one chosen in equilibrium by the skilled manager,

\[
loa_t \equiv E[dR|\pi^{ef}] - E[dR|\pi^{eq}, X_t, L_t] = E_t \left[ (\pi^{ef}(-) - \pi^{eq}(X_t, L_{t}, \zeta)) (\lambda - \delta \zeta) \right],
\]

where $\pi^{ef}(\zeta) = \begin{cases} +1 & \zeta^M \\ -1 & \zeta^R \end{cases}$ is the portfolio choice that maximizes the funds expected returns.

Proposition 6 can be illustrated graphically by Figure 2. The figure plots the equilibrium expected returns $\alpha^s + \pi(X, \zeta^R) (\lambda - \delta \zeta)$ and expected short-term performance $\alpha^s + \pi(X, \zeta^R) \lambda$ for a manager of an open-ended fund in a state where the reversal strategy is profitable. Case (A) can be seen in the right of the plot. The manager’s reputation is high enough that her portfolio is insensitive to changes in her reputation. In this region the manager maximizes expected returns and her expected short-term performance is very poor. For intermediate reputation levels (case (B)) we have a negative (positive) relationship between reputation and long-term (short-term) performance. In this case the manager trades off short-term reputation growth against long-term reputation reductions. As the manager’s reputation decreases and she approaches liquidation, short-term reputation growth becomes progressively more valuable than long-term growth, tilting the manager portfolio towards short-term oriented strategies. When the manager’s short-term concerns become strong enough, the manager portfolio becomes maximally tilted towards the short-term. In case (C), expected returns are minimum and short-term performance is maximum.

Proposition 7. (Limits to arbitrage, manager horizon, and lockup maturities)

Under the conditions of propositions 11 and 12,
(A) Incentives to pursue the long-term reversal arbitrage are increasing in the distance to liquidation (open-ended funds)

(B) Incentives to pursue the long-term reversal arbitrage are increasing in the distance to liquidation for managers with high enough reputation (lockup funds)

(C) Incentives to pursue the long-term reversal arbitrage are increasing in the lockup maturity.

Long-term contracts shape manager choices by changing the value a manager puts on short-term reputation changes. The restriction on redemption guarantees that the manager will not be liquidated even if her reputation drops below the level at which investors would like to pull out. This insurance with respect to short-term reputation drops leads the manager valuation to be less sensitive to changes in her reputation. This reduction in reputational concern feeds back into a more long-term oriented portfolio. This initial change in portfolio driven by the change in the lockup maturity is further amplified by investors’ learning equilibrium response.

2.3 Investors

Following similar steps as I did for the manager problem it can be shown that the investor value function is homogeneous of degree one in fund size and satisfies a similar HJB equation. It should be intuitive that if the fund returns are independent of fund size, the per dollar investors’ valuations and investors’ decisions should also be unrelated to fund size. In the Appendix Proposition 11 proves this intuition formally. The only important difference is that the fund return dynamics \(dR\) are different under the investor information set. It is a mixture of the return distribution under both types of managers, where the mixture weight is the manager’s current reputation. In the particular case in which dividend policy is set to keep fund size constant\(^\footnote{\(\beta_d = (1 - k)\) and \(\mu_d = r\)}\), investor HJB can be written as:

\[
0 = r + (1 - k)(\alpha(\Omega_t, \pi^a) - f) - \rho V + \frac{1}{2} V_{XX} \left(\Delta_t^B\right)^2 + \\
+ \delta_{p,t}(1 - V) + \delta_e(V(X_t, L^e) - V) + \delta_{\zeta} \left(\mathbb{E} \left[\int V(X_t + \Delta_t^\zeta z, L) d\Phi(z) dz\right] - V\right)
\]

\[(24)\]

\(^\footnote{\(\beta_d = (1 - k)\) and \(\mu_d = r\)}\)
Equilibrium expected returns, short-term performance, and limits to arbitrage

The figure illustrates how the skilled manager changes her behavior as a function of her reputation (X-axis). On the left we have the liquidation threshold. Managers with high reputation invest optimally in the long-term reversal strategy. Their funds have high expected returns, but low expected short-term performance. As they lose reputation they progressively tilt their portfolio towards short-term strategies, investing less in the long-term reversal. The flat areas in the two extremes reflect the fact the manager portfolio is constrained. High reputation managers would like to invest more in the long-term reversal than leverage constraints allow, while low reputation managers would like to invest even less.

Investors’ optimal policy consists of choosing the smallest \( X_L \) manager reputation at which they are willing to invest (and divest, since there are no transaction costs) when the shares are liquid, and the smallest manager reputation \( X_{L^s}(T) \) at which they are willing to invest when the shares are illiquid for a maturity \( T \). When choosing these policies investors have to factor in how large the skilled manager expected returns are, how much lower than his best outside opportunity is the bad manager, how quickly any rents from positive reputational news will be competed away, and how rapidly they anticipates learning about their manager.

**Proposition 8.** (Liquidation policy)

Assume there is no learning during crashes \( (\omega \to \infty) \) and,
(A) let \( \alpha(\Omega^a_t, \pi^a) \) be increasing in \( X \), then the liquidation policy \( (X_L \uparrow) \) is increasing in the degree of competition in the market for skill \( (\delta_a \uparrow) \).

(B) let the market for skill be competitive enough, then the liquidation policy in increasing in the amount of limits to arbitrage.

Besides its implications for equilibrium reputation dynamics, an important advantage of micro founding the investor behavior is that I can use the wedge between their investment policies for restricted and non-restricted shares to recover information about perceived entrenchment costs. This lockup premium measures the average net performance difference across funds that are currently *attracting capital*. The model says that managers with longer lockups need to have higher expected net alpha to attract capital at the time of the initial investment.

**Definition 4.** Let \( X_{L^a}(T) \) be the investment threshold for a fund with lockup maturity \( T \). The lockup premium is the difference in net expected return that makes investors indifferent between the lockup fund and an open-ended fund: \( \Lambda(T) \equiv (1-k)(E[dR|X_{L^a}(T),T] - E[dR|X_L,0]) \).

If we hold the funds expected returns constant, it is intuitive that this premium is positive and increasing in the lockup maturity. An investor in a shorter lockup fund can always replicate any investment in a longer lockup fund, so for the same alpha they must be better off. This logic implies a higher net alpha for lockup funds with longer lockups. But the introduction of lockups has positive effect on the gross fund returns. Does this effect amplify the initial entrenchment effect? For similar arguments as in Berk and Green [2004] any difference in gross performance in lockup funds is competed away exactly to the lockup premium implied by entrenchment. The argument takes the differences in gross performance as given. It could even be that lockup reduces gross performance, and the model would still predict a positive lockup premium. This quantity is purely determined by investor perception of how costly a lockup restriction is and does not say anything about the value added by a lockup contract. As long as the market for manager skill operates competitively, any difference in gross performance between funds beyond those dictated by difference in contract maturities, will eventually accrue to the manager.

**Proposition 9.** *(Lockups, Entrenchment costs and the Lockup premium)*
Let the market for skill be competitive \((\delta_a \to \infty)\), assume there is no learning during crashes \((\omega \to \infty)\), and fix the manager portfolio choice as a function of reputation, then

(A) The lockup premium is increasing in short-term performance signal-to-noise ratio.

(B) The lockup premium is increasing in the cost of investing with the bad manager.

3 Quantitative Implications of the Model

This section calibrates the model using previous results in the literature. My objective here is three-fold. First, I would like to show that the investor learning channel explored in this paper is a quantitatively relevant driver of limits to arbitrage. Second, I would like to characterize how a manager’s choices are impacted by the maturity of the contract. Third, I would like to evaluate whether the trade-off between entrenchment and limits to arbitrage can quantitatively explain the contract maturities observed in the data. All these objectives require solving the system six of integro-differential equations numerically, so I would like to use values that are reasonable for the hedge fund universe.

3.1 Calibration

The model has several parameters that can be transported from previous empirical work. Table 1 summarizes all the calibrated values. I calibrate the management fee (1%) and performance fee (20%) to match the median fees in the hedge fund universe as documented in Agarwal et al. [2009]\(^\text{15}\). I calibrate the fund (Brownian) idiosyncratic volatility to be consistent with the estimates in Agarwal et al. [2009] who documents a total volatility of 11%. There is no readily available estimate for hedge fund return volatility during crashes, and since I think of crashes in the model as an approximation to an event that unfolds not instantaneously, but during a time-period of abnormally high volatility when portfolios cannot be rebalanced, I set it to be consistent with the volatility during a three-month period, giving us \(\omega = 11\% \sqrt{0.25} \approx 6\%\).

I choose the crash premium \((\lambda)\) to match the properties of simple put-option writing strategies that match well the average hedge fund returns as shown in Jurek [2011] and-

\(^{15}\text{Page 2231, Table1}\)
Mitchell and Pulvino [2001]. Jurek [2011] shows that a strategy which mechanically sells out of the money puts with a fixed delta and a leverage of 2, can reproduce some key aggregate features of hedge fund returns. They estimate that this strategy produces an alpha of 7% if one uses a standard CAPM to risk adjust returns. Mitchell and Pulvino [2001] study this implied loading on catastrophe risk in the context of the Merge Arbitrage strategy. They document that this strategy has an average return of 13%, of which 3.5% can be explained by an implicit put writing strategy on the overall market. I interpret this 3-7% “alpha” as compensation for risk that is unaccounted for by off-the-shelf risk adjustments. Since agents are risk-neutral in the model, this alpha matches the crash premium perfectly. To be conservative I calibrate the crash premium towards the lower end of their estimates at $\lambda = 3.5\%$.

I assume that the momentum strategy is profitable $p^M = p^R = 1/2$ half of the time, which implies $E[\zeta] = \frac{\zeta^R + \zeta^M}{2}$, and since the static momentum strategy generates no value, this implies $\frac{\zeta^R + \zeta^M}{2} = \frac{\lambda}{\delta_\zeta}$. Time variation ($\zeta^R - \zeta^M$) in the crash size combined with crash intensity ($\delta_\zeta$) determines the returns to market timing in the model. This amounts to asking how much potentially explorable time-variation in expected excess returns skilled managers observe in the data. In the limit where $\zeta^R = \zeta^M$, the model has nothing to say about limits to arbitrage, since all the manager’s skill comes from selection. I will choose the time variation to be consistent with timing skill of 6% per year. The intensity could be parametrized using the number of large volatility spikes in the market or the number of events that can be clearly identified as a crash. This intensity parameter measures how long-term reversal strategies are, a key determinant of limits to arbitrage in the model. I set it to $\delta_\zeta = 0.33$, with a large crash happening on average every six years ($\frac{1}{\delta_\zeta \times p^R}$). I will also consider a high $f$ ($\delta_\zeta = 0.5$), and a low value ($\delta_\zeta = 0.2$). Given the crash premium, the crash size time variation, and the crash intensity, we can compute the size of the implied large crash. The baseline case implies a crash event of $|\zeta^R| = 24\%$. Considering that we are in a risk-neutral world this is a very modest crash. By comparison, the average hedge fund had a realized return of $-20\%$ in 2008 [Jurek, 2011].

The speed at which managers capture the rents of positive news about their skill is controlled by the rate of arrival of awareness shocks to outside investors. Agarwal et al. [2009] and Aragon [2007b] show evidence that funds without a lockup restriction deliver a zero net of fees alpha on average. This is consistent with a very competitive market for
manager skill [Berk and Green, 2004]. Consistent with this evidence I set the arrival rate to $\phi = 4$. On average, positive news are diluted within 3 months.

To evaluate the quantitative relevance of the learning mechanism I impose a constraint on how far from being liquidated the average manager is. If, for example, the average manager is always very close to being liquidated, the model can produce larger distortions, since distortions increases when the manager horizon is the shortest. However, this would produce very high liquidation rates as the manager would be liquidated with probability one. I assume entry reputations to be normally distributed and calibrate the average reputation of a new manager to be consistent with a first year attrition rate of 18% as documented in Brown et al. [2001]. I hold the reputation standard deviation constant at 0.4, but this plays almost no role in the analysis.

The remaining parameters to be calibrated are the selection skill for both the skilled ($\alpha^s$) and the bad manager ($\nu$). In order for the model to work, the bad manager has to destroy value relative to the investor’s best outside option and the gap in selection skill has to be large enough so that abnormal positive performance is interpreted as good news about the manager type. The first condition restricts the bad manager selection skill being negative ($\nu < 0$). If investors outside option is larger than the risk-free rate than the model can accommodate bad managers who do not destroy value ($\nu \geq 0$). What matters for the model is how much a bad manager under-performs relative to the investor’s outside investment opportunity and how much the skilled manager over-performs. If both the skilled and the bad manager under-perform, the problem becomes uninteresting as no investor would ever invest with any manager. If both over-perform, the problem would also go away as all managers would be able to attract funding regardless of their reputation, eliminating any liquidation risk and limits to arbitrage. The last condition imposes that the short-term performance signal to noise ratio is positive at the efficient portfolio choice $\Delta_{ef}^{df} > 0$. This parameter restriction imposes that the difference in selection skill has to be at least as large as the crash premium $\alpha^s - \nu > \lambda$. If this condition did not hold, a manager would have incentive to lose money to boost their reputation, which would not be a very robust equilibrium. Ideally, I would like to have two moments related to entrenchment costs and speed of learning to match these two quantities. The first natural empirical quantity to use is the lockup premium. Aragon [2007b] shows that funds with a lockup restriction have a annual net alpha between 4 and 7% larger than funds that have no redemption restrictions. More recently, Agarwal et al.
Table 1: Calibration

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performance fee</td>
<td>$\kappa$</td>
<td>20%</td>
</tr>
<tr>
<td>Management fee</td>
<td>$f$</td>
<td>1%</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>$\rho, r_f$</td>
<td>1%</td>
</tr>
<tr>
<td>Brownian volatility</td>
<td>$\sigma$</td>
<td>9%</td>
</tr>
<tr>
<td>Crash volatility</td>
<td>$\omega$</td>
<td>6%</td>
</tr>
<tr>
<td>Probability of $\zeta^R$</td>
<td>$E[1_{\zeta^R}]$</td>
<td>50%</td>
</tr>
<tr>
<td>First-year attrition</td>
<td>-</td>
<td>18%</td>
</tr>
<tr>
<td>Crash intensity</td>
<td>$\delta_\zeta$</td>
<td>0.5</td>
</tr>
<tr>
<td>Crash premium</td>
<td>$\lambda$</td>
<td>4%</td>
</tr>
<tr>
<td>Bad manager alpha</td>
<td>$\nu$</td>
<td>-5.5%</td>
</tr>
<tr>
<td>Selection skill</td>
<td>$\alpha^s$</td>
<td>9.5%</td>
</tr>
<tr>
<td>Timing skill</td>
<td>$2\delta_\zeta p^R p^M \Delta \zeta$</td>
<td>6%</td>
</tr>
<tr>
<td>Arrival of offers</td>
<td>$\delta_o$</td>
<td>4</td>
</tr>
</tbody>
</table>

[2009] provided estimates between 3% to 4%. This lockup premium has a direct counterpart in the model: it is the difference in net of fee alpha that an investor demand to be locked up. Since the median fund lockup is one year in their sample, I compute in my model the difference in expected net alphas at which an investor would be indifferent between investing in an open-ended fund versus a fund with a one year lockup. In the model, this lockup premium is increasing, in both the speed investors expect to learn about their manager and the cost of being stuck with a bad manager. In Figure 3, I show the lockup premium for different combinations of $\alpha^s$ and $\nu$. The model needs bad manager waste to be at least -4% (high selection skill case) to hit a lockup premium of 4%. If the skilled manager selection skill is low, the model need bad manager alpha to be as low as -8%. It should be obvious from Figure 3 that the lockup premium does not identify both quantities. A calibration that uses lower selection combined with “worse bad managers” will produce shorter optimal contracts relative to a calibration with the same lockup premium with a higher selection skill but a higher $\nu$. Given this trade-off, I calibrate $\alpha^s$ to equal 6% and $\nu$ to equal -5%. This parametrization produces a premium of 3.8%. Alternative combinations featuring lower $\alpha^s$ and $\nu$ or slightly higher $\alpha^s$ and $\nu$ are certainly possible, as long $\nu$ is not too high.

Table 2 shows the result of simulating a large cross-section of funds using the calibrated parameters. By construction, the model matches the first year attrition ratio of 18% in
This figure shows the relationship between the one-year lockup premium and the gross abnormal performance of skilled and unskilled managers. On the x-axis we have the bad manager alpha ($\nu$), with the different lines showing the comparative statics for different levels of selection skill $\alpha^s$. All the remaining parameters are from Table 1. The Target line depicts the lockup premium measured in the data. The lockup premium is defined in Definition 4.

The first year, produces an average difference in net alphas between 1 year lockup funds and open-ended funds of 3.8%\(^{16}\), and generates a fund level volatility of 12%, all of which roughly match their data counterparts. Funds have an average gross return of 6.3%, and a net average return of 3%.

The median lockup is 13 months, which compares with a median lockup maturity in the data of one year (conditional on the fund having a lockup provision). Aragon [2007b] documents that about 17% of hedge funds in the TASS database have a lockup provision and around 55% demand a notice period of at least one month before investors can redeem their

\(^{16}\)Note that this quantity does not match the calibrated premium exactly because in the calibration exercise I calibrated the difference in net returns for managers that reputations just large enough to open a one lockup fund versus an open-ended fund. The data, however, is about average difference in net-alphas over-time, and that is exactly what this 3.8% is.
shares. Of the 17% of funds with lockups, 14% have a one year lockup maturity, with the remaining 3% spread out, with maturities ranging from one month to 30 months. Redemption notice periods are concentrated in the one month horizon (50%), with the remaining funds having notice periods from 15 days to two months. As investors can anticipate a lockup expiration, redemption notice periods shorter than a lockup provision are not binding for locked up investors. The model generate maturity choices consistent with the data if we focus on funds with lockup provisions, but it fails to generate enough short contracts at the same time. This is consistent with the signaling issues raised in Stein [2005]. It also consistent with other types of heterogeneity (not related to the initial manager reputation), for example manager style, driving the maturity choices.

In this baseline calibration, skilled managers have total skill of 12%, of which 6% is timing skill, the portion of manager skill subject to limits to arbitrage. If all managers were forced to operate under an open-ended contract the average expected returns of a skilled manager in 10.35% in the first year of operation, 1.65% short of the maximum expected return feasible. limits to arbitrage is reduced to 0.53% under the optimal maturity choice, and to 0.6% if all funds have a one year lockup. The average fund skewness is 0.07, but varies substantially in the cross-section with the top 20% fund having a skewness of 0.72 and the bottom 20% a skewness of -0.4. Numbers that are in the range of the estimates in the literature for different hedge fund strategies [Agarwal, 2003].

3.2 The dynamics of limits to arbitrage

A revealing way to explore the dynamics of the model is to follow the possible paths of a typical new entrant. Figure 4(a) shows the impulse response for the average new manager. It plots two alternative histories of managers that are in a reversal state. The lucky manager experiences a superb year and performs one standard deviation better than she expected, while the unlucky one has a dismal year and performs one standard deviation below expected. Both histories are conditional there being on no market crashes. The upper left plot shows how on paths with no crashes, the unlucky manager’s reputation rebounds after the first year, while the reputation of the lucky manager slowly drifts down. This mean reversion in reputations is a consequence of the different degrees of aggressiveness with which both managers pursue the long-term opportunity. Limits to arbitrage sharply increases for the
unlucky manager, while it decreases for the lucky one (upper right plot), going from 1.5% to 3.6% per year at the end of an unlucky year, and then slowly reverting back. By the end of year three, the unlucky manager still expects to have 0.5% lower expected return than when she first entered and roughly 1.3% lower expected return than the lucky manager. These managers are ex-ante identical, but the different return histories induce them to make different choices. For the unlucky manager, the difference in expected returns is compensated by a faster reputation build up due to their higher expected short-term performance. As emphasized throughout the paper, investors understand the manager’s incentives and in equilibrium adjust the way they learn about their manager. Investors’ endogenous learning behavior shows up in the short-term performance benchmark $\mu^dB_t$ and how much of performance surprises investors attribute to skill $\Delta^dB_t$. The two plots in the bottom of Figure 4(a) show that these two quantities reinforce the original shocks. The benchmark becomes higher for the under-performing manager and short-term performance becomes more informative. Both effects reinforce the manager’s incentive to prioritize short-term performance at the expense of expected returns.

While Figure 4(a) explores alternative histories given possible return realizations for realized paths without crashes, a more informative way to study the quantitative relevance of the mechanism is to study the system dynamics across all possible histories. Figure 4(b) plots the expected limits to arbitrage for the average new entrant after $t$ years of survival. The plot factors in all endogenous dynamics of the system, expected reputation build up due to short and long-term performance, and selection effects that cut the left tail of underperforming managers. For the average new entrant, limits to arbitrage start high and slowly decrease as surviving managers build their reputations. The distortion is quantitatively large. Limits to arbitrage amounts to an average of 1.9% initially, implying that roughly one third of the manager timing skill is wasted due to limits to arbitrage. One important disciplining aspect of the calibration is the new entrant attrition ratio, which implies that the average new manager cannot start too close to the liquidation threshold where limits to arbitrage are larger, otherwise it would imply much higher attrition ratios than are consistent with the data. Another important aspect of modeling learning and reputation dynamics explicitly is that now we can say something about how quickly these distortions fade out over the manager career. Due to the high attrition rates imposed by the baseline calibration, limits to arbitrage start high but ease relatively quickly, with a half life slightly longer than one
Figure 4: Limits to Arbitrage Dynamics

Figure (a) displays the impulse response functions with respect to a positive and negative one-year one-standard-deviation shock to fund performance. The continuous line plots the faith for the lucky manager, the dashed line for the unlucky manager. The three additional panels show the limits to arbitrage, short-term benchmark, and short-term performance informativeness for the two alternative histories. Figure (b) displays how limits to arbitrage evolve over time. On the x-axis we have the number of years since manager entry, and on the y-axis we have the average limits to arbitrage conditional on survival. In the left panel I show how limits to arbitrage changes with short-term performance informativeness. The center panel shows how it changes with the frequency of the crashes. The right panel shows the limits to arbitrage for different informativeness of crash performance.

An important dimension for the level and evolution of limits to arbitrage is how learning takes place. Investors can learn from short-term performance and long-term performance. Learning from short-term performance is smooth and frequent, and is the typical channel analyzed in the literature that followed Berk and Green [2004]. Learning from long-term performance is lumpy and infrequent, and takes palace during the large and infrequent market crashes. A manager who positions her portfolio to increase the funds expected short-term performance, knows her reputation growth next interval will be higher than otherwise would have been with very high probability. On the other hand a manager that positions the
portfolio to do well during crashes knows that for any short time horizon it is unlikely that this high expected return bet will translate into an increase in her reputation. Since in the model and in reality, long-term strategies typically have a negative carry, performing well during crashes implies a poorer performance in the short-term. This logic explains why the strength of the performance signal matters a great deal. Comparing the different cases in Figure 4(b), we see that as the signal-to-noise ratio $\Delta_{dB}$ increases from 0.66 to 0.99, the initial level of limits to arbitrage increase from 1.3 to 3.3. Entry managers reputations become more volatile and their marginal incentives to boost short-term performance increase, since investors rationally attribute more of it to skill than luck. Standard intuition would suggest that the large distortions which occur when learning is strong should be compensated with a steeper decline due to the faster reputational build up. In Figure 4(b) we can see that the distortion die out at a similar pace. In fact the specification with stronger performance informativeness ends up being more persistent, with a half-life of 15 months, in contrast to 13 months for the baseline case. This intuition fails in the model because when learning is stronger, the manager value function becomes less concave, what implies that short-term reputational concerns take longer to ease out. The dynamics links the level and the intertemporal growth of limits to arbitrage.

The long-term performance channel is impacted by two different parameters, the crash frequency $\delta_\zeta$ and the signal-to-noise ratio of long-term performance $\Delta_{dN}$. Figure 4(b) shows limits to arbitrage dynamics for three different crash frequencies, the baseline ($\delta_\zeta = 0.33$) and $\delta_\zeta = \{0.2, 0.5\}$. In this case, static and dynamic effects reinforce each other. The slower resolution of uncertainty of the long-term reversal strategy leads managers to increase their preference towards the short-term strategy, increasing the initial level of limits to arbitrage. The differences are large, limits to arbitrage increases from 1% to 4% as the horizon of the strategy increases from 2 to 5 years. For the long horizon case, limits to arbitrage is more than 25% of the manager timing skill even five years after the manager entry. In contrast to learning driven by short-term performance, in the case of faster learning from long-term performance results in a reduction in the manager reputation volatility, because investors anticipate that the manager is pursuing the long-term strategy more aggressively. This lower volatility counteract the increase in the expected reputation growth. The result is that

\[\text{Like for short-term performance, long-term performance signal-to-noise ration is an equilibrium object, so the comparative statics regards different } \Delta_{dN} \text{, the signal-to-noise ratio at the efficient portfolio choice.}\]
managers build reputation faster and their reputation concerns die out faster, the two forces reinforce each other leading to a decrease in the persistence with the increase in the learning speed. The half-life of the distortion goes from 2 years to six months as the strategy horizon goes from five years to two years.

Given reasonable parameters, the model generates very large and relatively persistent distortions for strategies with an horizon longer than three years (baseline case). The attrition rate in the industry is especially important to determine limits to arbitrage. It this pins down the horizon of the typical manager, which drives the sensitivity of limits to arbitrage to the strategy horizon. For example in the baseline case there is a 60% probability that the strategy will not pay out during the first year. Given the calibrated attrition ratio of 18% in the first year, this implies that the average new entrant has at least an 11% probability of being liquidated before the strategy pays out. This probability increases to 15% if the strategy horizon increases to five years.

The dynamics of learning during crashes also depends on how informative manager performance is during a crash. In Section 2, I focus on cases where either learning during crashes was perfect $\Delta_{ef}^{dN} \to \infty$ or not present $\Delta_{ef}^{dN} \to 0$. These two extremes were convenient because they effectively shut down the role of long-term reputation concerns on the manager’s portfolio choice. In the first case, the skilled manager understands that as long as she keeps her expected crash performance slightly better than that of a bad manager, investors will learn that she is skilled. This extreme is obviously an artificial consequence of the ability to perfectly detect a jump in a continuous time environment, but it makes the point that when resolution of uncertainty is lumpy, at the portfolio that maximizes expected returns the marginal long-term reputational concerns will always going to be weak. Long-term reputation concerns only discipline the manager once her expected crash performance is low enough, so that the expected after-crash reputation becomes sensitive to the portfolio choice. This motivates the comparison in Figure 4(b), where both low and high informativeness induce much larger limits to arbitrage than the baseline case. In both cases, marginal long-term reputational incentives are weak ($\Delta_{ef}^{dN} = 1$) and under weak learning, the performance during a crash barely impacts the manager’s reputation even when investors have to most optimistic beliefs about the manager portfolio. In equilibrium investors anticipate these weak incentives and learning collapses to zero. In the case of high informativeness ($\Delta_{ef}^{dN} = 100$), marginal long-term reputational incentives are weak, but only at $\pi = -1$. With such an aggressive
portfolio and such a high Sharpe ratio, the manager knows that investors will learn she is skilled with probability one. So from the perspective of her long-term reputation she can afford to reduce the aggressiveness which she pursues the long-term strategy before facing any risk of impacting her after-crash reputation. The baseline case features both a lower level of limits to arbitrage and a lower persistence than the two extremes. The “no-learning” case features a half-life longer than five years, the strong learning case a half-life of 2 years. Limits to arbitrage are lower for moderate levels of long-term performance informativeness because in this case the disciplining impact of long-term reputational concerns is larger. The steeper decline is a consequence of the equilibrium reduction in short-term performance informativeness. Even though the three different specifications share identical short-term performance parameters, the one with intermediate long-term performance informativeness makes better choices, which feeds back into less informative short-term performance.

This analysis highlights how limits to arbitrage dynamics have an intricate relationship with how learning takes place. Managers known for being skilled stock pickers will face a much steeper uphill climb when exploring a long-term strategy. This property of the equilibrium suggests that some sort of strategy segmentation should be desirable. But one also needs to keep in mind that bad managers would have strong incentives to claim they are long-term oriented themselves, since it involves less short-term performance evaluation. The optimal segmentation/bundling of strategy frequency styles likely involves some mix of both, but with long-term oriented managers bundling with relatively weak stock pickers. They need to promise, and deliver, a high enough short-term performance so that the claim of being skilled is not vacuous, but they should not promise too much so that investing in long-term oriented strategies becomes impossible.

In addition to the natural result that limits to arbitrage increasing with the horizon of the strategy, the comparative statics on the long-term performance signal-to-noise ratio can be interpreted as saying that limits to arbitrage will be more severe for economy-wide events, if these are events during which assets have relatively low idiosyncratic risk. Less aggregate bets, for example a play on a firm going bankrupt, should carry more idiosyncratic risk. In such cases the manager understands that she will only get the credit if she really goes all in.

Figure 5(a) shows that if liquidated managers are substituted by new managers with reputations and types sampled from the the calibrated reputation distribution the economy
converges to a steady state in roughly ten years. This assumes that crash states and events are idiosyncratic sampled for each fund, what is a very restrictive condition. Figure 5(b) shows what happens in paths that a crash did not happen, we can see that within three to five years limits to arbitrage more than doubles, reaching 230 bps. This increase in limits to arbitrage is a consequence of a lower average reputation as the average skilled manager lose reputational capital. Skilled manager lose reputational because they initially bet aggressively in the long-term reversal strategy. The benefit of the increase in limits to arbitrage is the increase in short-term performance, that eventually is large enough to stabilize the reputation distribution in a lower level.

### 3.3 Lockups and limits to arbitrage

In section 2.2 we saw that longer lockups improve manager incentives to invest in the high expected return long-term reversal strategy. In this section I show that this effect is quantitatively important and study how the lockup effectiveness depends on different model parameters. In Figure 6(a) I start by comparing the portfolio choices of a manager of an open-ended fund with those of a manager of a one year lockup fund. For high reputations both managers maximize expected returns, as we can see in the left region of the plot. As
their reputations decrease, their portfolios start to diverge. Both become more exposed to limits to arbitrage distortions, since liquidation risk will increase for both of them. The key difference is that liquidation risk is less sensitive to decreases in reputation when the fund has a lockup provision. As we move further past the liquidation threshold, the open-ended fund manager is liquidated and limits to arbitrage pressures start to ease on the lockup fund manager. This happens not because liquidation risk is low, but because it is so high that it is insensitive to any short-term performance boost achievable before lockup expiration. When the manager is underwater her best hope is to focus on the long-term strategy and hope it pays out before investors pull out. In Figure 6 (b) I show the difference in expected short-term performance and short-term performance informativeness across these two different managers. These quantities represent the endogenous performance metrics that investors use to evaluate their managers. Everything else equal, a higher expected short-term performance, implies that the manager’s reputation is more likely to trend downwards, increasing fund liquidation risk. Performance informativeness shapes manager incentives through two distinct channels. More learning implies that the manager’s reputation is more volatile.

On the quantitative side, the calibration implies large reductions in limits to arbitrage for contract maturities in the range of the ones observed in the data. A fund with a one-year lockup contract experiences a reduction in limits to arbitrage from 1.6% to 1% per year in the first year of fund operation. Figure 6 (b) shows that the marginal benefits of longer contracts are strongly dependent on the horizon of the manager strategies, with larger potential benefits when the horizon of the strategy is longer.

In the model, a reduction in limits to arbitrage is the main benefit of a lockup provision. Other potential benefits of restricting withdrawals relate to the illiquidity of the assets in which the manager invests. It might be that there are no limits to arbitrage, but that a fund with a longer lockup is better positioned to capture an illiquidity premium [Cherkes et al., 2009]. Absent informational issues the manager could get the investors to make the right calculation by pricing the assets at the price that the manager would be able to sell. As long as investors who are leaving the fund receive the actual sale value of the fund assets, the remaining investors will have no incentive to get out. It is easy to see how any amount of asymmetric information regarding the manager’s skill would disrupt such an arrangement, giving the manager incentives to keep the fund share value of dropping too low, what will ultimately feedback in investors pulling out too fast.
Figure 6: Lockups and Limits to Arbitrage

Figure (a) compares the equilibrium across a fund with a one-year lockup to an otherwise identical manager of an open-ended fund. The open-ended fund is set to a maturity of one month, which is the typical frequency of performance release in the hedge fund industry. The upper-left panel shows both managers’ equilibrium investment in the long-term reversal strategy (in state $\zeta^R$). The three other quantities are a direct function of this equilibrium choice. In the upper-right we have the equilibrium limits to arbitrage, in the lower-left the short-term reputation benchmark, and in the lower-right the short-term performance informativeness, which is the diffusion volatility of the manager’s reputation. In Figure (b) I plot the limits to arbitrage for a new manager as a function of the lockup maturity. The different lines show the results for different horizons of the long-term reversal strategy. The baseline case is 3 years ($\delta_\zeta = 0.33$), and I also show the results if the expected crash arrival between 2 and 5 years ($\delta_\zeta = 0.5$ and $\delta_\zeta = 0.2$).

The benefit of the lockup arises exactly from the interaction between some form of illiquidity and asymmetric information about the manager. In the model, illiquidity is captured by these Poisson crashes. But the model could easily be modified so that these crashes are events during which fund assets can be sold at their true fundamental value, having to sell the assets at a discount if choose to sell before. So limits to arbitrage should not be seem as an alternative to illiquidity as a rational for long-term contracts, but a complement.

3.4 The Lockup Premium

The model generates a relationship between equilibrium net-alpha and the maturity of the lockup provision. In the model, this premium (the difference between equilibrium net-alphas across a lockup fund and an open-ended fund) measures the entrenchment costs as perceived by investors, and is a function of the cost of investing with a bad manager alpha, the learning
speed, and the degree of competition in the market for skill. The learning speed determines how fast the manager’s track-record will reflect her true skill. The faster learning unfolds, the sooner investors expect to either have their capital returned in case the news are good, or to be locked is a contract with a manager that they know it is likely to be bad.

To understand this key model mechanism it is useful to think about the following example. Supposing investors know that everyone will learn $t$ instants from now the true type of their manager, how high would the lockup premium have to be to convince investors to write a contract of maturity $T$? Suppose expected returns of a skilled (bad) manager is $\alpha (\nu)$. Assume the risk-free rate and the investor best outside option is zero, and that there are no performance fees. Assume also that the market for skill is competitive and rents are extracted once investors learn the manager type with probability $\phi$. In this case the investors get $1 + (1 - \phi)(\alpha - f)T + \phi(\alpha - f)t$ with probability $a$ (in case the manager is skilled), enjoying net expected returns $\alpha - f$ up to time $t$ and receiving their capital back with probability $\phi$. Investors get $1 + T(\nu - f)$ with probability $(1 - a)$ (in case the manager is bad). Assuming $t < T$, we can see that for investors to break even at contract inception we need $a((1 - \phi)(\alpha - f)T + \phi(\alpha - f)t) + (1 - a)T(\nu - f) \geq 0$. In the limit that the contract maturity goes to zero this condition becomes $(\alpha - f)a + (1 - a)(\nu - f) \geq 0$, so the expected net alpha is zero and investors invest with a fund manager as long as she has a reputation higher than $a \geq \frac{\nu}{\alpha - \nu}$.$^{18}$ In the case of a lockup shorter than the timing of resolution of uncertainty, the condition is identical. Since $t > T$, investors do not expect to learn any new information before lockup expiration. In the interesting case where $T > t$, and we have $a \geq \frac{T(\nu - f)}{((1 - \phi)(\alpha - f)T + \phi(\alpha - f)t) + T(\nu - f)}$, and the lockup premium is $\Lambda = \frac{T(\nu - f)}{((1 - \phi)(\alpha - f)T + \phi(\alpha - f)t)(\alpha - \nu) + (\nu - f)}$, which increase in the contract maturity$^{19}$, the speed of resolution of uncertainty, the cost of investing with a bad manager $(-\nu)$, the degree of competition in the market for skill $\phi$. So investors demand a premium in the sense that they only invest with managers who currently have reputations which deliver a high expected net return. The premium determination in the model is more complicated than the above expression because the speed of resolution of uncertainty is endogenous to the manager’s choices and several model primitives, and learning is not binary as in the example.

$^{18}$Where I am assuming for this example that rent extraction is perfect after lockup expiration, this shuts down the possibility of having negative alphas due to the option value of future skill improvements.

$^{19}$Since $(f - \nu) > 0$, $(\alpha - \nu) > 0$, and $\alpha - f > 0$. 

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Figure 7(a) shows that the lockup premium increases from 1.4 to 12% per year as the lockup increases from three months to nine years. The increase is steep at first, but it slowly flattens as the maturity is increased. This concavity is a consequence of the reduction in learning, as investors expect to learn much more about their manager in the first year of investment, then between years size and seven. In fact, investors know that if, by year 5, they are still invested with the same manager, it is because the manager is bad. Because if the managers were good, she would very likely received an outside offer within this period.

We saw in the calibration (Figure 3) that the lockup premium is very sensitive to the bad manager alpha, going from 2 to 6% as the bad manager alpha goes from -2 to 8%, for the baseline case. It is important to emphasize that decreases in $\nu$ also increase the learning speed, which further amplifies the cost of being locked up. This picture indicates that the lockup premium is a useful quantity for learning about investors beliefs regarding the left tail of managers’ skills. However, as we discussed in the last paragraph it also depends on any other parameter impacting the speed of information, in particular $\sigma$, $\alpha^s$, $\delta$, $\zeta^R$, and $\omega$, which impact either the short-term or the long-term performance informativeness. In Table 2 in the Appendix I show the comparative statics for these different parameters. Under the baseline calibration it is typically the case that parameter changes which produce increases in $\Delta dB_{ef}$ will generate an increase in the lockup premium ($\sigma \downarrow, \alpha^s \uparrow$), and quantities that produce an increase in long-term performance informativeness will lead to an increase in the lockup premium ($\omega \downarrow, \delta \uparrow, \zeta^R \uparrow$), but it is not always the case since equilibrium changes in portfolio choice can undo the first round effects.

Competition in the market for skill increases the reputation investors require in order to invest in a locked up fund, and the threshold at which they liquidate a fund. As the market becomes more competitive, investors expect better than expected managers to accept outside job offers faster. This leads them to expect fast dilution by new investors. Similar effects play out in the determination of the liquidation threshold. When outside offers are very infrequent, investors are willing to stick with a manager of lower reputation because they anticipate that any expected loses they are suffering now can be more than compensated if they receive good news about their manager in the future. If the news is bad they can always redeem later. As the market becomes more competitive (outside offers more frequent), current investors capture less of the upside of positive news. Quantitatively we see in Figure 7(b) that when outside offers arrive on average once a year, investors in open-ended funds only
Figure 7: Lockup Premium

Figure (a) shows the level and the slope of the lockup premium as a function of the lockup maturity. The concavity is a direct consequence of the convexity of learning. Maturity increases for short contracts are more costly than for longer ones because investors expect to learn more in the beginning of the relationship than later. The lockup premium is defined in Definition 4, and it is the difference in net abnormal expected returns between a manager of a fund with $T$ years lockup maturity and an open ended fund at the minimum reputation at which the manager attracts investors. Figure (b) shows the lockup premium as a function of the intensity of competition in the market for skill. On the x-axis we have the expected frequency of arrival of outside offers, what measures the degree of competition for managerial skill. The continuous line plots the net alpha of an open-ended fund at the liquidation threshold, the dashed line, the lockup premium.

Liquidate a manager if its expected net alpha is lower than -1.4%, however, as the frequency of offers increases to once a month, this threshold increases to -0.4%. In the limit, as the market becomes perfectly competitive, the net alpha goes to zero as in Berk and Green [2004]. The lockup premium also increases from 3.8 to almost 4%, as the outside offer frequency increases from once every two years to once a month. Competition increases the costs of long-term contracts as it makes entrenchment more frequent.
3.5 Optimal Lockups

If one focuses on funds that have a lockup provision the model produces maturity choices consistent with the observed in the hedge fund data. Aragon [2007b] documents maturity choices as long as 30 months, with the bulk of the managers that have a lockup choosing a one year maturity. Figure 8 shows that optimal choices range from one up to 36 months depending on the manager reputation. The calibrated distribution of entry reputations imply a median contract choice of 13 months, which is in line with the data if we condition on funds that have a lockup provision.

The model logic results in a perverse (but efficient) relationship between the benefits of longer contracts and their feasibility. The right panel of Figure 8 shows the valuation gains of a fund manager who chooses the optimal lockup versus an open-ended contract. For low reputations, valuation gains increase steeply with reputation. These managers have reputations very close to the liquidation threshold and the ability to lock in longer contracts has a very strong effect in decreasing liquidation risk. Gains are large, but managers are constrained as they choose the maximum maturity feasible. Valuation gains start to decrease as the reduction in liquidation risk is less important to managers with higher reputations. They are far from the liquidation threshold and naturally less exposed to the risk of liquidation. For manager with high enough reputations, liquidation risk is low, limits to arbitrage go to zero and they chose short-term contracts. Quantitatively we see that a manager who chooses a one year lockup has a valuation gain of roughly 10% over the value of managing an open-ended fund, and limits to arbitrage goes from 1.8% to 0.7%, an increase in fund expected returns of 1.1% per year.

Now let me discuss how maturity choice is impacted by different parameters. An important driver of lockup choices is the amount of timing skill that the manager has, since only timing skill is exposed to limits to arbitrage. Figure 9 (a) shows that the optimal lockup increases from 7 to 15 months as timing skill goes from zero to 9% per year. The same figure shows that while limits to arbitrage increase less than they would if all managers were forced to manage an open-ended fund, it increases by less due to the increase in lockup maturity. Due to entrenchment costs, the optimally chosen lockup only partially offsets the increase in limits to arbitrage distortions. In this case lockup choices mitigate the increase in limits

\[20\text{I assume the open-ended fund has a maturity of 1 month, which is the typical return reporting frequency for hedge funds.}\]
Figure 8: Optimal Lockup Maturity
The left panel shows the optimal lockup maturity as a function of manager reputation at entry (dashed line) and the reputation density among new managers (continuous line). The right panel shows the valuation (percentage) gains of choosing the optimal lockup maturity relative to managing an open ended fund. The two additional lines show the limits to arbitrage that a manager of such reputation faces when the fund is open-ended and when the fund has an optimal maturity. An open-ended fund is calibrated to have an maturity of one-month as in the standard frequency of performance information release in the hedge fund industry.

to arbitrage due to higher timing skill. Lockup choices have a similar beneficial role with respect to changes in the horizon of the reversal strategy. In Figure 9 (b) in the left plot, the continuous line shows that the optimal lockup maturity increase from 13 to almost 15 months as the strategy horizon increase from 2 to 5 years. As in the case of timing skill, optimal maturities are longer when limits to arbitrage distortions are larger.

In the other two panels of Figure 9 (b) we consider parameter changes for which optimal choices imply shorter contracts exactly when limits to arbitrage are larger. The effect for the alpha of the bad manager is particularly large. As the bad manager alpha goes from -2 to -8% per year, optimal maturities decrease from 50 to 6 months. At the same time limits to arbitrage increase from close to zero to more than 4% per year. Managers choose shorter contract due to higher entrenchment costs in parametrizations for which they would benefit the most of longer maturities due to larger limits to arbitrage. A weaker but similar
effect happens due to competition for manager skill (9 (b), right panel). As competition for manager skill increases, the manager’s incentive to boost short-term performance increases, leading to stronger limits to arbitrage. However, the increase in competition also makes long-term contracting more costly, making optimal contracts shorter exactly when limits to arbitrage are larger. The effect of competition in lockup maturities is substantial. An increase in the frequency of outside offers from once a year to once a month reduce optimal choices from 23 to 11 months.

3.6 Infrequent Redemption Periods

An important feature of lockup contracts that providing suggestive evidence the asset illiquidity story is not the whole story is the fact that the only initially lockup restricts withdraws. After the lockup expires, the shares become fully liquid. The illiquidity story would predict that a portfolio of expired lockup funds would be vastly different from a fund with unexpired lockups. Either way, this kind of arrangement would induce wasteful time-variation into the portfolio the fund holds. This feature of lockup contracts can be well understood in the context of my model. If investors were forced to choose between cashing out and renewing the lockup contract, in the same way infrequent redemption periods work, a longer period would both increase the manager horizon as the baseline lockup contract does, but also increase the reputation threshold for which investors pull their money out. This second force reduces the horizon of the manager. Figure 10 compares limits to arbitrage between two funds with a one year maturity contract. For the lockup fund shares become liquid after lockup expiration, while for the infrequent redemption fund, investors shares become liquid at expiration, but if not redeemed the contract is automatically renewed for another period. In the lower panel I compare the reputation thresholds at which the managers are liquidated. While the liquidation threshold increases with the infrequency of the redemption period, it does not increase with increases in the lockup maturity. As a result, limits to arbitrage distortion decay faster the with contract horizon in the lockup case.

The logic for this result follows from the manager first order condition in Equation 23, which shows that a steeper value function leads to more limits to arbitrage, and Proposition 4, which shows that valuations are steeper for reputations close to the reputation threshold.
3.7 High Water Marks

Most of hedge fund contracts do not feature symmetric performance fees as assumed in the model. An overwhelming majority of funds with high performance fees have a high water mark (HWM) provision that specifies performance payments only after cumulative fund returns (possibly adjusted for some benchmark, typically a risk-free interest rate) have reached a past peak, which is the previous high water mark. Introducing HWM in the model is challenging because in contrast to the literature on HWM, the threshold at which managers are liquidated is endogenous. The fact that investors are choosing optimally when to cash out implies that funds with different distances to the HWM would be liquidated at different thresholds, since, everything else constant, the further the fund net asset value is from the HWM, the more valuable the investment in the fund is for current investors, since the expected performance payments are smaller so investors expect a higher share of the overall pie. In addition to the liquidation threshold, the introduction of HWM leads to strong incentives for renegotiation. A manager with a high reputation that happen to be very far from the HWM can easily attract new investors if they resettle it. They often become “too high to overcome”. They are “too high” because a manager who is still respected can easily attract money to a new fund, and will not have an incentive to deploy her skill towards the losing fund.

Given these challenges, a full solution to a model featuring endogenous manager reputation, lockup contracts, an active market for skill and HWMs is left for future work. But in this section my aim is to give insight into how the results will change if the performance fees were asymmetric and payouts driven by a high-water-mark type contract. I will do that by assuming a fixed and exogenous liquidation threshold $X_L$. Taking this exogenous threshold as given, I will solve for the equilibrium portfolio choice. This approach can show the effect a HWM provision has on the portfolio choice and on the equilibrium limits to arbitrage, but it cannot answer questions regarding the optimal lockup maturity, since the liquidation threshold is exogenous. As we discussed in the calibration section, the liquidation threshold alone is largely inconsequential for the model analysis of limits to arbitrage; what matters is how far from the threshold the manager distorts her choices. This makes the properties of HWMs regarding limits to arbitrage easy to compare if we hold the liquidation threshold constant across the two. Of course the introduction of the asymmetry will likely increase the liquidation threshold when the fund is close to the HWM, and will likely reduce it when the
HWM is far away. However, this is not critical for the analysis since the model was calibrated by matching the rate of attrition across new entrants. If it is true that the HWM drastically changes the investors liquidation/investment policy, this will result in a different implied distribution of reputation among new entrants. The discipline of the calibration makes both results (in terms of the limits to arbitrage distortions that the different contracts induce) comparable even if we do not solve for the optimal liquidation threshold. Since the investors’ liquidation decision will be exogenous there is little to gain from modeling the market for skill, so I assume that there are no outside offers.

Let the fund HWM evolve as $dH = (r - f)dt$ when the fund net asset value (NAV) is below the HWM, and $dH = dNAV$ when the fund NAV is at the HWM. We have that when fund NAV increases by $\epsilon$, from $NAV = H$ to $NAV = H + \epsilon$, a performance fee of $\kappa\epsilon$ is paid out to the manager, fund NAV is reduced by $\kappa\epsilon$, and the HWM is reset to $H' = H + \epsilon$. Let $dH^e$ be this increase at the boundary. Instead of working with the $H$ process it is convenient to work with $h = H - NAV$, which follows $dh = (r - f)dt - dNAV_t$ when $dh > 0$, and $dh = 0$ when $h = 0$. Where $dNAV = dR - f dt$, the manager HJB follows,

$$0 = \max_{\pi} \left[ f + G_h(-\mu(\Omega^{m, \phi}_t, \pi)) - \rho G + G_X \frac{\Delta dB}{\sigma}(\mu(\Omega^{m, \phi}_t, \pi) - \mu_t)^dB + \frac{1}{2}G_{XX}(\Delta dB)^2 + \frac{1}{2}G_{hh}\sigma^2 - G_{hx}\sigma dB + \delta_t E \left[ (k[(\pi \phi + (1 - \phi))(-\zeta_t) + \omega - h_t]^+ + G(X_{t+}, L, Z^{m, \phi}_{t+}, h_{t+}) - G)\mid \Omega^{m, \phi}_t \right] \right],$$

where $h_{t+} = [h_t - ((\pi \phi + (1 - \phi))(-\zeta_t) + \omega)]^+$, and $X_{t+}$ and $Z^{m, \phi}_{t+}$ evolve identically as the baseline case. The boundary conditions imposed by the liquidation threshold and the HWM are $G(X_L, L^e, Z^{m, \phi}_t, h) = 0$ and $G_h(X, L^e, Z^{m, \phi}_t, h)|_{h=0} = -\kappa$. The intuition for the second boundary is as follows. We have that whenever the NAV raises above the HWM, a fraction $\kappa$ is paid out and the HWM is reset: $G(X_t, L^e, Z^{m, \phi}_t, -\epsilon) = G(X_t, L^e, Z^{m, \phi}_t, 0) + \kappa\epsilon$, taking the limit as $\epsilon \to 0$, we have $G_h(X, L^e, Z^{m, \phi}_t, h)|_{h=0} = -\kappa$. During jumps the analysis is similar, with any returns that pushing the fund performance past its previous peak being distributed to the manager and the HWM being resettled.

Before solving the model, I perform one more transformation. Let $z = \frac{\exp(h)}{1 + \exp(h)}$, which is a one-to-one mapping from $[0, \infty] :\to [1/2, 1]$, which is convenient for solving the HJB.
In this case we have that \( dz = z(1 - z)dh \) whenever \( z > 1/2 \), jump behavior is given by \( z' = \frac{e^{\delta} y}{e^{\phi} + 1} \), where

\[
y = \left[ \ln\left(\frac{z}{1 - z}\right) - \left(\pi \phi + (1 - \phi)(-\zeta R) + \omega\right) \right]^+,
\]

the HJB can be written as:

\[
0 = \max_\pi \left[ f + G_z z(1 - z)(-\mu(\Omega_t^m, \pi)) - \rho G + G_X \frac{\Delta t^B}{\sigma}(\mu(\Omega_t^m, \pi) - \mu_t^B) + \frac{1}{2} G_{XX} (\Delta t^B)^2 + \frac{1}{2} G_{zz} z(1 - z_0)^2 - G_{hxx} \sigma \Delta t^B z(1 - z) \right. \\
+ \delta_c (G(X_t, L, Z_t^m, z) - G) + \delta \zeta E \left[ \left[ k[(\pi \phi + (1 - \phi)(-\zeta_t) + \omega - \ln(z_t/(1 - z_t))]^+ + G(X_{t+}, L, Z_t^m, z_{t+}) - G_{t} \right] | \Omega_t^m \right] ,
\]

with modified boundary \( G_z(X, L, Z_t^m, z)(1 - z)|_{z=1/2} = -\kappa \).

I solve this problem using the case of an open-ended fund with a one-month redemption, as I did for the rest of the baseline calculations. In Figure 11(a) I plot the manager contract valuation against the manager reputation and the distance that the manages is from the HWM (in return space). As expected Figure 11(a) shows that a manager close to the HWM has a much more valuable contract as her performance compensation is close to at the money. But from the perspective of limits to arbitrage, we care about the ability of the HWM to improve the manager’s marginal incentives to invest in the long-term strategy. Formally what will determine the impact on limits to arbitrage is how the performance incentives in the symmetric performance case \( (k(\lambda + \delta \zeta R)) \), compare to the one under the HWM contract, which can be written as

\[
\Psi = G_z z(1 - z)(-\frac{\partial}{\partial \pi} \pi \lambda) + \delta \zeta k \frac{\partial}{\partial \pi} E \left[ \pi (-\zeta_t) + \omega - \ln(z/(1 - z)) \right]^+ | \Omega_t^m \right] + \\
\delta \zeta E \left[ G_z(X_{t+}, L, Z_t^m, z') z'(1 - z') \frac{\partial}{\partial \pi} [\ln(z/(1 - z)) - \pi (-\zeta_t) + \omega]^+ | \Omega_t^m \right].
\]

The first term of this expression is the impact of a higher short-term performance on the distance to the HWM. This term can be thought of as substituting \( k \lambda \). As we discussed, \(-G_z z(1 - z) \leq k \), so it always leads the manager to put less focus on short-term performance.
The second term is the performance payment the manager expects from a crash realization. This term takes the place of \(k\delta\zeta C^R\), and as the short-term term, this is always smaller than the marginal incentive provided by the symmetric fee. The intuition is identical as in option pricing, any call option price has an elasticity (delta) to the underlying (weakly) smaller than one. Or more directly one can recognize this term as nothing more than the derivative of a truncated normal with respect to its mean. There is also a third term absent from symmetric fee specification: the continuation value effect of reducing the distance to the HWM during a crash. So this third term is analogous to the first term, but for periods with crashes. Note that the third and second term are negatively related, with the third term being high when the performance option is very out of the money, so the manager expects crash performance to have tiny impact of direct performance payment, but exactly because the option is out of the money the manager expects that all extra performance will reduce her distance to the HWM. One could conjecture that they exactly balance each other out, since

\[
\frac{\partial}{\partial \pi} kE \left\{ -\ln(z/(1-z)) - [\pi(-\zeta_t) + \omega]^+ + [\pi(-\zeta_t) + \omega - \ln(z/(1-z))]^+ \right\} = k\zeta_t.
\]

That is, if the performance payment is not in the money, the impact on the HWM is. But because \(|G_z z^t(1-z')| \leq k\), the incentive to decreases the distance to the HWM will always be weaker than the direct performance incentive. As the manager is pushed away from the HWM her incentive to perform well during crashes is reduced and the maximum marginal incentive happens when the manager is close to it. In addition to making incentives weaker as the manager becomes further from the HWM this also makes marginal incentives decreasing on the portfolio weight \(\pi\), as small changes in the portfolio are unlikely to push the fund above the HWM if the initial portfolio has a low expected return. The hope is that the incentives become weaker but more positively skewed, so that the reduction in \(G_z z(1-z)\) more than compensates for the reduction in the level of incentives. In Figure 11(b) I plot the ratio \(\Psi_{\pi=-1}/k(\lambda + \delta\zeta C^R)\). I plot at \(\pi = -1\), because in this case the performance incentives are maximized under the HWM contract. The figure shows that, as expected, when the manager is close to the HWM her marginal incentives are identical to the symmetric fee case. For high reputation managers, an increase in the distance to the HWM reduces her performance incentives. Skewness incentives have no impact on a high reputation manager because she expects to manage the fund for a long time. As the manager reputation goes down, the skewness induced by the HWM starts to kick in and the HWM can lead to a performance incentive 40% higher than the symmetric fee for the type of positively skewed
bets being considered here. That is for a performance fee of 20%, however, the manager’s implicit marginal incentives can be as high as 28% when she is close to liquidation. This happens because the manager understands that she is more likely to be liquidated in this instance then if reaching the HWM in paths without jumps, so $G_z z(1 - z)$ becomes very small, which leads the manager to favor positively skewed strategies.

Figure 11(b) suggests that the HWM can lead to substantial reductions in limits to arbitrage as the pay for performance structure induces preferences for positively skewed bets exactly when the threat of short-term liquidation induces a preference for negatively skewed bets, which is the source of limits to arbitrage. Unfortunately, this improvement in marginal incentives only happens at the optimal portfolio. As the threat of liquidation pushes the manager to suboptimal portfolios, the likelihood that her return will be high enough to reach the HWM in the next crash is reduced, which results in weaker marginal incentives.

While casual intuition that the HWM induce a preference for skewness is basically correct, it only does so for low reputation managers and for portfolios which expected crash performance is high enough to make the performance option in the money. In Figure 11(c) I compare the limits to arbitrage for a fund with symmetric fees versus a similar fund with a HWM contract with the performance option at the money ($h = 0$) and a fund that needs a 50% return to reach the HWM ($h = 50\%$). A fund closer to the HWM has better portfolio choices when the manager has a high reputation, but makes worse choices when the reputation is low. A manager far from the HWM chooses better portfolios (relative to a manager close to it) for low reputations but worse ones when her reputation is high. The fourth line is the lower envelope of portfolio choices across all distances to the HWM for a given reputation level. There is only a small range of reputations for which the HWM does not perform worse than the symmetric fee, and as the upper envelope line shows it sometimes leads to substantially worse outcomes.

This is not meant to be a full analysis of HWM in a setting with learning. Interesting interactions should arise between the distance to the HWM, the investor decision to cash out, and the market for skill, but this is left for future work. But adding these new pieces is unlikely to impact the conclusion of this section. HWM contracts do not reduce limits to arbitrage in a quantitative relevant way relative to the symmetric case considered in the paper, and might even lead to substantially larger distortions. If anything, introducing it in the main analysis is likely to lead lockup provision to play an even larger impact in mitigating
these distortions, as the preference for positive skewness that the HWM contract induces among low reputation managers acts a complementary force to the reduction in preference for negatively skewed bets induced by the lockups. However, a high water mark contract alone is not enough to mitigate the distortion and generates quantitatively similar results to the main analysis with symmetric fees.

4 Conclusion

This paper presents a dynamic model where rational investor behavior leads managers to favor momentum type strategies at the expense of more profitable long-term bets. The paper highlights a fundamental delegation friction, our inability to distinguish in the short-run a solid long-term strategy from lack of skill, and shows that this friction generates large limits to arbitrage leading managers to be over-exposed to large market crashes. I show that limits to arbitrage is a robust feature of an environment with uncertainty about manager skill and time-variation in the horizon of the manager investment opportunities: a wide range of parametrizations leads to quantitatively large limits to arbitrage. Lockup maturities like the ones observed in the hedge fund data go a long way towards reducing limits to arbitrage, but they come with the cost of more managerial entrenchment. The commitment not to liquidate a manager, induced by a lockup provision, improves a manager’s ex-ante incentives to invest in long-term opportunities, but entrench bad managers ex-post. From this intuitive trade-off I show that the model has implications for the optimal contract maturity. This trade off is often perverse, with optimal lockups becoming shorter exactly for parameters which limits to arbitrage are larger. In this sense even optimally chosen lockup contracts are too short.

On the practical side, the model tells us to ask four questions when evaluating or designing a management contract: What is the typical horizon of the manager investment? How informative is performance about the manager ability? How competitive is the market for managers? What is the cost of lack of skill? While the model is applied to hedge funds the fundamental friction studied in this paper can be extended to the problem of compensating CEOs in financial and non-financial corporations. Whenever there is uncertainty about managerial skill, manager actions that are hard to evaluate, profitable opportunities that sometimes pay poorly in the short-run, and there is an active market for manager skill, the insights developed in this paper apply.
References


5 Appendix

5.1 Solution

5.1.1 Managers

Proposition 10. (Manager valuations are increasing in her reputation)

Let $\delta_a = 0$, $U = 0, f \geq 0, k \geq 0, \alpha^s > 0, \alpha^m \geq 0$, $\Delta_{ef}^{dB} > 0$, and fix the liquidation threshold $X_L$, and the manager portfolio $\pi$ then the skilled manager valuation is increasing in her reputation ($G_X > 0$).
Proof. If no outside offers are made ($\delta_a = 0$), then the manager reputation only impacts her valuation through the liquidation event. The conditions on fees and manager skill imply that the manager has more value alive than when liquidated. If one hold manager portfolio constant (and investors beliefs) it is trivial that for reputation levels $X_1 > X_2$ for any interval $\Delta$, $Pr(X_{t+\Delta} < X_L\mid X_1) < Pr(X_{t+\Delta} < X_L\mid X_2)(*)$. In short the distribution of $X_{t+\Delta}$ conditional on $X_1$ first order stochastic dominate the distribution conditional on $X_2$. The probability of fund liquidation is given by the first time the reputation reaches threshold $X_L$ and the lockup is expired. Since the lockup expiration process is independent of the initial reputation, the probability of fund liquidation will be increasing in probability of hitting $X_L$. Since $X_{t+\Delta}\mid X_1$ FOSD $X_{t+\Delta}\mid X_2$, it follows that the the hitting time probability is weakly smaller for $X_1$ than for $X_2$. It is intuitive that $X_{t+\Delta}\mid X_1$ FOSD $X_{t+\Delta}\mid X_2$, since both share the same distribution, to see it formally, $X_{t+\Delta}$ can be written a mixture of normally distributed random variables (see Ait-Sahalia [2004]).

$$p(X_{t+\Delta} < z\mid X, \zeta) = \sum_{n=0}^{1} e^{-\frac{\delta_a \Delta}{n!}} \Phi \left( \frac{z - \frac{\Delta d_B}{\sigma} (\alpha + \pi \lambda - \mu d_B) \Delta + \frac{\Delta d_N}{\omega} (\pi \zeta - \mu d_N) - X}{\sqrt{\frac{\Delta d_N \omega}{\Delta d_n \zeta}} + \Delta \left( \frac{\Delta d_B}{\omega} \right)^2} \right)$$

$$+ \sum_{n=2}^{\infty} \sum_{n_R=0}^{n-1} (p_R)^n (1 - p_R)^{n-n_R} e^{-\frac{\delta a \Delta}{n!}} \phi \left( \frac{z - \mu(1, n, n_R, \Delta) - X}{\sqrt{\frac{\Delta d_N \omega}{\Delta d_n \zeta} + \Delta \left( \frac{\Delta d_B}{\omega} \right)^2}} \right)$$

where

$$\mu(1, n, n_R, \Delta) = \frac{\Delta d_B}{\alpha + \pi \lambda - \mu d_B} \Delta + \frac{\Delta d_N \omega}{\pi \zeta - \mu d_N} + n_R \frac{\Delta d_N \omega R}{\omega} (-\pi \zeta + \mu d_N \zeta) + (n - 1 - n_R) \frac{\Delta d_N \omega M}{\omega} (-\pi \zeta + \mu d_N \zeta)$$

Since for each of these normal distributions we have $\Phi \left( \frac{z - \mu(n_0, n, n_R, \Delta) - X_1}{\psi(n_0, n, n_R, \Delta)} \right) < \Phi \left( \frac{z - \mu(n_0, n, n_R, \Delta) - X_2}{\psi(n_0, n, n_R, \Delta)} \right)$ if $X_1 > X_2$, it follows that $p(X_{t+\Delta} < z\mid X_1, \zeta) < p(X_{t+\Delta} < z\mid X_2, \zeta)$. Under these assumption
we can rewrite the manager valuation in (14) as 

\[ G = \int_0^\infty e^{-rt} E [k(dR_t - (f + r)dt) + f dt | X] (1 - \text{Prob}(\tau_l < t | X))dt, \]

that is the sum of the discounted expected cash-flows weighted by the probability of being alive. Since portfolio choice is assumed to be fixed across reputations, the expected cash flows are independent of \( X \), so the the manager valuation depends (negatively) on the probability of fund liquidation. Since we proved that this is decreasing in reputation, it follows that manager valuations are increasing in reputations \( (G_X > 0) \).

Proof of Proposition 4. When \( \omega \to 0 \) learning is perfect during crashes. The results hold more generally, but this “no learning in crashes” case is convenient as the manager HJB becomes a system of four ODE’s with constant coefficients, with boundary condition \( G(X_t, L^e, \zeta) = 0 \) for \( X \leq X_L \). The constructive proof is almost identical if learning is perfect, that is if the manager reputation jumps to \( \infty \) the first time a crash arrives. The four different ODES are for states \( \{\zeta, L\} \in (\{\zeta^R, \zeta^M\} \times \{L = L^e, L = L^a\}) \)

\[
0 = k(\alpha + \pi(\lambda - \delta_\zeta \zeta^R - f)) + f - rG + G_X \frac{\Delta dB}{\sigma} (\alpha + \pi \lambda - \mu^I dB) + \frac{1}{2} G_{XX} (\Delta dB)^2 + \delta_\zeta G(X_t, L^e, \zeta) - G
\]

Under the assumptions of the proposition, the only difference between the ODEs are the \( \zeta \) state that shows up only in \( k(\alpha + \pi(\lambda - \delta_\zeta \zeta^R - f)) + f \), and the transition form lockup to open fund. Let me start with the open fund case, since there are no outside offers the open fund never transitions back to lockup fund, then the ODE system can be decoupled from the lockup odes:

\[
0 = k(\alpha + \pi(\lambda - \delta_\zeta \zeta^R - f)) + f - rG(X, \zeta^R) + G_X(X, \zeta^R) \frac{\Delta dB}{\sigma} (\alpha + \pi \lambda - \mu^I dB) + \frac{1}{2} G_{XX}(X, \zeta^R) (\Delta dB)^2 + \delta_\zeta 0.5 (G(X, \zeta^M) - G(X, \zeta^R))
\]

\[
0 = k(\alpha + \pi(\lambda - \delta_\zeta \zeta^M - f)) + f - rG(X, \zeta^M) + G_X(X, \zeta^M) \frac{\Delta dB}{\sigma} (\alpha + \pi \lambda - \mu^I dB) + \frac{1}{2} G_{XX}(X, \zeta^M) (\Delta dB)^2 + \delta_\zeta 0.5 (G(X, \zeta^R) - G(X, \zeta^M))
\]

Let \( W(X) = G(X, \zeta^M) - G(X, \zeta^R) \), subtracting both equations we obtain

\[
0 = k\pi \delta_\zeta (\zeta^R - \zeta^M) - rW(X) + W_X(X) \frac{\Delta dB}{\sigma} (\alpha + \pi \lambda - \mu^I dB) + \frac{1}{2} W_{XX}(X) (\Delta dB)^2 - \delta_\zeta W(X)
\]
Check that $W(X) = \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z}$, satisfies this equation. This tells us that $G_X(X, \zeta^M) = \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z} + G_X(X, \zeta^R)$. The system simplifies to one ODE,

$$0 = k(\alpha + \pi(\lambda - \delta_z \zeta^R) - f) + f - rG(X, \zeta^R) + G_X(X, \zeta^R)\frac{\Delta dB}{\sigma}(\alpha + \pi \lambda - \mu^L dB) + \frac{1}{2}G_{XX}(X, \zeta^R)(\Delta dB)^2 + \delta_z 0.5 \left( \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z} \right)$$

Which homogeneous solution can be easily found to be:

$$G(X, \zeta^R) = \frac{k(\alpha + \pi(\lambda - \delta_z \zeta^R) - f) + f + 0.5\delta_z \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z}}{r} + K_1 e^{X \eta_1} + K_2 e^{\eta_2}$$

with $\eta_1 < 0 < \eta_2$. There are two relevant boundary conditions $G(X_L, \zeta^R) = 0$ and $\lim_{X \to \infty} G(X, \zeta^R) = \frac{k(\alpha + \pi(\lambda - \delta_z \zeta^R) - f) + f + 0.5\delta_z \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z}}{r}$, the value that the manager would earn if managing the fund forever. The second boundary implies $K_2 = 0$, since $\eta_2 > 0$. $K_1$ is determined by:

$$\frac{k(\alpha + \pi(\lambda - \delta_z \zeta^R) - f) + f + 0.5\delta_z \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z}}{r} + K_1 e^{X \eta_1} = 0$$

then

$$G(X, \zeta^R, L^c) = \begin{cases} \frac{k(\alpha + \pi(\lambda - \delta_z \zeta^R) - f) + f + 0.5\delta_z \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z}}{r} (1 - e^{(X - X_L) \eta_1}) & \text{if } X > X_L \\ 0 & \text{if } X \leq X_L \end{cases}$$

It is trivial to check that $\text{Sign}[G_{XX}] = \text{Sign}[G \times (-n_2)] < 0$. So managers of open-ended funds are always averse to reputational risk. Now lockup funds,

$$0 = k(\alpha + \pi(\lambda - \delta_z \zeta^R) - f) + f - rG(X, \zeta^R, L^a) + G_X(X, \zeta^R, L^a)\frac{\Delta dB}{\sigma}(\alpha + \pi \lambda - \mu^L dB) + \frac{1}{2}G_{XX}(X, \zeta^R, L^a)(\Delta dB)^2 + \delta_z 0.5 \left( G(X, \zeta^R, L^a) - G(X, \zeta^R, L^a) \right)$$

$$0 = k(\alpha + \pi(\lambda - \delta_z \zeta^M) - f) + f - rG(X, \zeta^M, L^a) + G_X(X, \zeta^M, L^a)\frac{\Delta dB}{\sigma}(\alpha + \pi \lambda - \mu^L dB) + \frac{1}{2}G_{XX}(X, \zeta^M, L^a)(\Delta dB)^2 + \delta_z 0.5 \left( G(X, \zeta^M, L^a) - G(X, \zeta^M, L^a) \right)$$

Following identical steps as for open-ended fund we have that $G(X, \zeta^M, L^a) - G(X, \zeta^R, L^a) = \frac{k\pi \delta_z (\zeta_R - \zeta_M)}{r + \delta_z}$.
Manager \(G\) then the value of a lockup fund is of the form

\[
G(X, \zeta^R, L^e) = \frac{k(\alpha + \pi(\lambda - \delta_\zeta^R - f) + f + 0.5\delta_\zeta^R \frac{k\pi\delta_\zeta^R(\zeta^R - \zeta^M)}{\sigma + \delta_\zeta^R}}{X} + K_1 e^{X\eta_3} + K_2 e^{X\eta_4}
\]

Below the liquidation threshold \(G(X, \zeta^R, L^e) = 0\), then

\[
G(X, \zeta^R, L^e) = \frac{k(\alpha + \pi(\lambda - \delta_\zeta^R - f) + f + 0.5\delta_\zeta^R \frac{k\pi\delta_\zeta^R(\zeta^R - \zeta^M)}{\sigma + \delta_\zeta^R}}{X} + K_1 e^{X\eta_3} + K_2 e^{X\eta_4}
\]

with boundary conditions \(\lim_{X \to \infty} G(X, \zeta^R) = \frac{k(\alpha + \pi(\lambda - \delta_\zeta^R - f) + f + 0.5\delta_\zeta^R \frac{k\pi\delta_\zeta^R(\zeta^R - \zeta^M)}{\sigma + \delta_\zeta^R}}{X} + K_1 e^{X\eta_3} + K_2 e^{X\eta_4}\), the value of managing the fund if the manager is liquidated for sure once the lockup expires, this implies \(K_1 = 0\). So below \(X_L\) we have that \(\text{Sign}[G_{XX}] = \text{Sign}[K_2 \times (n^2_4)]\). Since \(\eta_4 > 0\) and \(K_2 > 0\), since we know that \(G(X, \zeta^R, L^a)\) is increasing in reputation, we have that \(\text{Sign}[G_{XX}] > 0\). So managers with reputation below the liquidation threshold that have their investors locked up like reputational risk. To solve for the fund lockup value above the threshold I use the same strategy and solve the valuation of the difference between the value-function, let \(W(X, \zeta^R) = G(X, \zeta^R, L^a) - G(X, \zeta^R, L^e)\)

\[
0 = -rW(X, \zeta^R) + W_X(X, \zeta^R) \frac{\Delta dB}{\sigma}(\alpha + \pi\lambda - \mu^B) + \frac{1}{2} W_{XX}(X, \zeta^R) (\Delta dB)^2 - \delta_c W(X, \zeta^R)
\]

which solution is of the form :

\[
W(X, \zeta^R) = K_3 e^{X\eta_3} + K_4 e^{X\eta_4}
\]

and the roots \(\eta_3 < 0 < \eta_4\) are the same roots from the value function below the threshold. So above the threshold we have \(G(X, \zeta^R, L^a) = G(X, \zeta^R, L^e) = K_3 e^{X\eta_3} + K_4 e^{X\eta_4}\), since as \(X \to \infty\), the manager is not liquidated the value of the open-ended fund has to converge to the value of the lockup fund \(\lim_{X \to \infty} G(X, \zeta^R, L^e) = G(X, \zeta^R, L^e)\), this implies \(K_4 = 0\). Let \(G_{\infty} = k(\alpha + \pi(\lambda - \delta_\zeta^R - f) + f + 0.5\delta_\zeta^R \frac{k\pi\delta_\zeta^R(\zeta^R - \zeta^M)}{\sigma + \delta_\zeta^R}\) be the value of the infinite lived manager, then the value of a lockup fund is of the form

\[
G(X, \zeta^R, L^a) = \begin{cases} \frac{G_{\infty}}{r} (1 - e^{(X - X_L)\eta_1}) + K_3 e^{X\eta_3} & X > X_L \\ \frac{G_{\infty}}{r + \delta_c} + K_2 e^{X\eta_4} & X \leq X_L \end{cases}
\]

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Where it $K_3$ and $K_2$ can be solved for by imposing value-matching ($\lim_{x \to X_L} G(x, \zeta^R, l^a) = \lim_{x \to X_L} G(x, \zeta, l^a)$) and smooth-pasting ($\lim_{x \to X_L} G_X(x, \zeta^R, l^a) = \lim_{x \to X_L} G_X(x, \zeta, l^a)$). See Dixit [1993] for a discussion why these are the relevant boundary conditions for a transitional boundary. This gives us a system of two equations and two unknowns:

\[
\frac{G_\infty}{r}(1 - e^{(x_L - x_L)\eta_1}) + K_3 e^{x_L\eta_3} = \frac{G_\infty}{r + \delta_c} + K_2 e^{x_L\eta_4}
\]

\[
-\frac{G_\infty}{r}(\eta_1 e^{(x_L - x_L)\eta_1}) + \eta_3 K_3 e^{x_L\eta_3} = \eta_4 K_2 e^{x_L\eta_4}
\]

\[
K_3 = e^{-x_L\eta_3} \left( \frac{G_\infty}{r + \delta_c} + \frac{-G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_c} \right)}{(\eta_4 - \eta_3)} \right) 
\]

\[
K_2 = e^{-x_L\eta_4} \left( \frac{G_\infty}{r + \delta_c} \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_c} \right) \right) 
\]

Above $X_L$, $G_{XX} = -\frac{G_\infty}{r}(\eta_1^2 e^{(x_L - x_L)\eta_1}) + \eta_3^2 e^{(x_L - x_L)\eta_3} \left( \frac{G_\infty}{r + \delta_c} + \frac{-G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_c} \right)}{(\eta_4 - \eta_3)} \right)$. Since it can be shown that $\left( \frac{G_\infty}{r + \delta_c} + \frac{-G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_c} \right)}{(\eta_4 - \eta_3)} \right) > 0$ and $\eta_1, \eta_3 < 0$, we see that the risk aversion can alternate between positive and negative depending which root dominates. These roots are function of the model primitives and are given by:

\[
\varrho(\delta_c) = \sqrt{\mu^2 + 2(r + \delta_c)\sigma^2}
\]

\[
\eta_1 = \frac{-\mu - \varrho(0)}{\Delta dB \sigma}
\]

\[
\eta_3 = \frac{-\mu - \varrho(\delta_c)}{\Delta dB \sigma}
\]

\[
\eta_4 = \frac{-\mu + \varrho(\delta_c)}{\Delta dB \sigma}
\]

where $\mu = (\alpha + \pi\lambda - \mu^{l, dB})$. It is easy to see that $\eta_3 < \eta_1 < 0 < \eta_4$. This implies that for reputations large enough the risk aversion term will dominate $\lim_{x \to \infty} \frac{G_{XX}(x, \zeta^R, l^a)}{e^{(x_L - X)^2/2}} = -\frac{G_\infty}{r} \eta_1^2 \leq 0$. Since $\eta_3 < \eta_1 < 0$ holds we have that either $|\frac{G_\infty}{r}(\eta_1^2)| > \eta_3^2 \left( \frac{G_\infty}{r + \delta_c} + \frac{-G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_c} \right)}{(\eta_4 - \eta_3)} \right)$, which case the manager is risk averse for any $X > X_L$, or $|\frac{G_\infty}{r}(\eta_1^2)| < \eta_3^2 \left( \frac{G_\infty}{r + \delta_c} + \frac{-G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_c} \right)}{(\eta_4 - \eta_3)} \right)$. In this case the manager is risk-loving for values above $X_L$ and below $X^*$ where $X^*$ is defined by:

\[\]
\[-\frac{G_\infty}{r} (\eta_1 e^{(X^*-X_L)\eta_1}) + \eta_3 e^{(X^*-X_L)\eta_3} \left( \frac{G_\infty}{r + \delta_e} + \frac{G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_e}\right)}{(\eta_4 - \eta_3)} \right) = 0 \]

\[X^* - X_L = \frac{1}{\eta_1 - \eta_3} \ln \left( \frac{r \frac{\eta_2}{G_\infty \eta_1}}{\frac{G_\infty}{r + \delta_e} + \frac{G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_e}\right)}{(\eta_4 - \eta_3)} \right) \]

Proof of Proposition 7. Proposition 11 shown conditions under which reputation concerns are decreasing in the lockup maturity. Proposition 12 show us that when the long-term reversal strategy is the most profitable the manager first order condition is:

\[k(\lambda - \zeta R \delta \zeta) + G_X \frac{\alpha + pR(\pi - 1)\lambda - \nu}{\sigma^2} \lambda + \frac{(\pi - 1)|R|^2 \delta \zeta E \left[ \int G_X (X + z, L, \zeta) \frac{dz}{\sqrt{\omega^2 (\pi - 1)}} \right] d\Phi}{\frac{dz}{\sqrt{\omega^2 (\pi - 1)}}} \]

In the case of no crash-learning ($\omega \to \infty$), this expression becomes:

\[k(\lambda - \zeta R \delta \zeta) + G_X \frac{\alpha + pR(\pi - 1)\lambda - \nu}{\sigma^2} \lambda \]

With the manager maximizing returns ($\pi = -1$) if it is negative. Note that $\frac{\alpha + pR(\pi - 1)\lambda - \nu}{\sigma^2} > 0$, $k(\lambda - \zeta R \delta \zeta) < 0$ So all we need to show is that $G_X$ is decreasing in these different dimensions. (A) Proposition 11 implies $G_{XX} < 0$ for any $X > X_L$ in the case of open ended funds, so the result follows. (B) Proposition 11 implies $G_{XX} < 0$ for high enough reputations, so the result follows. (A) Proposition 12 shows $\frac{\partial G_X (\cdot | T)}{\partial T} < 0$ it follows that a manager with a longer lockups has stronger incentives to pursue the long-term reversal strategy.

Proof of Proposition 5. Let $\delta = \frac{1}{T}$, let the manager reputation be above the liquidation threshold $X_L$. From proposition 10 we have:

\[G_X = -\eta_1 \frac{G_\infty}{r} e^{(X-X_L)\eta_1} + \eta_3(\delta_e) \left( \frac{G_\infty}{r + \delta_e} + \frac{G_\infty \eta_1 + \eta_3 \left( \frac{G_\infty}{r + \delta_e}\right)}{(\eta_4(\delta_e) - \eta_3(\delta_e))} \right) e^{(X-X_L)\eta_3(\delta_e)} \]

where I made explicit that roots $\eta_3$ and $\eta_4$ depend on the lockup expiration frequency. We need to show that $\frac{\partial G_X}{\partial \delta_e} > 0$ to prove the result. Differentiating the above expression I obtain, \[^{21}\text{Identical argument for perfect learning during crashes ($\omega \to 0$)}\]
\[
\frac{\partial G_X}{\partial \delta e} = (X - X_L)e^{-\frac{(X - X_L)(\mu + \sigma(\delta e))}{\Delta dB}} V \left( \mu^3 + 2(\delta_e + r)\mu \sigma^2 + (\mu^2 + 2(\delta_e + r)\sigma^2) \varphi(0) + (\mu^2 - 2r \sigma^2 + \mu \varphi(0)) \varphi(\delta_e) \right) \\
+ 1 - e^{-\frac{(X - X_L)(\mu + \sigma(\delta e))}{\Delta dB}} \frac{V \Delta dB \sigma \left( \mu(\mu + \varphi(0)) - 2r \sigma^2 \right)}{2r(\Delta dB)^2 (\mu^2 + 2(\delta_e + r)\sigma^2)^{3/2}}
\]

Note that for large reputation the first term dominate, while for reputation close to liquidation the second term dominates. The first term is positive as long \( \mu > 0 \). So \( \frac{\partial G_X}{\partial \delta e} > 0 \Rightarrow \frac{\partial G_X}{\partial T} < 0 \) for high enough reputations. The second term has a positive constant term and a negative term that is decreasing (in absolute value) in the reputation. Note that this second term goes to zero as the lockup maturity goes to zero \( (\delta_e \rightarrow \infty) \). So this shows that reputation concerns are decreasing with increase in lockups when lockups are short-enough, there is a \( T^* > 0 \), such that for any \( T < T^* \), \( \frac{\partial G_X(T)}{\partial T} < 0 \). So we have shown so far that reputation concerns are decreasing in lockup maturity for any maturity for high enough reputations, and for any reputation for short enough maturitie s. to prove the general result we need conditions under which

\[
G_\infty \Delta dB \sigma^3 \left( \mu + \sqrt{\frac{\mu^2 + 2r \sigma^2}{\sigma^2}} - 2r \right) < 1
\]

Which holds easily for reasonable parameters. For instance, volatility is around 0.1, signal-to-noise ratio around 1, drift around 0.1, and the total fund value \( V \) no larger than 10. With such numbers the expression has value of \( \approx 0.01 \). We can also frame this as a condition on the amount of idiosyncratic risk. Recall that \( \Delta dB = \frac{\alpha + \pi \lambda - (\lambda + \nu)}{\sigma} \). The expression is clearly increasing in \( \sigma \) and holds for \( \sigma \rightarrow 0 \). So given the model primitives we can define \( \sigma^* \) implicitly as

\[
G_\infty (\alpha + \pi \lambda - (\lambda + \nu)) \left( \mu + \sqrt{\mu^2 + 2r \sigma^*} - \frac{2r}{\sigma^*} \right) = 1
\]

So \( \frac{\partial G_X(T)}{\partial T} < 0 \) for any \( \sigma < \sigma^* \). This condition is not an important limitation because everything else constant reputation concerns are decreasing in idiosyncratic volatility. So lockups reduce reputation concerns when they are large.

\[\square\]
5.1.2 Investors

Following similar steps as I did for the manager problem it can be shown that the investor value function is homogeneous of degree one in fund size and satisfies a similar HJB equation. It should be intuitive that if the fund returns are independent of fund size, the per dollar investors’ valuations and investors’ decisions should also be unrelated to fund size. Proposition states this intuition formally. Contrasting the investor HJB in equation (28) with the manager HJB (22), we see that because reputation is a martingale under the investor information set, the term that accounts for expected changes in reputation drops off. Note that this martingale property also holds for non-local reputational changes as we can from the last line of equation that implies that $E[X_t|X_t] = X_t$. Another important thing to notice is that expected returns and consequently expected fund growth are a weighted average of expected returns under the different types of managers, where the weight is the manager reputation is probability space.

**Proposition 11. (Investors’ policy)**

Let investors beliefs about manager policies $\pi^I$ be homogeneous of degree zero in fund size and let $a(X) \equiv E[\phi|X] = \frac{e^X}{e^X + 1}$ be the manager reputation in the probability space, and define the liquidation threshold to be the manager reputation such that $X_L|V(X_L, L^e) = 1$, and let the lockup investment threshold to be $X_{L^a}|V(X_{L^a}, L^a) = 1$, then $\forall\{X, L\} \in \{-\infty, \infty\} \times \{L^a, L^e\}$ we have:

1. **Investors liquidation policy** $l^I$ is homogeneous of degree zero in fund size and is of the threshold type with:

$$l^I(X, L) = \begin{cases} 
1 & \text{if } X < X_L \\
0 & \text{if } L = L^a \text{ or } X \geq X_L
\end{cases}$$

2. **Investors offer policies** are homogeneous of degree zero in fund size and have the follow-
ing form:

\[
O^I(X) = \begin{cases} 
1 & X \geq X_L \\
0 & X \leq X_L 
\end{cases}
\]

\[
L^I(X) = \begin{cases} 
L^a & X \geq X_{L^a} \\
L^e & X < X_{L^a} 
\end{cases}
\]

\[
B^I(X) = \beta \left[ V(X, L^I(X)) - 1 \right]^{+}
\]

where \( V(X, L) = 1 = V(X, L^a) \).

3. If \( I^I(X, L) = 0 \), then investors valuation respects the following HJB:

\[
0 = \mu_d + \beta_d (r - f + a(X_t) \Delta^d \sigma + \nu + \lambda) - \rho V + \frac{1}{2} V_{XX} (\Delta^d)^2 + 
+ V (r + (1 - k - \beta_d) (\mu + f + a(X_t) \Delta^d \sigma) - \mu_d) + V_X (1 - k - \beta_d) \sigma \Delta^d 
+ \delta_{p,t} (1 - V) + \delta_c (V(X_t, L^e) - V) + \delta_{\zeta} \beta_d E \left[ a(X_t) \Delta^d \omega - \zeta \right] 
+ \delta_{\zeta} \left( E \left[ \int \left[ (1 - k - \beta_d) (\Delta^d \omega - \zeta + z \omega) + 1 \right] V(X_t + \Delta^d z, L_t) d\Phi(z) dz \right] - V \right),
\]

(28)

4. If \( I^I(X, L) = 1 \), then investors valuations respect the following boundary condition

\[ V(X, L) = 1 \]

Proof. The first step of this proposition is to apply Ito to the manager value function defined in equation (14). This gives us an HJB. We then conjecture that investors and manager policies are homogenous of degree zero in fund size, and that investors valuations are of degree one in fund size. This conjecture proves to be correct as we obtain equation (28) that does not depend on fund size. It follows that the per dollar investor valuation does not depend on fund size. What implies that the offer policies and liquidation policy also do not depend on fund size.

Proof of Proposition 8. (A) A more competitive market for skill means \( \delta_a \uparrow \). The liquidation policy satisfies \( V(X_L, L^e) = 1 \). Optimal outside investors behavior implies \( \delta_{p,t} = \delta_a \) when \( V(X, L^e) > 1 \) and zero otherwise. Since an offer arrival gives investors a capital loss of \( 1 - V \), and the investor never has a capital gain since no offers are made when \( V(X, L^e) < \)
1. Everything else constant an increase in the rate of offer arrival decreases the investors valuation. Since we can write since at the liquidation threshold the manager has to be indifferent between a share in the fund and one unit of cash, it follows that the liquidation threshold has to increase. (B) More limits to arbitrage implies smaller expected returns $\alpha(\Omega^m_1, \pi^m_1)$, what everything else constant implies smaller valuations. But more limits to arbitrage also implies more learning from short-term performance $\Delta^{dB}$↑. Since the liquidation decision is a option on the manager skill this effect increases fund investor value. As $\delta_a \to \infty$ this volatility effect goes to zero as outside offers bid on any positive news. The expected return effect dominates with the liquidation policy increasing with limits to arbitrage. Under the assumption that there is no learning during crashes the HJB becomes an ODE for any reputation higher than the liquidation threshold ($X \geq X_L$):

$$0 = r + (1 - k)(a(X)\alpha(\Omega^m_1, \pi^m_1) + (1 - a(X))\nu - f) - \rho V + \frac{1}{2}V_{XX}(\Delta^{dB})^2 + \delta_a(1 - V)$$

with boundary conditions $V_X(X_L, L^e) = 0$, and $V(X_L, L^e) = 1$. In the case the skilled manager portfolio choice is constant this equation has solution:

$$X_L = \frac{(1-k)e^{x(\alpha(\Omega^m_1, \pi^m_1) - \nu)}}{1 + e^x} + r + \delta_a(1 - k)(\nu - f) + \frac{1}{\delta_a + r} (e^x)^{\frac{1}{2}} \sqrt{\frac{8r + (\Delta^{dB})^2 + 8\delta_a}{(\Delta^{dB})^2}} (1 + e^x)^{-1} K_1$$

where $K_1$ is a constant that is a function of model primitives. The liquidation threshold $X_L = \frac{a_L}{1 + e^{x_L}}$, where constant $a_L$ is given by

$$a_L = \frac{f - \nu}{\alpha(\Omega^m_1, \pi^m_1)} \left( \frac{1}{\sqrt{1 + \frac{8r + (\Delta^{dB})^2 + 8\delta_a}{(\Delta^{dB})^2}}} - (\nu - \sqrt{\frac{8r + (\Delta^{dB})^2 + 8\delta_a}{(\Delta^{dB})^2}} - 1)^{-1} 2f \right)$$

From the above expression it is easy to see that a decrease in expected returns $\alpha(\Omega^m_1, \pi^m_1)↓$ will lead to an increase in the liquidation threshold, but an increase in the performance informativeness will lead to a decrease in the liquidation threshold. Since limits to arbitrage in the model decreases expected returns but increase performance informativeness the overall result is ambiguous. The expression also shows that as the market for skill becomes more competitive, the impact of learning goes away and $\lim_{\delta_a \to \infty} a_L = \frac{f - \nu}{\alpha(\Omega^m_1, \pi^m_1) - \nu}$.

Note that in this case fund gross alpha at the liquidation is equal to the management fees,
\[
\frac{f-\nu}{\alpha(\Omega^m_1, \pi^m_1) - \nu}(\alpha(\Omega^m_1, \pi^m_1) - \nu) + \nu = f. \]
As we discussed before, in this case only expected returns matter. So in this limit case limits to arbitrage increases the liquidation threshold, since the threshold is decreasing in expected returns. Since the threshold is a continuous function of \(\delta_a\) it follows that for any combination of parameters there is a high enough \(\delta_a\) such that the expected return effect dominates.

Proof of Proposition 9. Under the above conditions the investor HJB becomes an ODE for for any reputation lower than the investment threshold \((X \leq X_{L^a})\)

\[
0 = r + (1-k)(a(X)\alpha(\Omega^m_1, \pi^m_1) + (1-a(X))\nu - f) - \rho V + \frac{1}{2}V_{XX} (\Delta d)^2 + \frac{1}{T}(1-V)
\]
with boundary conditions \(V_X(X_{L^a}, L^a) = 0\), and \(V(X_{L^a}, L^a) = 1\). The solution is almost identical as the liquidation threshold

\[
a_{L^a}(T) = \frac{f-\nu}{\alpha(\Omega^m_1, \pi^m_1) - \nu}(\alpha(\Omega^m_1, \pi^m_1) - \nu) - \nu + \left(\sqrt{\frac{8r + (\Delta d)^2 + \frac{1}{8}}{(\Delta d)^2 + \frac{1}{8}}} + \frac{1}{T}(1-V)\right)\Bigg)^{-1} 2f
\]

The lockup premium can be written as:

\[
\Lambda(T) = a_{L^a}(T) (\alpha(\Omega^m_1, \pi^m_1|T) - \nu) - a_L (\alpha(\Omega^m_1, \pi^m_1|0) - \nu)
\]

\[
\frac{f-\nu}{\alpha(\Omega^m_1, \pi^m_1) - \nu}(\alpha(\Omega^m_1, \pi^m_1) - \nu) - \nu + \left(\sqrt{\frac{8r + (\Delta d)^2 + \frac{1}{8}}{(\Delta d)^2 + \frac{1}{8}}} + \frac{1}{T}(1-V)\right)\Bigg)^{-1} 2f
\]

Note that the portfolio choices (and expected returns) of a skilled manager under an open-ended contract can be different from the portfolio of a skilled manager that manages a lockup fund. So skilled managers gross returns can increase or decrease with lockup maturity. The proposition takes as given any such differences. (A) Differentiating this lockup premium with respect to short-term performance signal-to-noise ratio we have that \(\text{Sign}\left[\frac{\partial \Lambda(T)}{\partial \Delta d}\right] = \text{Sign}[a_{L^a}(T)(\alpha(\Omega^m_1, \pi^m_1) - \nu) - 1].\) Now recognize that \(a_{L^a}(T) \geq \frac{f-\nu}{\alpha(\Omega^m_1, \pi^m_1) - \nu},\) which follows
from \( \left( \frac{8r + (\Delta dB)^2 + s a^L}{(\Delta dB)^2} \right) \leq 1 \), \( \left( \frac{8r + (\Delta dB)^2 + s a^L}{(\Delta dB)^2} \right) \leq 1, \) \( a(\Omega^{m1}, \pi^{m1}) > 0, \) and \( f \geq 0 \). This implies \( a_L^L(T) \frac{(a(\Omega^{m1}, \pi^{m1}) - \mu)}{\mu} \geq 1 \) and \( \frac{\partial a(T)}{\Delta a^L} \geq 0. \) (B) Differentiating this lockup premium with respect to the bad manager alpha we have \( \text{Sign}[\frac{\partial a(T)}{\Delta a^L}] = \text{Sign}[a_L^L(T) - 1] = -1, \) as long as the liquidation threshold is below one, what is always true under the relevant assumptions. This says that increases in the bad manager alpha reduce the lockup premium. It follows that increases in the costs of investing with the bad manager, imply larger lockup premium.

5.2 Numerical implementation

I apply the finite-difference method to solve the integro partial differential equations. To solve for optimal policies I sequentially iterate until the value function converges. Three value functions (the good manager’s \((G)\), the bad manager’s \((B)\) and the investor’s \((V)\)), three choice variables (the good manager’s portfolio choice \((\pi^{\zeta_R})\) , investor’s investment policy \((a_L^L, a_L^{L^a})\) ) are determined jointly. Equilibrium offers \( B^I \) are determined once we have the investor valuation \( V \). The state space consists of manager reputation in probability space \((a \in [0, 1])\) , strategy state \((\zeta \in \{\zeta^R, \zeta^M\})\), and the lockup status \( L \in \{L^a, L^e\}\). I solve in the probability space instead of the log-likelihood space which I derived most of the results since it is more convenient as the reputation grid ranges from 0 to 1 instead of minus infinity to plus infinity.

First I hold constant the good manager’s portfolio choice at the efficient choice \( \pi(\zeta^R, a, L) = -1 \) and iterate to find what I denote the efficient solution \((G_{\text{eff}}, B_{\text{eff}}, V_{\text{eff}}, a_{L_{\text{eff}}}^L, a_{L_{\text{eff}}}^{L^a})\). The efficient liquidation threshold \( a_{L_{\text{eff}}}^L \) is the lower bound to the equilibrium liquidation threshold \( a_{L_{\text{eq}}}^L \). Starting from the efficient policies I iterate on the HJB, but this time solving for the optimal portfolio \( \pi(\zeta^M, a, L) = 1\) at each step. Recall that \( \pi(\zeta^M, a, L) = 1, \forall a, L \)

More specifically, the iteration procedure can be divided in steps.

1. Given choices \( \pi^{i-1} \) solve for \( a_{L_{\text{eff}}}^L, a_{L_{\text{eff}}}^{L^a}, V^{i} \) such that \( V^{i}(a_{L_{\text{eff}}}^L, L) = 1, V^{i}(a_{L_{\text{eff}}}^{L^a}, L^a) = 1 \), and \( V \) satisfies the investor HJB.

2. Given \( a_{L_{\text{eff}}}^L, a_{L_{\text{eff}}}^{L^a}, V^{i} \) and \( \pi^{i-1} \) solve for \( G^{i} \) such that the manager valuation satisfies the HJB and \( G^{i}(\zeta, a_{L_{\text{eff}}}^L, L) = 0 \).
3. Given $G^i$ solve for $\pi^i$ using proposition 6

4. If $|G^i - G^{i-1}| < \epsilon$ and $|B^i - B^{i-1}| < \epsilon$ and $|V^i - V^{i-1}| < \epsilon$ and stop. If not repeat .

When solving for the efficient solution, I skip step 3 as the portfolio choice is always held at $\pi(\zeta^R) = -1$. 

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Figure 9: Optimal Lockup Maturity: comparative statics
Figure (a) shows how limits to arbitrage and the optimal lockup maturity changes with manager timing skill. The figure shows that while the reduction of limits to arbitrage due to the optimal maturity choice increase with timing skill, managers with more timing are still subject to larger distortion even when choosing lockups optimally. Figure (b) shows how limits to arbitrage in the open-ended fund case and optimal lockup maturity change with respect to the horizon of the strategy, the expected “abnormal” performance of the bad manager, and the degree of competition. Sometimes contract maturity plays a stabilization role, with longer contracts being chosen when distortions are larger (strategy horizon comparative statics), but often it plays a perverse role, with shorter contracts being chosen as limits to arbitrage become larger (entrenchment and competition).
Figure 10: Lockups versus Infrequent Redemption periods
The top figure shows limits to arbitrage as a function of contract maturity for a fund with a lockup and a fund with an infrequent redemption window. The figure in the middle compares the reputation threshold at which both managers are liquidated. While the liquidation threshold for a lockup fund is insensitive to the contract maturity, since the fund automatically becomes open when the manager’s reputation is not high enough to renew the contract, the same logic is not true for infrequent redemption periods, as investors are forced to decide between cashing out and reinvesting the money for a new black out period. The bottom figure compares the valuation gains for the manager of writing a contract of maturity $T$. The gains are percentage gains relative to the valuation of an open fund. All the results are averages across reputations using the distribution of reputation across new entrants.
Figure 11: High Water Marks

Figure (a) shows the valuation of the hedge fund management contract for a skilled manager as a function of how far she is from her HWM (in return space) and as a function of her reputation (in probability space). Figure (b) compares the marginal performance incentives to investing in the long-term reversal strategy for a manager subject to a symmetric performance fee (baseline) against a manager subject to a HWM. The ratio is computed at the efficient portfolio $\pi(\zeta^R) = -1$. A ratio larger than one indicates regions where the HWM provides stronger marginal incentives.

Figure (c) compares the equilibrium limits to arbitrage for different reputations. It plots limits to arbitrage when the managers is at the HWM (dashed black line), when it is a 50% return from the the HWM (dashed red), and the upper and lower envelope for each reputation. The lower (upper) envelope is the lowest (highest) limit to arbitrage across all possible distances to the high water mark.
Table 2: Quantitative implications of the model

The Baseline column show results for the parameters in Table 1. Quantities are computed by solving the model and simulating the model for 5 years and a large cross-section of funds (50,000). The results are the average across funds. The remaining columns show the same quantities for alternative parameters. In the top row we have the parameter (for example $\alpha^m$), and the alternative values at which the model was solved (3% and 9% for the $\alpha^m$ case).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Baseline</th>
<th>$\alpha^m$</th>
<th>$\nu$</th>
<th>$\sigma$</th>
<th>$\delta^m$</th>
<th>$\delta^b$</th>
<th>$\delta^c$</th>
<th>12.000</th>
<th>1.000</th>
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</thead>
<tbody>
<tr>
<td>Average initial reputation</td>
<td>0.544</td>
<td>0.614</td>
<td>0.482</td>
<td>0.690</td>
<td>0.387</td>
<td>0.584</td>
<td>0.516</td>
<td>0.665</td>
<td>0.501</td>
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<tr>
<td>Liquidation threshold</td>
<td>0.387</td>
<td>0.446</td>
<td>0.434</td>
<td>0.533</td>
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<td>0.407</td>
<td>0.453</td>
<td>0.370</td>
</tr>
<tr>
<td>Median Maturity (Months)</td>
<td>13.333</td>
<td>12.000</td>
<td>14.000</td>
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<td>Lockup premium (%)</td>
<td>3.714</td>
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<td>4.007</td>
<td>5.826</td>
<td>1.552</td>
<td>2.790</td>
<td>4.489</td>
<td>3.959</td>
<td>2.971</td>
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**Optimal Maturity**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Baseline</th>
<th>$\kappa$</th>
<th>$\nu^R$</th>
<th>$\omega$</th>
<th>$\lambda$</th>
<th>$\phi$</th>
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<tbody>
<tr>
<td>Average initial reputation</td>
<td>0.544</td>
<td>0.614</td>
<td>0.482</td>
<td>0.690</td>
<td>0.387</td>
<td>0.584</td>
<td>0.516</td>
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</tr>
<tr>
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**Open (Maturity=1 Month)**

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